Calculus of Variations and Integral Equation
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Lecture. #27

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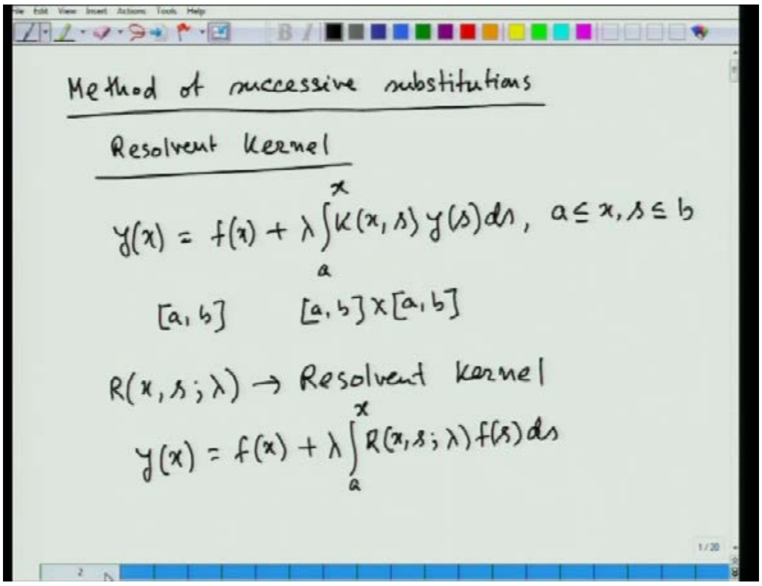

Welcome viewers to the seventh lecture on integral equation under the series of lectures of NPTEL program. We have already discussed various methods for solving volterra integral equation of second kind, which are of non homogenous type. In this lecture, we are going to discuss the last method, that is, specify within the series of lectures; the method is called method of successive substitution. And actually, we are going to solve this problem by using the concept of Resolvent kernel. So, method just we are going to discuss that is method of successive substitutions.

And main concept involved in this technique is actually known as Resolvent kernel. If we recall from the earlier lectures that in some methods, we have started with zero with order approximation that is y 0(x) equal to f(x) or y 0(x) equal to x or y 0(x) equal to 1. And we have calculated successive iterates y n, such that, this y n converges to the solution of the given problem. Also in case of series solution method, we have assume the existence of the solution in terms of a power series, then we have calculated C 1, C 2, C 3, and so on; and finally, we have obtain the solution to the given problem.

Now, in all those problems we never done anything with the kernel. Now, in this technique what we are going to do, first of all we are assuming that this is the given equation y(x) equal to f(x) plus lambda times integral a to x k(x,s) y(s) ds, where a less than equal to (x,s) less than equal to b. And already from the previous discussion, you can recall if f(x) is continues; over the close interval (a,b) and k(x,s) is continues over the square domain (a,b) cross (a,b), then solution of this particular problem exists whatever method we consider. Now, here we are in been going to work on the kernels involved with the given problem, we construct some iterative kernel.

And then, taking some of those iterated kernel, we find out the Resolvent kernel; and Resolvent kernels will give us the solution of the given problem. So, first of all I am just writing the notation for Resolvent kernel, and then I will justify how this things comes into the picture. If we denote that R(x, s, lambda) this stands for Resolvent kernel, if this stands for Resolvent kernel then solution to the given problem can be obtained as y(x) equal to f(x) plus lambda times integral a to x R of x s lambda f(s) ds. So, this clearly shows that if we know the Resolvent kernel, then, with the help of the known function f(s), we can find out the solution to the given problem. Now, question is from where we can find out this Resolvent kernel R(x,s).

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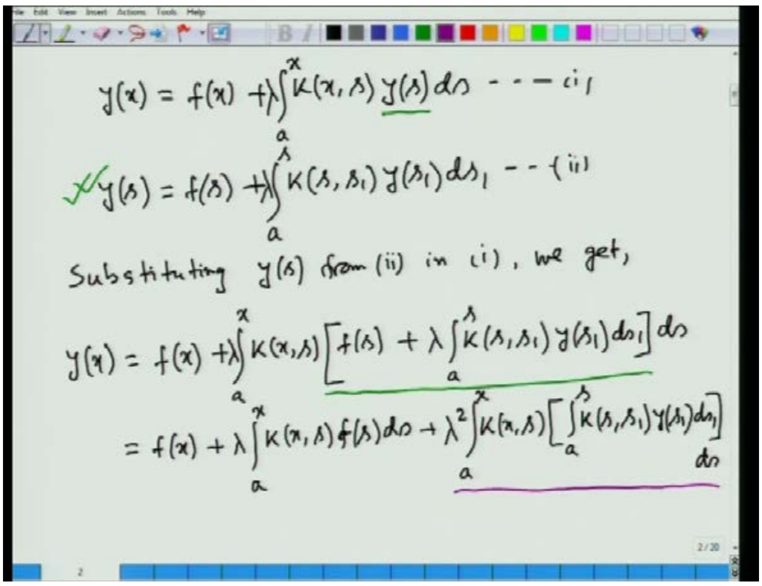

So, just we look at the given equation y(x) is equal to f(x) plus integral a to x k of (x,s) y(s) d s; this is the given equation. Now, we are going to replace this y(s) which is express in terms of this type of expression, and from there we can see that f(x) will be involved under the integral sign. And again we are left with one y(s) term, and successively will be updating this particular process with the new expression, we are getting for y x. So, first of all using this expression, we can write y(s) this will be equal to f(s) plus integral a to s k of (x,s) 1 y(s 1) ds1. First expression was y(x) equal to this, like this first integral equation or in the first expression y(s) was involved.

Now, here we have written y(s) in this format, using this particular given problem that is y(x) equal to this 1. Now, we are going to substitute this y(s) on to the right hand side of first expression call it 1, this is 2. So, then substituting y(s) from 2 in 1, we get… So, reducing this expression for y(s) into 1, and we can find y(x) is equal to f(x) plus integral a to x k of x comma s, then y(s) is now f(s) plus extremely sorry I forgot lambda here.

This is lambda plus lambda integral a to s k of s comma s 1 y(s 1), this entire expression with respect to ds 1 ds. So, just try to understand here, this y(s) we have used this expression, and this y(s) is equal to this entire expression. So, from here we can write this is equal to f(x) plus lambda times integral a to x k of (x,s) f(s) ds plus lambda square integral a to x k (x,s), then integral a to s k of s comma s 1 y(s 1) ds 1, and then ds. So, now, we just try to concentrate upon this particular integral. And our target is we are going to write this integral as iterated kernel of k with y(s), and then its integral that is actually our main target how this can be done.

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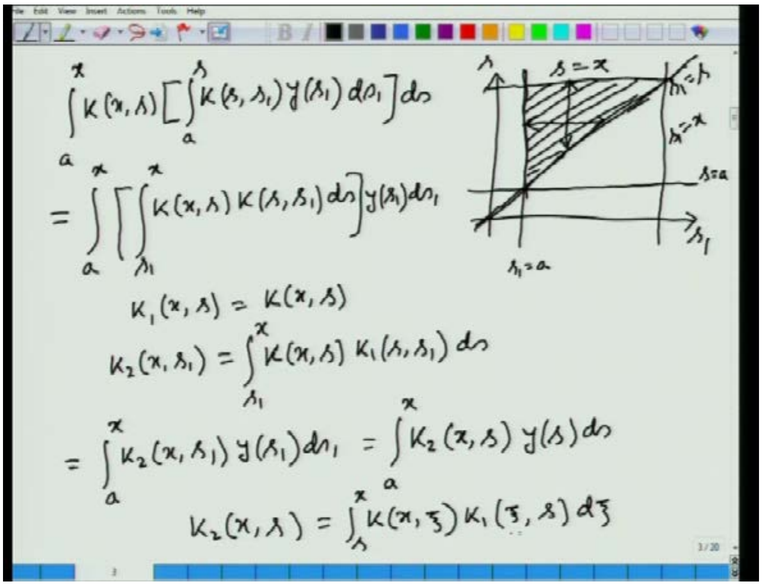

So, we can write this as integral a to x f k of x comma s, then integral a to x k s comma s 1 y(s 1) ds 1 ds. Now, our main target is just change of order of integration. So, if we look at this region of integration over s 1 s plain, this is s 1 direction, this is s direction. So, this particular line as by equation s 1 equal to a, this line as the equation s equal to a. So, first of all we are integrating with respect to s 1 from a to s 1 equal to s. Now, you can see this is the line s 1 equal to s. So, s 1 varies from s 1 equal to a up to s 1 equal to s.

This is the range of indentation of s 1, and then s varies from a to x. So, that means, if we consider this line as s equal to x. So, this is actually our desired region of integration, and now we are actually intended to inter change the order of integration. So, that means, first of all we are going to write integration with respect to x. So, this will be k (x,s) k(s, s 1) with respect to ds, because this y(s 1) is free from s.

So, in this region the range of s 1, s is going to be from s equal to s 1 to s equal to x, on this line we are actually having s 1 equal to s. So, that means, s is ranging from this limit s equal to s 1 up to s equal to x. So, this range will be s 1 to x, this entire integral multiplied with ds 1 y(s 1) . And finally, range of s 1 will be from this line up to this line, and you can easily verify equation of this line is going to be s 1 equal to x, because coordinator of this particular point is (x,s). So, this will range from s 1 equal to a to s 1 equal to x. So, this will be the range.

So, ultimately we have this particular expression. Now, we can define this quantity k 1(x,s), this is equal to k(x, s); and k 2(x,s 1) from here is integral s 1 to x k of x comma s, and this will be k 1(s,s 1) ds. So, if we use this particular notation k 2 (x, s 1) as this integral, then this expression becomes this is equal to integral a to x k 2(x,s 1) y(s 1) ds 1. Now, this s 1 is a dummy failure; this s 1 is a dummy failure. So, therefore, without any loss of generate, we can write this is equal to a to x k 2(x,s) y(s) ds, this is our expression that is a to x k 2(x,s) y(s) ds.

Now, once we write k 2(x,s) here. So, then in order to put this as in terms of k 2(x,s) we can write k 2 (x,s) that is equal to integral s to x k(x,psi) multiplied by k 1(psi,s) d psi. Just look at the change of variables, here we are going to write k 2(x,s).

So, s 1 is replaced by s. So, on the right hand side we have to replace s 1 by s. So, this limit is going to s, and this s 1 is s here; and then the dummy variable s is change to psi. So, ultimately it results in k(x,psi), k 1(psi,s) d psi, now at this point 1 question may comes in your mind, that here I have denoted k (x,s) is equal to k(x,s), k 1(x,s) equal to k(x,s). Now, why I am writing k 2(x,s 1), I have written here k (x,s), k 1(s,s 1). So, why not both of them are k 1; that means, k 1(x, s), and k 1 x s comma s 1 or why I am not writing k 1 x comma s comma k s comma s 1. This point will be clear, if we look at some next points, because in this way we are actually going to define the iterated kernels. In this way that is k 1(x,s) equal to k(x,s), then k 2 (x,s) equal to this one, in this way we can define the iterated kernel.

So, now our main target is that at this point from the previous slide you can recall, we was intended to reduce this particular double integral. And already we have reduced it, so that means, this integral this one is now simply equal to a to x k 2 (x,s) y(s) ds. Now, if we go back to the original expression from where we have started that is y(x) equal to this one.

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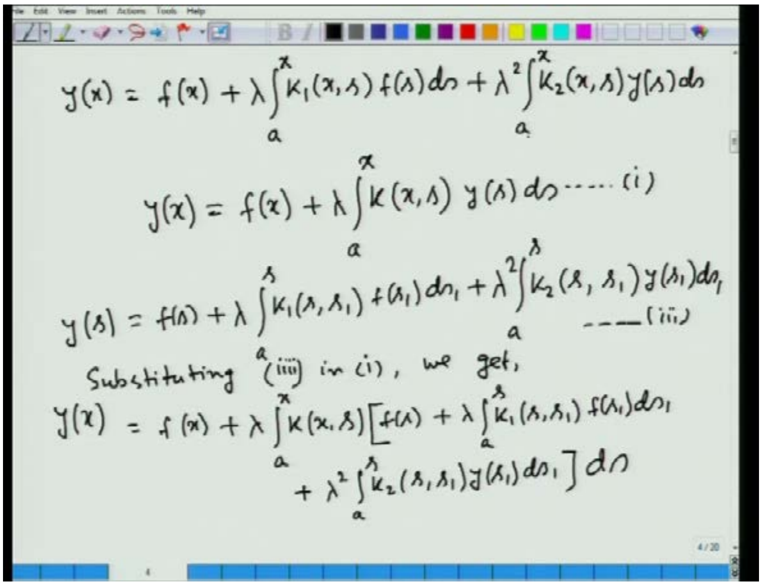

So, ultimately we are having y(x) this is equal to f(x) plus lambda times integral a to x k 1(x,s) f(s) ds plus from the last night you can recall this will be a to x k 2(x,s) y(s) ds plus lambda square integral a to x k 2(x,s) y(s) ds, this is expression for y(x). Now, again we will be substituting this expression into this particular equation y(x) equal to f(x) plus lambda times integral a to x k(x,s) y(s) ds. So, now, this y(s) will be replaced by the expression for y, what we have obtained here. So that means, we can write y(s) this is equal to f(s) plus lambda times integral a to s k 1(s,s 1) f(s 1) ds 1 plus lambda square integral a to s 1 k 2(s,s 1) y(s 1) ds 1.

So, this is the expression for y(s); this expression will be substituted here. If you substituted here, then you will be having y(x). So, this is one, all this expression as free. So, substituting 3 in 1, we get. This y(x) equal to f(x) plus lambda integral a to x k of(x,s), then y(s) will be replaced by f(s) plus lambda times internal a to x k 1(s,s 1) f(s 1) ds 1 plus lambda square integral a to s k 2(s,s 1) y(s 1) d s1 times ds. So, this is the expression.

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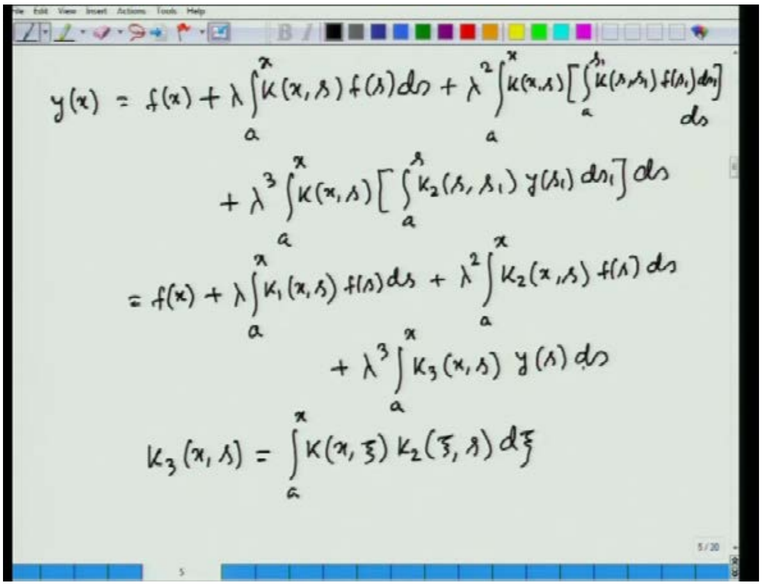

Now, if we rearrange all this terms, then will be having y(x) this is equal to f(x) plus from the first term will be having k(x,s) f(s) ds, because this term does not involve any s 1 term. So, this will be plus lambda times integral a to x k of (x,s) f(s) d s plus second term will be this lambda multiplied with lambda is lambda square, and then this double integral. So, this will be plus lambda square integral a to x k of (x,s), then integral a to s 1 k(s,s 1) f(s 1) ds 1 ds plus lambda cube, that is the last term; this lambda into lambda square. So, lambda cube k(x,s), and then this integral.

So, it will be plus lambda cube integral a to x k of (x,s), then integral a to s k 2(s,s 1) f(s 1) , this will be y(s 1) ds 1 ds. Now, you try to recall that this second integral is nothing but what we have obtain in the last step that a to x, only thing is y(s 1) is now here replaced by f of s 1 instead of y(s 1), we have f of s 1. So, if we replace this, in this particular way, then we will be having this expression is equal to f(x) plus lambda times integral a to x, now I am writing here k 1 (x,s) f(s) ds plus lambda square integral a to x, it will be as usual k 2 (x,s) f(s) ds plus lambda cube integral a to x. And here, we can write this as k 3(x,s) y(s) ds, where this k 3(x,s) this is equal to integral a to x k(x,psi) k 2(psi,s) d psi, this will be the hard iterate kernel. So, if we proceed in this way.

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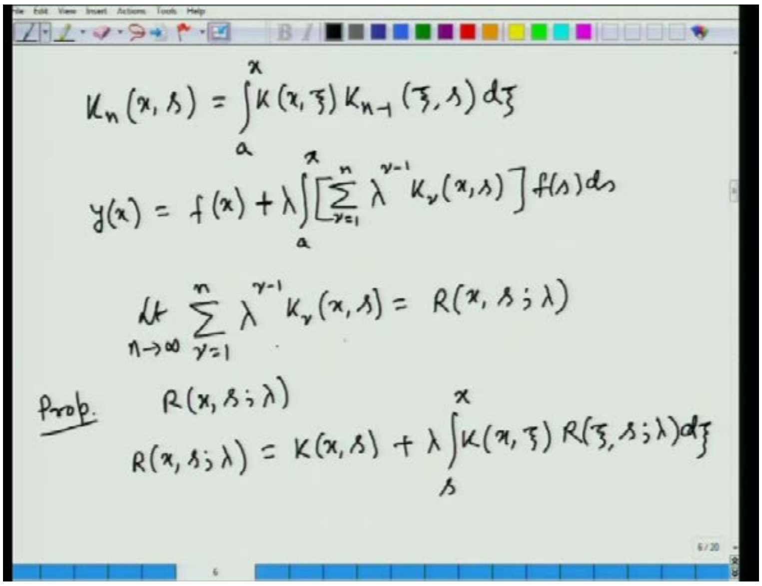

So, every time will be having the expression for y(s) from this kind of last iteration, and substituting into the original equation will be having iterated kernels of the form which are inguinal define by k n (x,s) is equal to integral a to x k(x,psi) k n minus 1(psi,s) d psi. So, proceeding in this way ultimately after n steps will be arriving at the result y(x) is equal to f(x) plus lambda times integral a to x summation nu running's from 1 to n lambda to the power nu minus 1 k nu (x,s); this multiplied with f(s) ds. And for that timing, if we assume that this summation converges as intends to infinity, then the some of this particular series is nothing but our Resolvent kernel; that is lambda to the power nu minus 1 k nu (x,s), this is equal to Resolvent kernel x s lambda.

Now of course, at a later stage will be proving this converges, now before going to that I just try to draw your attention towards this particular expression, that how we are getting this one. And this can be easily verified, that if we back substituted; the expression for n equal to 1, 2, 3, so you will be able to verify that whatever expression we have obtain for y(s) at the first step then y 2(x), and so on. So, ultimately you will be having this expression for general s.

Now, at this point we are assuming this series converges, converges uniformly and to a continues function this one, where this is called the Resolvent kernel, and this iterated kernels k nu (x,s), that is defined by this one. Now, before proceeding further are going to discuss anything about the solution of volterra equation using this Resolvent kernel, first of all we state a property of this particular Resolvent kernel. This is an important prosperity of Resolvent kernel, that is Resolvent kernel are x, s, lambda; this particular Resolvent kernel satisfies integral equation that is x, s, lambda is equal to k (x,s) plus lambda integral s to x k of x comma psi R of psi s lambda d psi.

Now, the question is from where we are getting this relation; actually this relation can be obtain easily from this definition of Resolvent kernel assuming it converges uniformly to the some function.

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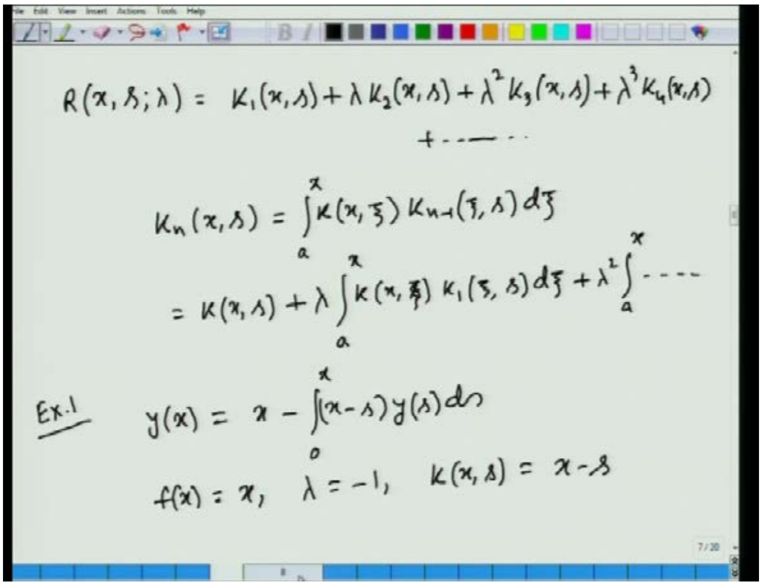

But this some function is actually our Resolvent kernel, because as per definition, we can write that R of x, s, lambda this is equal to k 1 (x,s) plus lambda k 2 (x,s) plus lambda square k 3(x,s) plus lambda cube k 4(x,s) plus dot dot up to infinity. With these series we can recall the formula that is k n (x,s) that was actually defined by integral a to x k(x,psi) k n minus 1(psi,s) d phi, this was the expression alright.

Now with this expression, if you just rearrange this term k 1 is k (x,s) plus lambda times this k 2 (x,s) will be this integral a to x k of (x,s) k 1(psi,s) d psi plus lambda square integral a to x and so on. Then, immediately you will able to verify this results is true, now we try to solve one problem using this particular the method of Resolvent kernel. So, first of all we consider the usual known problem, what we have solve in some other methods. Now, we are just going to check whether this method is give us the same solution or not.

That is k y(x) equal to x minus integral 0 to x x minus s y(s) ds. So, first of all will be finding out Resolvent kernel, and then using the formula involving Resolvent kernel will solve the integral equation. So, here f(x) this is equal to x lambda, this is equal to minus 1 and k (x,s) this is equal to x minus s, this is actually the kernel.

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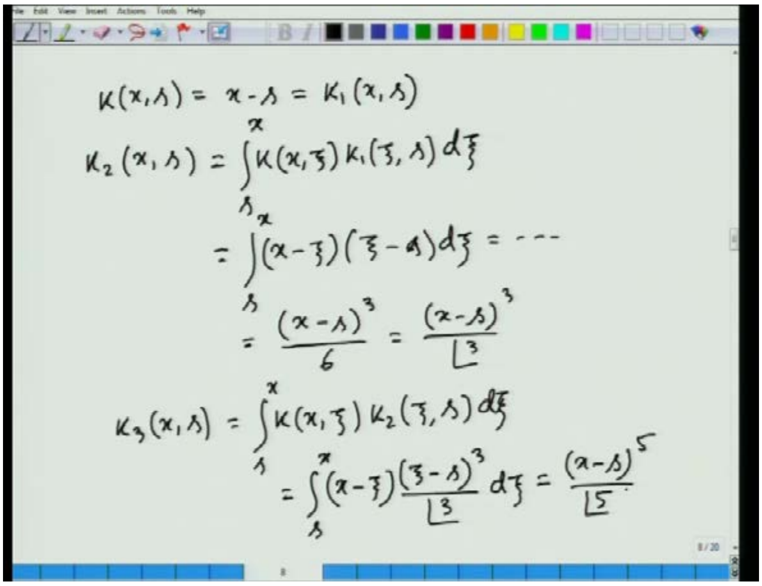

So, using the formula for finding Resolvent kernel first of all k (x,s), that is equal to x minus s, and this is nothing but k 1 (x,s). Next, k 2 (x,s) this is by definition integral s to x k of(x,psi) k 1(psi,s) d psi. So, this is equal to integral s to x k(x,psi) will be x minus psi, and k 1(psi,s) is nothing but k(psi,s). So, this will be psi minus s d psi, and after some calculation you can find this will be x minus s whole cube divide by 6, and of course, we can write it as this is equal to factorial 3.

And in the next step, if you calculate k 3(x,s), this will be integral s to x k of(x,psi) k 2(psi,s) d psi, because this is the formula for calculating the iterated kernel, and then substituting this expression s x it will be x minus psi, then psi minus s whole cube divided by factorial 3 d psi. And this will be equal to x minus s whole to the power 5 divide by factorial 5.

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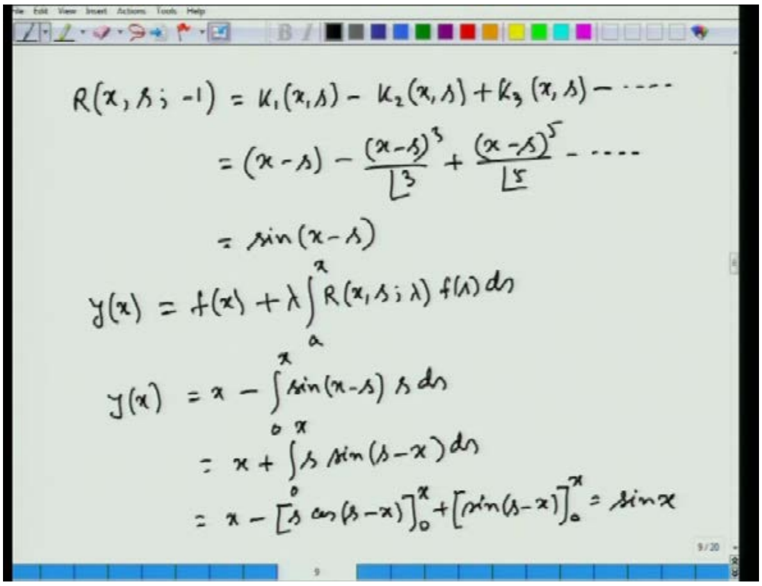

So proceeding in this way, we will be having the Resolvent kernel x, s, minus 1; this is equal to k 1(x,s) minus k 2 (x,s) plus k 3(x,s) minus dot dot. This alternative plus minus are coming from this is k 1(x,s), and this is lambda k 1(x,s) with lambda equal to minus 1 then plus lambda square k 3(x,s) as lambda equal to minus also this will be plus. Next, one will be plus lambda cube k 4(x,s), so this is minus 1, and so on.

So, next sign will be minus, and after substituting will be having x minus s minus x minus s whole cube divided by factorial 3 plus x minus s whole to the power 5 divided by factorial 5 minus dot, dot, and this is nothing but sin of x minus s. Now, if we recall the required solution that was given by y(x) equal to f(x) plus lambda times integral a to x R of(x, s, lambda) f(s) ds. So, for the given problem, the required solution will be y(x) is equal to x minus integral 0 to x, because for the given problem a equal to 0; this R x s comma minus 1 this is sin of x minus s, then then f(s) is equal to s.

So, s d s and after rearranging the term, you can write this is 0 to x s sin of s minus x ds, and after integration this will be x, then you have to perform the integration by integration by parts, it will be minus s cosine s minus x limit from 0 to x, and then plus sin of s minus x this limit from 0 to x, and after simplification you will be having this is equal to sin x. Now, at this point it may come comes in your mind for some other type of kernel, if they are little bit complicated. Then it will be difficult to calculate this iterative kernels, but fortunately they are add some particular methods, whenever this kernel can be expressed as a polynomial of s of degree n minus 1, then we can use a short cut method to find out the Resolvent kernel. So, what is that short cut method.

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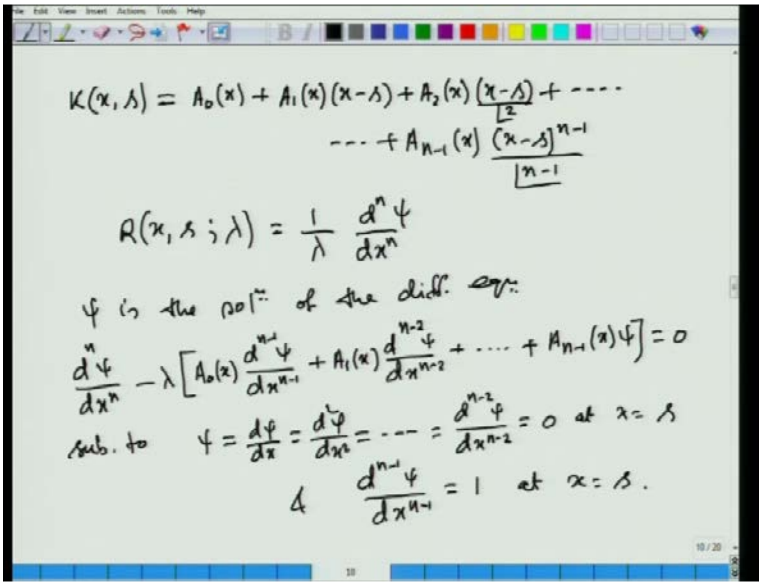

So, first of all we are assuming that kernel is a n minus 1 f degree polynomial in s, and that can be written as, A zero x plus A 1 x into x minus s plus A 2 x into x minus s plus dot, dot, up to A n minus 1 x times x minus s whole to the power n minus 1 by factorial n minus 1. So, of course, we need some sort of exercises, in order to find out A 0(x), A 1(x), A 2(x), because up to A n minus 1(x), because this kind of format is required. If we are able to put the kernel into this particular format, then we can say that R x s lambda can be obtain from 1 by lambda d n d x n of psi.

Where psi, this is the solution of the differential equation d n psi d x n minus lambda A 0(x) d n minus 1 psi d x n minus 1 plus A 1(x) d n minus 2 psi d x n minus 2 plus dot, dot, up to A n minus 1(x) psi; this is equal to zero subjected to the condition that psi equal to d psi d x equal to d 2 psi d x 2 equal to dot, dot up to d n minus 2 psi d x n minus 2, this is equal to 0, at x equal to s and d n minus 1 psi d x n minus 1, this is equal to 1 at x equal to s.

So, that means, if we are able to express k (x,s) in this particular format, then we can adopt this particular method. And so the point is that I mention this one, and previously consider this example just to show that whether this Resolvent kernel sin x minus s can be obtain from the given problem, that where we know that lambda equal to minus 1 and the k (x,s) is equal to x minus s.

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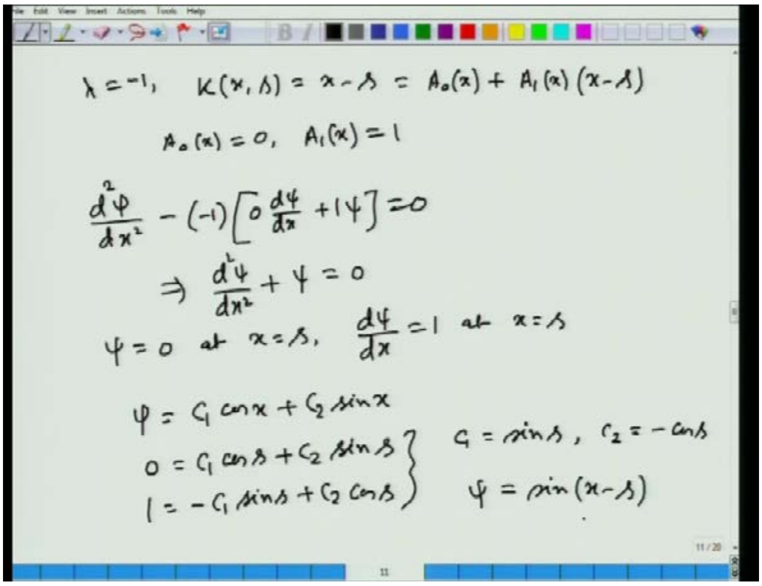

So, taking this lambda is equal to minus 1, and k (x,s) is equal to x minus s and comparing it with the form A 0(x) plus A 1(x) into x minus s; you can easily find this A o(x) is equal to 0, and A 1(x) this is equal to 1. So, whenever A 0(x) equal to 0, and A 1(x) equal to 1. So, now we can write down this differential equation; this differential equation will be d 2 psi d x 2, because is a polynomial of degree 1. So therefore, order of the differential equation will be the second order differential equation, then minus here lambda equal to minus 1, this lambda equal to minus 1. So, minus of minus 1 and then, 0 d psi dx plus 1 times psi equal to o, imply d 2 psi d x 2 plus psi, this is equal to 0 with the condition that psi equal to 0 at x equal to s, and d psi d x -this is equal to 1 at x equal to s.

So, immediately solution of this second order differential equation, that is d 2 psi d x 2 plus psi equal to 0 will be psi equal to C 1 cosine x plus C 2 sin x, this is the expression for psi, then using this condition you can find 0 equal to C 1 cosine s plus C 2 sin s, and again using this condition 1 will be equal to minus C 1 sin s plus C 2 cos s. If we solve this system of equation, then you will be having C 1 this is equal to sin s and C 2, this is equal to minus cos s. And therefore, psi is equal to sine of x minus s. So, that means, the Resolvent kernel what we have obtain in terms of this expression.

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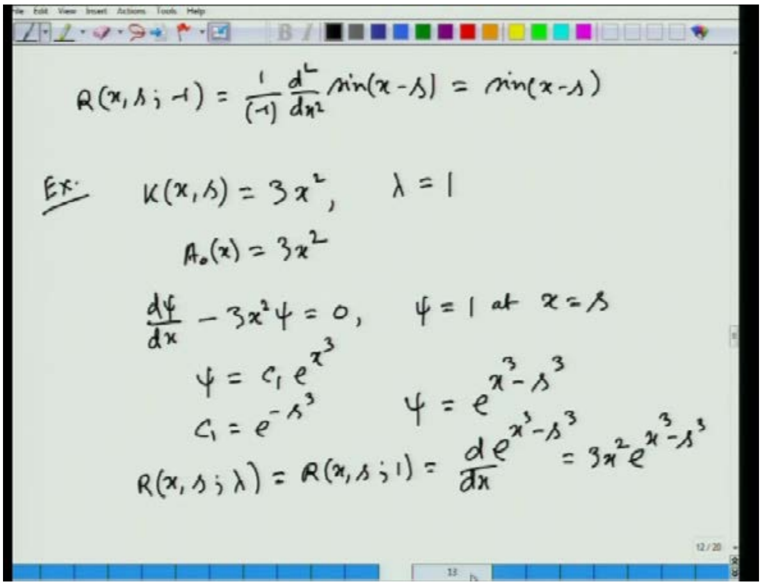

Now, we have to find out this one, because here only we have find out psi, and from this definition R(x, s, minus 1) this is equal to 1 by minus 1 d 2 d x 2 sin of x minus s; that will be equal to sin of x minus s. And after substituting into the given problem, will be having the solution.

Next, we consider one more example to find out the Resolvent kernel. Here k (x,s) this is equal to 3 x square, and lambda equal to 1. So, in this problem actually we are having this A 0(x), this is equal to 3 x square, and no other term involving s. And therefore, the required differential equation will be d psi d x minus 3 x square psi, this is equal to zero with psi equal to 1 at x equal to s; this is the expression. And after integration you will be having psi equal to C 1 e to the power x cube using this condition at x equal to s psi equal to C 1 e to the power x cube we can find C 1 this is equal to e to the power minus s cube.

And after substituting here, we can find psi equal to e to the power x cube minus s cube; and therefore, the Resolvent kernel x s lambda that is actually equal to x, s, 1. So, this is equal to simply d d x of e to the power x cube minus s cube. So, this is equal to 3 x square e to the power x cube minus s cube.

Now, before completing today's lecture, I want to discuss one more problem where you can easily understand why this kind of Resolvent kernel method is little bit useful, because if we recall the previous example, where Resolvent kernel was sin x minus s.

You can think about that this problem can be solved more easily by using laplace transform method. So, why I am considering this method? This is only applicable for some specific type of integral equation, those are volterra integral equation of second kind that involve kernel will prompted us that it would be better to consider the method of Resolvent kernel in order to solve the problem.

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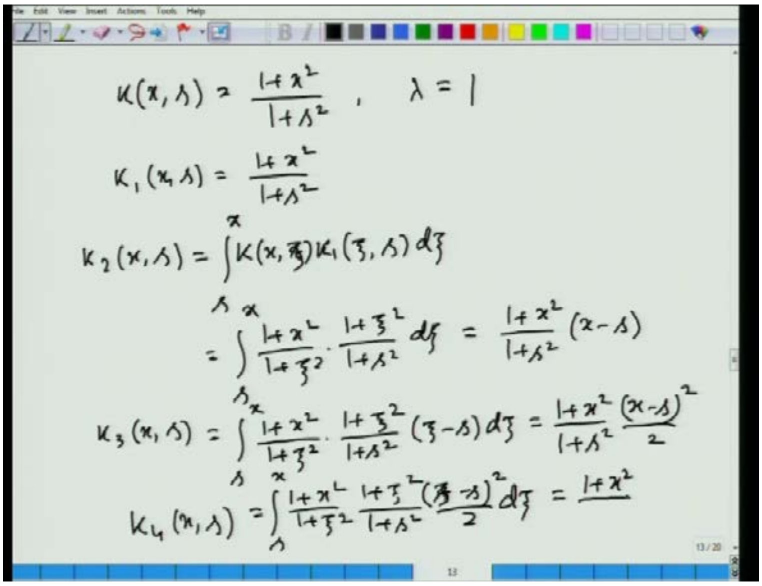

If you just have a look at the kernel, then we will be able to understand clearly, suppose k (x,s) is equal to 1 plus x square divided by 1 plus s square. So, if the integral equation involve this kind of expression, then it would be little bit difficult to apply other type of methods. So, with lambda equal to 1, we can try to calculate the Resolvent kernel.

Now, in this case we do not have any possibility to consider this the method of differential equation to find out the Resolvent kernel, whether you can directly calculate from here k (x,s) that is equal to 1 plus x square by 1 plus s square, then k 2(x,s) this is equal to integral s to x k(x,psi) k 1(psi,s) d psi. So, this is equal to s to x 1 plus x square by 1 plus psi square into 1 plus psi square divided by 1 plus x square d psi.

So, this will be equal to simply 1 plus x square by 1 plus s square times x minus s, then k 3(x,s), this will be integral s to x k(x,psi). So, that means, 1 plus x square by 1 plus psi square into k 2(psi,s). So, it will be 1 plus psi square divided by 1 plus s square times psi minus s d psi, and this will be equal to 1 plus x square by 1 plus s square x minus s whole square by 2. And in the next step, if you calculate k 4(x,s), this will be equal to 1 plus x square by 1 plus psi square s to x 1 plus psi square by 1 plus s square times psi minus s this square by 2 d psi, this will be equal to 1 plus x square by 1 plus s square x minus s whole cube divide by factorial 3.

So, ultimately the required Resolvent kernel x s comma 1, this is equal to 1 plus x square divided by 1 plus s square times 1 plus x minus s plus s minus x whole square by 2 plus x minus s whole cube by factorial 3 plus dot dot. So, this will equal to 1 plus x square by 1.