Calculus of Variations and Integral Equation
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Module No. # 01
Lecture No. # 25

Welcome viewers to the 5 th lecture of series of NPTEL lectures on Integral Equations. In the last lecture, we have discussed about the successive approximation method for solving, volterra integral equation of second kind.

Now, in today's lecture, we are going to discuss about two different techniques of solving volterra integral equation of first kind; and one method is Laplace transform method and second one is the series solution method. So, just for a quick recapitulation I start with the definition of Laplace transform, because we have to be specific about the notations we are going to use to solve the volterra integral equation of the first kind.

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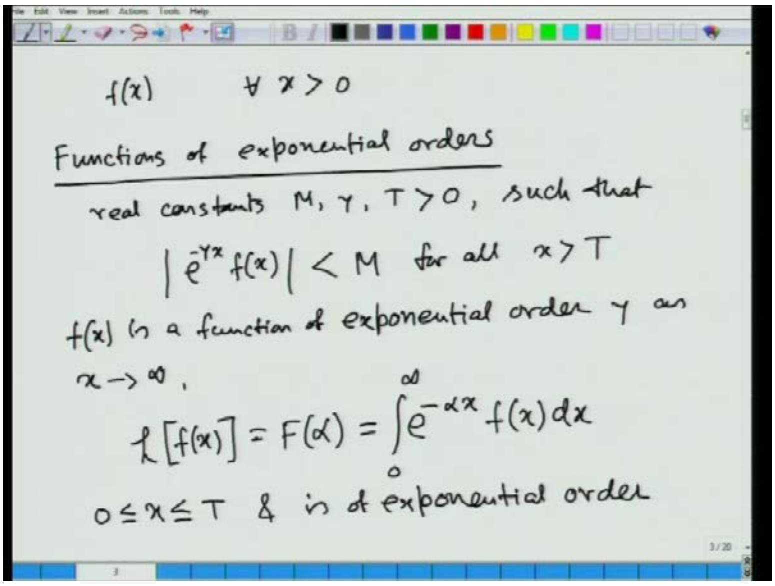

So, suppose f x is a function, which is defined for all x greater than 0 and this function is of exponential order.

Now, before going to the definition of Laplace transform we try to understand, what is the meaning of functions of exponential order? This function f x which is defined for all x greater than 0, now if there exist; real constants capital M, gamma and T; these are all greater than 0 such that, the function f x when multiplied with e to the power minus gamma x, then it is modulus is less than M for all x greater than capital T; f x is a function, which is defined for all x greater than 0, there exist three positive constants; one is M, second is gamma, third is capital T; such that, e to the power minus gamma x f x is modulus is less than M for all x greater than equal to T.

If this happens then, we can say f x is a function of exponential order is a function of exponential order and order of this given by gamma as x tends to infinity or briefly we can say, this function is of exponential order. Actually, if a real valued function f x for x greater than 0, if satisfies the exponential order criteria, then only its Laplace transform exist. And Laplace transform of this function f x is denoted by L of f x. And oftenly, we denote these by F of alpha and is defined by integral 0 to infinity e to the power minus alpha x f x d x, where alpha is a positive real constant.

Now, convergence of these integral depends upon the exponential order of the function. And actually, if this function f x is sectionally continuous over the interval 0 less than equal to x less than equal to T and is of exponential order and is of exponential order then, this Laplace transform exist.

Now, once we use this Laplace transform to convert f x to F alpha; that means, we are relating this function of x to a real variable function that is constant of parameter alpha then of course, using the inverse Laplace transform we can get back this function f x from, where we have obtained this F alpha.

Now, most of the time we use the (( )) of Laplace transform table that is F alpha and f x capital F alpha and f x such that, when we considered the inverse Laplace transform of F alpha will be get back the function, f x.

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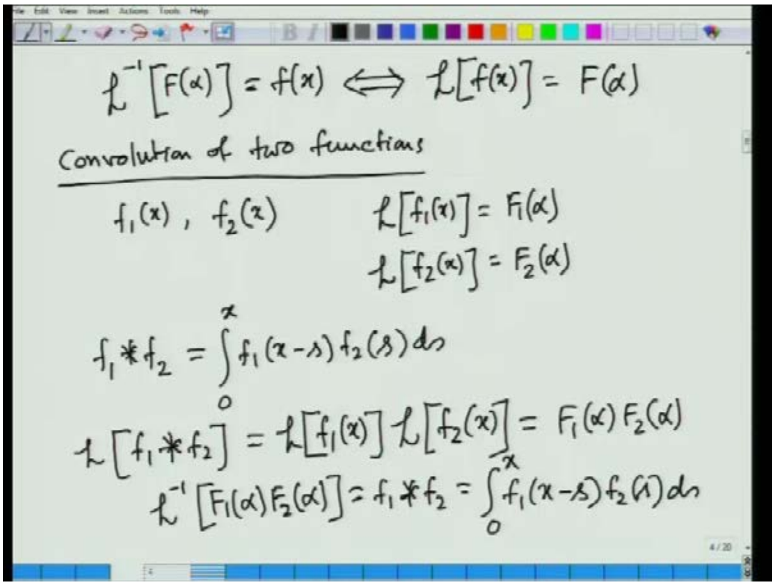

And according to the notation we use L inverse F alpha that is equal to f x. So, this actually implies an implied by, if L of f x this is equal to F alpha then, L inverse F alpha this is equal to f x. Now, we are going to use this particular Laplace transform in order to solve, volterra integral equation of some special type and before going to that, I just like to recall another idea that is convolution of two functions convolution of two functions.

Let, f 1 x and f 2 x; these are two real valued functions, defined for all x greater than 0 and both of them satisfies the criteria for exponential order. And the Laplace transforms are denoted by L of f 1 x this is equal to say, F 1 alpha and L of f 2 x this is equal to F 2 alpha, these are the Laplace transform of f 1 x and f 2 x.

Now, convolution of these two functions is denoted by f 1 star f 2 and defined by integral 0 to x f 1 x minus s f 2 s d s, this is actually called convolution of two functions; f 1 x and f 2 x, which is denoted by f 1 star f 2. You have to keep in mind, this convolution properties actually commodity, we can easily prove f 1 star f 2 is equal to f 2 star f 1 just by changing the variable, if we substitute x minus s equal to u, you can easily prove, this is equal to 0 to x integral f 2 x minus s times f 1 s d s.

Now, interesting point is that, if we consider the Laplace transform of the convolution of these two functions; f 1 star f 2 then, it will give us the product of the Laplace transform of the functions, L of f 1 x multiplied by L of f 2 x.

So, that means this is going to be F 1 alpha multiplied by F 2 alpha. And most important part is that, if we have a function of alpha; which can be expressed as product of two functions of alpha like F 1 alpha and F 2 alpha such that, from the table we know what at the functions for which F 1 alpha and F 2 alpha at the Laplace transform then, inverse Laplace transform of F 1 alpha and F 2 alpha will be the convolution of two functions; f 1 and f 2.

So, that means L inverse F 1 alpha multiplied by F 2 alpha, this is equal to f 1 star f 2, so that is equal to integral 0 to x f 1 x minus s f 2 s d s, this is actually formula related with convolution of two functions and the Laplace transform.

Now, we are going to use these Laplace transform method in order to solve volterra equation of second kind, but of course, for those volterra equation of second kind, if lower limit is starting from 0; that means, instead of a; if we have 0 and the kernel of the function is actually a function of difference of two variables that is x minus s, then only we can apply this Laplace transform method.

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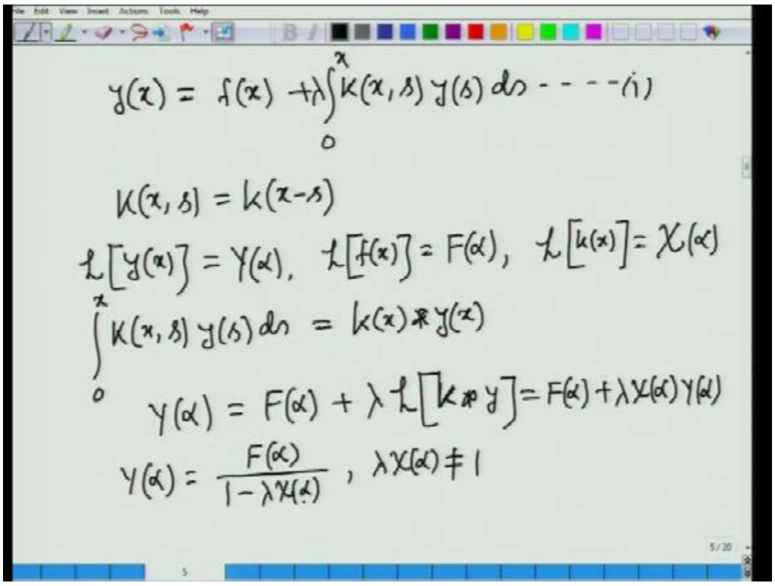

So, to be specific in terms of mathematical notations this is your given integral equation, y x equal to f x plus lambda integral 0 to x K of x comma s y s d s. So, first criteria is lower limits should be equal to 0. Now, if capital K x this is actually a function of small k of x minus s, if this condition is satisfied by the kernel of the function, then only we can apply the Laplace transform method.

So, in this case you have to keep in mind, the applicability of this technique to find out the solution of the integral equation depends upon the lower limit whether it is 0 or not number 1. And secondly, kernel is of actually particular type k of x minus s and if this condition is satisfied then, first I described here how to proceed to find out the solution of the problem.

Suppose, Laplace transform of y x this will be denoted by a capital Y alpha as usual Laplace transform of f x is denoted by capital F of alpha and Laplace transform of small k x this will be denoted by chi of alpha, this is the notations. And actually you can try to understand that, whenever K x comma s is a function of the form k of x minus s then, integral 0 to x K x comma s y s d s this is nothing but, the convolution of the function small k x and the unknown function; y s. So that means, specifically we can write if K x comma s satisfies this criteria then, integral 0 to x K x comma s y s d s, this is actually called to convolution of k x star y x.

So, then we can take Laplace transform of the given integral equation. So, taking Laplace transform of this equation, call it 1; we can get Y alpha is equal to F alpha plus lambda times Laplace transform of convolution of k star y, this is small k keep in mind not capital K and this is equal to actually F alpha plus lambda times chi alpha multiplied by Y alpha. So, solving for Y alpha we can find Y alpha this is equal to F alpha divided by 1 minus lambda chi alpha with the hypothesis that, lambda chi alpha this is not equal to 1.

So, once we have this result that is Y alpha is equal to F alpha by 1 minus lambda times kapa alpha then, we can solve it by using inverse transform inverse Laplace transformation method to get.

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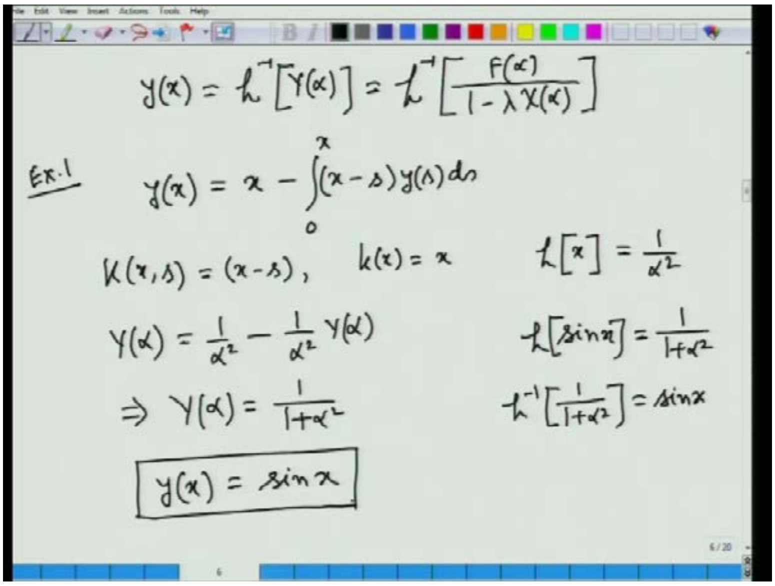

y x is equal to L inverse of Y alpha and that is equal to inverse Laplace transform of F alpha divided by 1 minus lambda chi alpha. So, once we are able to find out inverse Laplace transform of this algebraic expression in terms of alpha then, actually we can find the solution of the given problem as y x.

Now, we look at some illustrative example to understand this method. You can recall that, these problem we have solved earlier by using other technique, that is y x is equal to x minus integral 0 to x x minus s y s d s.

So, clearly you can understand that K x comma s is actually x minus s. So, if we write in terms of small k x, so then small k x is nothing but, x. And here, from the structure of the equation f x is equal to x, k x equal to x. So, we need only the knowledge about the at this moment Laplace transform of x and that is nothing but, 1 by alpha square.

So, using these result Laplace transform of x equal to 1 by alpha square, if we take the Laplace transform the given integral equation then, we can find Y alpha; this is equal to 1 by alpha square minus 1 by alpha square multiplied with Y alpha, this is the result. Actually this is coming from the concept that Laplace transform of integral 0 to x x minus s y s d s means; we are taking the Laplace transform of convolution of two functions; x and y x.

Now, Laplace transform of x is 1 by alpha square, Laplace transform of y x we assumed to denote it by capital Y alpha then, Laplace transform the integral 0 to x x minus s y s d s, these expression results in 1 by alpha square Y alpha.

So, solving for Y alpha we can find Y alpha, this is equal to 1 by alpha square and from the table we can find L of sin x; that is Laplace transform of sin x is equal to 1 by alpha square and hence, L inverse 1 by 1 plus alpha square that is equal to sin x. And hence, taking inverse Laplace transform of Y alpha we can find, y x this is equal to sin x. This is actually solution of the given integral equation that is first example.

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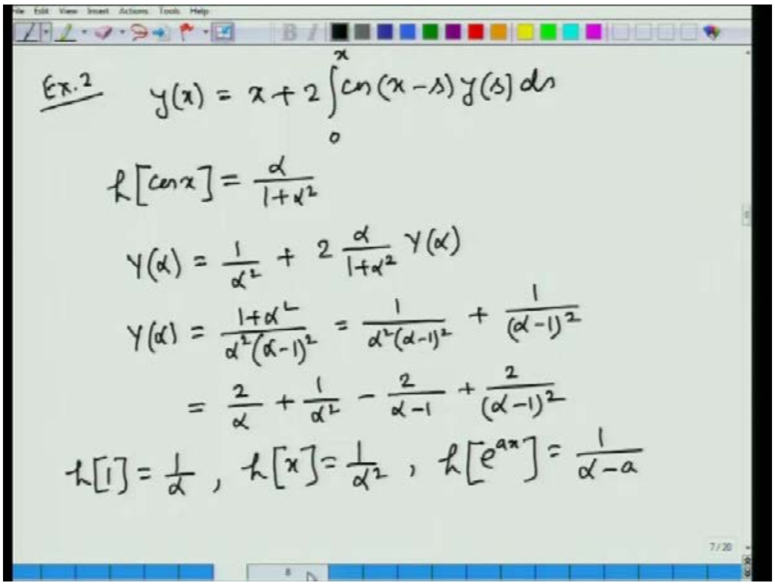

Now, we consider one more example, example 2 I am going to consider this example for the reason, that here in order to apply inverse Laplace transform; we have to utilize the first shifting property of inverse Laplace transform. So the first of all, we write the problem; problem is y x equal to x plus 2 integral 0 to x cosine of x minus s y s d s.

Now, already I have mentioned that Laplace transforms of x is 1 by alpha square. And here, I can mention Laplace transform of cosine x, this is equal to alpha y by 1 plus alpha square, this is the Laplace transform of cosine x; this is needed because, here 0 to x cosine x minus s y s d is again convolution of two functions, that is cosine x and y x.

So, taking Laplace transform of the given integral equation, we can write Y alpha that is equal to 1 by alpha square plus 2 alpha by 1 plus alpha square this multiplied with Y alpha. And transferring this particular term on to the left and after rearranging, we can find Y alpha this is equal to 1 plus alpha square divided by alpha square into alpha minus 1 whole square, and we can rearrange this term first into the form that is 1 by alpha square times alpha minus 1 whole square plus 1 by alpha minus 1 whole square.

And using the method of partial fractions, we can write this is equal to after some algebraic calculation, this will be equal to 2 by alpha plus 1 by alpha square minus 2 by alpha minus 1 plus 2 by alpha minus 1 whole square; so, this will be the expressions.

Now, we need this results that is L of 1 equal to 1 by alpha. Actually, inverse Laplace transform of 1 by alpha will be then 1. Already we know the result that is Laplace transform of x that is equal to 1 by alpha square. So, inverse Laplace transform of 1 by alpha square will be x. And thirdly, Laplace transform of e to the power a x this is equal to 1 by alpha minus a; so, that means inverse Laplace transform of 1 by alpha minus 1 is going to be e to the power x. And now, we have to be careful for the inverse Laplace transform for 1 by alpha minus 1 whole square.

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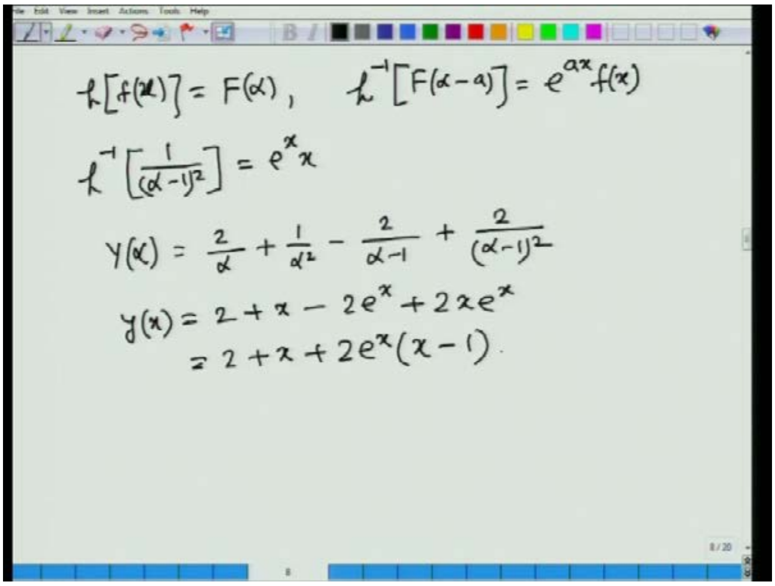

And here actually we need to apply the first shifting property of inverse Laplace transform; it states that, if L of f x this is equal to F alpha, then inverse Laplace transform of F alpha minus a, this is equal to e to the power a x multiplied by f x. So, actually we apply this result in order to find out this inverse Laplace transform of 1 by alpha minus 1 whole square.

So, if we consider, if alpha equal to 1 by alpha square, then 1 by alpha minus 1 whole square is coming out to be f of alpha minus 1. So, with a equal to 1 and then, using this first shifting property; we can write this is equal to e to the power x into x, because this e to the power x coming from the part e to the power a x and this x is stands for a f x because, this 1 by alpha square is actually Laplace transform of x. So, L inverse 1 by L minus 1 whole square is equal to x e to the power x.

So, therefore, from Y alpha is equal to 2 by alpha plus 1 by alpha square minus 2 by alpha minus 1 plus 2 by alpha minus 1 whole square. If we take the inverse Laplace transform, then we will be having y x this is equal to 2 inverse Laplace transform 1 by alpha is 1 plus x here minus 2 e to the power x plus 2 x e to the power x. So, answer will be required answer is 2 plus x plus 2 e to the power x into x minus 1. So, this is the solution for the given volterra integral equation of the first kind.

Next, we considered one more example; this is little bit interesting only for the reason that, when we solve this equation then, you can find that solution of the given equation is completely related with the non homogeneous part of the integral equation involved with the given problem.

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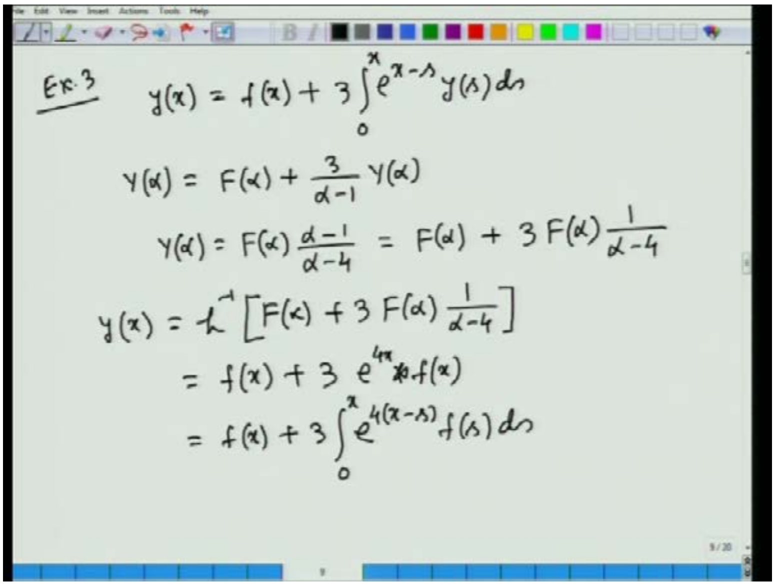

This example 3; it states that, y x is equal to f x plus 3 integral 0 to x e to the power x minus s y s d s. So, here e to the power x minus s is (( )), so that means first function f 1 x is e to the power x and second function is y s. And already we have noticed that, Laplace transform of e to the power a x is equal to 1 by alpha minus a (Refer Slide Time: 24:10). So, therefore, Laplace transform of e to the power x is going to be 1 by alpha minus 1.

So here, if we take the Laplace transform of both the sides, this is actually convolution of e to the powered x and y x. So, taking Laplace transform we can find Y alpha, this is equal to F alpha plus 3 divided by alpha minus 1 this multiplied with Y alpha this one.

Now, if we simplify it, then we can find Y alpha; this is equal to F alpha times alpha minus 1 divided by alpha minus 4 and writing the numerator into the form alpha minus 4 plus 3 we can find from here, that is F alpha plus 3 F alpha multiplied with 1 by alpha minus 4. So now, F alpha comes into this part.

And from here, if we take the inverse Laplace transform of the both sides, then y x is equal to inverse Laplace transform of F alpha plus 3 F alpha multiplied with 1 by alpha minus 4. So, this is equal to f x because, we have denoted the Laplace transform of f x by capital F alpha. So, inverse Laplace transform of capital F alpha will be, f x plus 3 convolution of e to the power 4 x with f x, this is actually the convolution of this two functions.

So, writing the formula for convolution of two functions; we can find, this is equal to f x plus 3 integral 0 to x e to the power 4 x minus s f s d s. So, this problem is little bit interesting only from that for the reason that, given integral equation is y x equal to this one.

Now, once this f x is known this f x is known, so result of the integral equation is completely depends upon this integral. So, once f x is known, so substituting f x here we can evaluate this integral and then, this will gives as the desired result for the given problem we can evaluate this integral and then this will gives us the desired result for the given problem.

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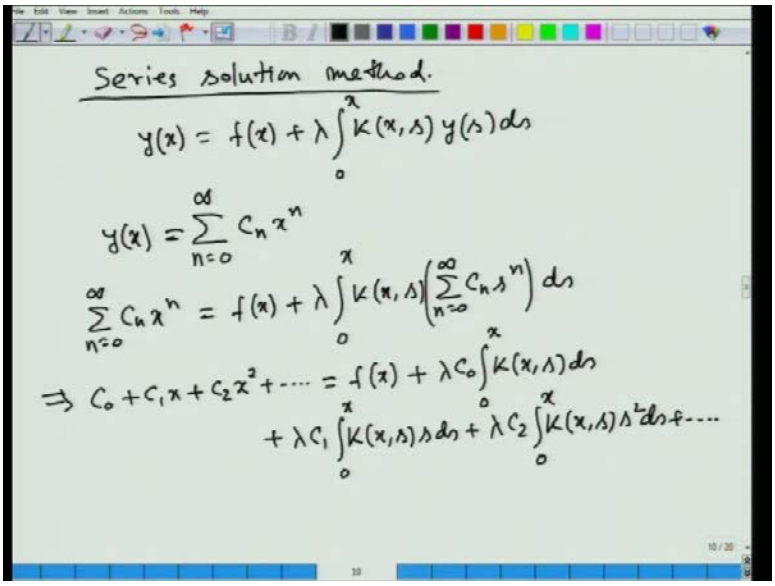

Next, we consider the series solution method (No audio from 27:10 to 27:19); series solution method for volterra integral equation of the second kind. Again in this case, we restrict ourselves problems of the type y x equal to f x plus lambda times integral 0 to x K of x comma s y s d s.

So, here only lower limit 0, this is required and there is no restriction for y x comma s; that means, no particular format for this kernel is required as we have seen that, in case of applying Laplace transform is need to be K x comma s is a function of x minus s. But, at a later stage I will make some remark that in which cases we can think about solution of the integral equation into the with the help of series solution method. So, the point is that, we are assuming that solution of this equation can be expressed as a power series around x equal to 0.

So, we are assuming solution in the form, y x is equal to sigma in runnings from 0 to infinity C n x to the power n, this is our targeted form of the solution, so that means if we are going to solve this equation by series solution method then, we have to substitute this expression into the given equation then, we can integrate these K x comma s multiplied by the series it will be actually converted into term by term integration and then, using the Taylor series expansion of effects we have to solve for C 0, C 1, C 2 and so on.

And actually we have to derive some recurrence formula from where once we know the values of some initial C 0, C 1, C 2; using the recurrence formula will be able to calculate all C n’s. Now, the method is, if you substitute the series into this given problem. So, it is equal to n running from 0 to infinity C n x to the power n, this is equal to f x plus lambda times integral 0 to x K x comma s sigma in runnings from 0 to infinity C n s to the power n, this is expression for y s d s.

And this actually implies will be having C 0 plus C 1 x plus C 2 x square plus dot dot, this is equal to a f x plus lambda times C 0 integral 0 to x K x comma s d s plus lambda times C 1 integral 0 to x K x comma s s d s plus lambda times C 2 integral 0 to x K x comma s s square d s plus dot dot dot.

Now, using the continuity of this kernel K x comma s over the square domain 0 to say some beta cross 0 to beta we can find that, the summation and integral sign can be interchanged. And then, after expanding will be having this type of infinite series consist of summation of these integrals.

And after expressing effects in terms of a Taylor series, we can get this expression which is valid for all x then, been collecting the coefficients of equal power suffix; we can find a system of equation. And solving the system of the equation once you find out C 0, C 1, and C 2 and so on, then we can have the solution for the given integral equation. So, in order to understand this method we consider two examples.

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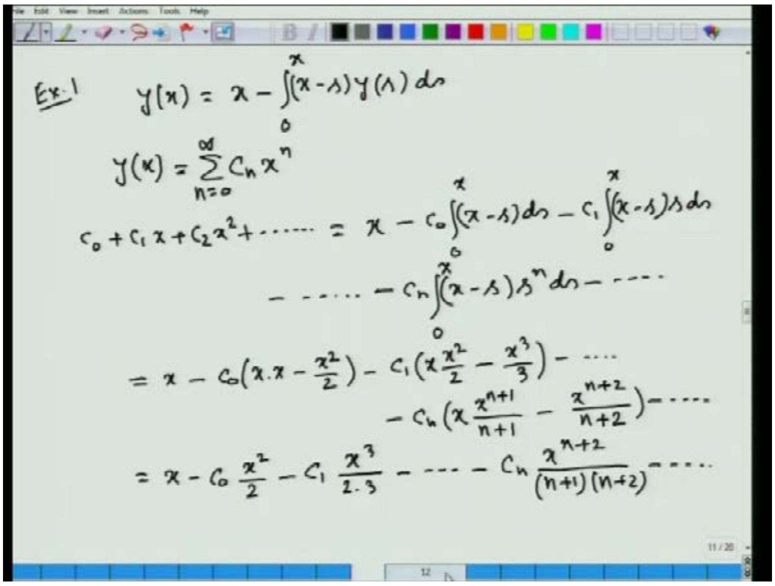

First example is y x, this is equal to x minus integral 0 to x x minus s y s d s. So, of course, the this equation we just solve with help of Laplace transform method and just to verify that, whether we are getting same solution or not; we going to apply here the series solution method.

Now, you can see that, if we use series solution method here, so substituting the series that is sigma in runnings from 0 to infinity C n s to the power n; we have to integrate the integrals of the form 0 to x x minus s d s then, 0 to x x minus s multiplied by s d s. So, in general, that means 0 to x x minus s multiplied by s to the power (( )).

So, for each in ranging from 0 to infinity will be integrating the integrand that is x into s to the power n minus s to the power n plus 1, it is very easy to integrate. So, this gives us some sort of indication that, although we can apply the Laplace transforms method to solve this equation. And then, we can going to apply this series solution method whenever this integrand is easy to handle or it will be easy to integrate this integrand coming out after substituting sigma C n s to the power n.

So, here substituting this series y x is equal to sigma in runnings from 0 to infinity C n x to the power n, we can find on the left hand side; C 0 plus C 1 x plus C 2 x square plus dot dot, these are the terms. And here we do not need to think about further, because f x is simply x and then, minus C 0 integral 0 to x x minus s d s minus C 1 integral 0 to x x minus s s d s minus dot dot, general term is C n integral 0 to x x minus s multiplied with s to the power n d s minus dot dot, here this should not be plus whether it will be minus here (Refer Slide Time: 34:42).

After integration we can find this is equal to x minus C 0 x into x minus x square by 2 minus C 1 this is x into s, so that means x into x square by 2 minus (( )), so that is x cube by 3 minus dot dot; in this way general term will be, minus C n x into x to the power n plus 1 by factorial n plus 1 minus x to the power n plus 2 divided by n plus 2 minus dot dot. So, this is actually the general term, what will be getting after integration.

So, this is actually equal to x minus C 0 x square by 2 minus C 1 x cube by 2 into 3; and here as a general term will be having minus C n x to the power n plus 2 divided by n plus 1 multiplied with n plus 2 I am sorry here it will be n plus 2 (Refer Slide Time: 36:15).

So, term after C 1 it is clearly it will be minus C 2 x to the power 4 divided by 3 into 4 and so on minus dot dot. So, if we equate first few terms on the left hand side; we have C 0, there is no constant term on the right hand side.

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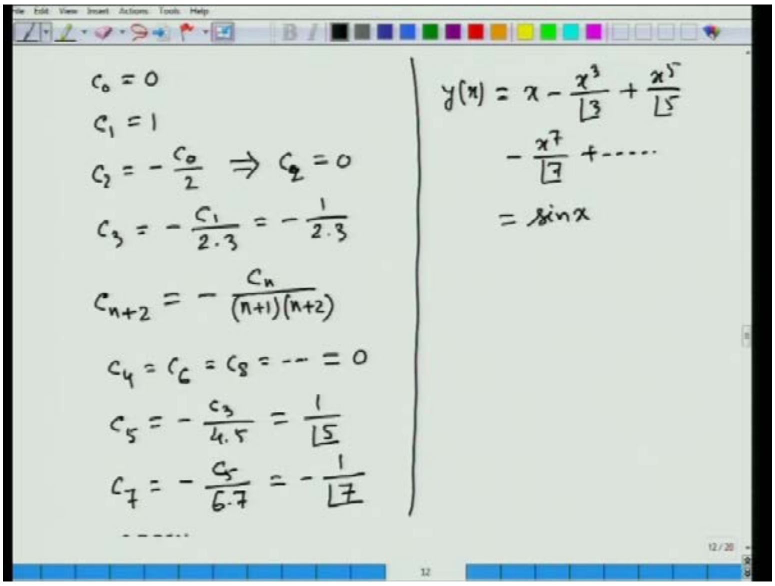

So, therefore, C 0 this is equal to 0. So, this result we are getting by equating constant term from both the sides. Next, if you look at the coefficient of x on the left hand side, co efficient of x is C 1 and on the right hand side coefficient of x is 1. So, therefore, equating the coefficient of x; we can find C 1 this is equal to 1. Next, square term C 2 x square this is equal to minus C 0 x square by 2 (Refer Slide Time: 37:24). So, that means C 2, this is equal to minus C 0 by 2 this implies, C 2 equal to 0, because C 0 equal to 0.

Similarly, if we equate the coefficient of cubic term here we have C 3 x cube and on the right minus C 1 x cube by 2 into 3 e. So, therefore, C 3 is equal to minus C 1 by 2 into 3. So, this is equal to minus 1 by 2 into 3. And recurrence formula can be obtain in this way; on the left hand side, coefficient of x to power n plus 2 is actually C n plus 2. And here we have already observed that, coefficient of x to the power n plus 2 on the right hand side, is minus C n divided by n plus 1 multiplied by n plus 2.

So, therefore, equating the coefficient of x to power n plus 2 from both sides, will be having minus C n divided by n plus 1 into n plus 2. So, using this recurrence formula and the results; C 0 equal to 0, C 2 equal to 0, we can find C 4 equal to C 6 equal to C 8, all these coefficients exactly equal to 0.

We have already obtained C 3 then, C 5, this is going to be minus C 3 divided by 4 into 5. So, after substituting this expression minus 1 by 2 into 3 we can write, this is equal to 1 by factorial 5 then, C 7 will be minus C 5 divided by 6 into 7. So, this is equal to minus 1 by factorial 7 and so on.

So, if we substitute these expressions into the series that, we have assume, then we can find y x; this is equal to x minus x cube by factorial 3 plus x to the power 5 by factorial 5 minus x to the power 7 by factorial 7 plus dot dot. So, that means we are getting the mac loadings infinite series expansion for sin x. So, therefore, by using the method of series solution; we are getting the solution y x is equal to sin x, for the given problem.

Last example that I am going to consider here, that is little bit interesting only for the reason that, although we are assuming a infinite series solution for the given problem, but after solving the problem; you can see that, the solution is actually a polynomial.

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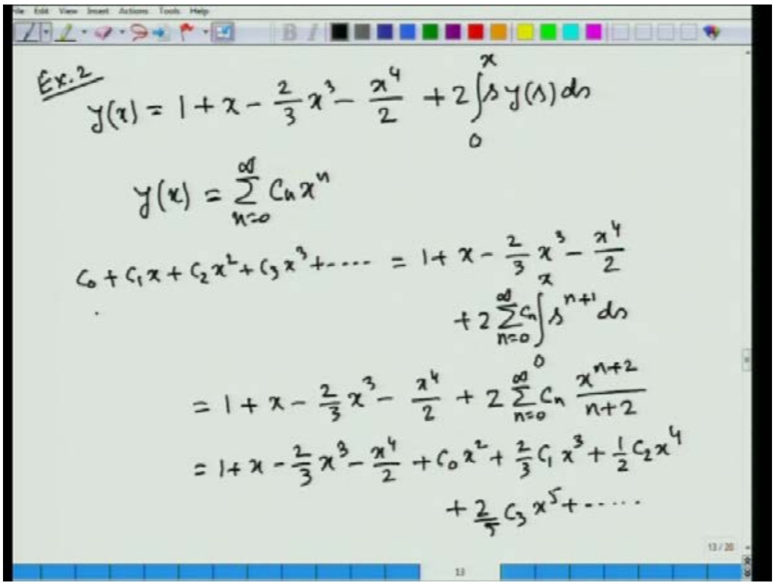

And the problem is, y x this is equal to 1 plus x minus two-third x cube minus x to the power 4 by 2 plus 2 integral 0 to x s y s d s. Again, look at the kernel involved with this particular problem, it is only s here. So, that means if we use this particular series assumption about the existence of series solution that is 0 to infinity C n x to the power n.

So, after substitution the general term actually we need the integral of s to the power n plus 1. So, this is very easily easy to integrate and obtain the result into the closed form. So, most of the time will be adapting the series solution technique, whenever after multiplying s to power n by the kernel involve with the integral equation; it is easy to handle, that is the some sort of you can say, a suggestion where we can apply the series solution method.

Now, if we substitute this series into these given problem, (( )) will be having C 0 plus C 1 x plus C 2 x square plus C 3 x cube plus dot dot; this is equal to 1 plus x minus two-third x cube minus x to the power 4 by 2 plus 2 sigma in runnings from 0 to infinity integral 0 to x s to the power n plus 1 d s.

So, after integration this is coming out to be 1 plus x minus two-third x cube minus x to the power 4 by 2 plus 2 sigma in runnings from 0 to infinity C n x to the power n plus 2 divided by n plus 2, here I have missed the C n term; this is C n (Refer Slide Time: 43:16). So, that means we have this particular series.

Now, this problem is little bit interesting from two sides; number 1 upto x to the power 4 we have one sort of relations between these (( )) and for x to the power n plus 2 by n plus 2 whenever n plus 2 is greater than 4, then will be having another set of relations.

So, first of all we have to find out first four constants; C 0, C 1 upto C 4 carefully because, on the right hand side; we have these expressions involve with the part f x and rest of the the part does not interfere here. So, that is no f x and rest the part does not interfere here. So, that means no turn from x will interfere with the series whenever the index is high and higher.

So, this is equal to 1 plus x minus two-third x cube minus x to the power 4 by 2 and if we substitute, n equal to 0 here; so, these series start it is giving contributions from x square and onwards; for x square will be having C 0 x square substituting n equal to 0, will be having x square from here. Substituting, n equal to 1 will be having two-third C 1 x cube, this is the second term from the series then, plus half C 2 x to the power 4, this is the term we get by substituting n equal to 2 plus rest of that term will be of the form 2 by 5 C 3 x to the power 5 and so on.

Now, if we just compare the constant terms from the both sides.

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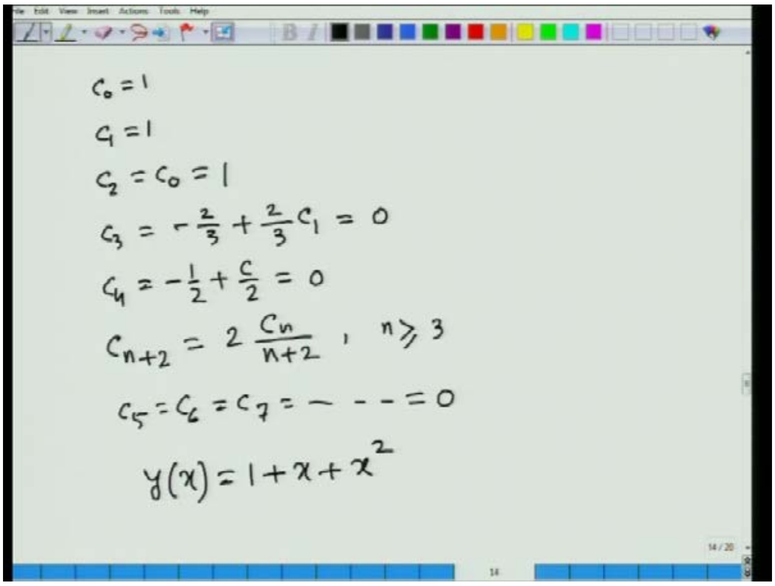

So, first of all we will be having C 0 this is equal to 1, because on the left hand side; it is C 0 and on the right hand side; we have only 1 here. Then, collecting the coefficient of x from both sides we can find on the left it is C 1; on the right, it is x only.

So, therefore, C 1 this is equal to 1. Next, we look at the coefficient of x square term. Here, it is C 2 and in this f x part; there is no x square term and x square term is coming from these parts (Refer Slide Time: 45:59). So, that is C 0. So, coefficient of x square on the right hand side, it is C 0; on the left hand side, it is C 2. So, he equating we can find C 2, this is equal to C 0 and already we have obtained C 0 equal to 1, so this C 2 coming out to be 1.

Next, we collect the coefficient of x cube on the left hand side; coefficient of x cube, this is equal to C 3, on the right hand side; f x part contains minus two-third x cube. So, minus two-third is coming from here and here will be having plus two-third C 1. So, this is minus two-third plus two-third these multiplied by C 1; already we know the value of C 1. So, after substituting the C 1 value will be having this is identically equal to 0.

Last but one, that is coefficient of x to the power 4, on the left hand side it is C 4. So, from left hand side; will having C 4 and from the right hand side; this is minus half and plus C 2 by 2, here coefficient of x to the power 4 is C 2 by 2 (Refer Slide Time: 47:16).

So, this gives minus half plus C 2 divided by 2, we already have C 1 equal to 1. So, this is equal to 0. So, two consecutive terms C 3 and C 4 this is equal to 0. Now, on the rest of the terms; that is terms of the form x to the power 5 and higher, on the left hand side we will be having C 5, C 6 and so on. These are the coefficients of x to the power 5 x to the power 6 and so on.

And on the right hand side; this is nothing but, 2 C n divided by n plus 2, this is the coefficient of x to the power n plus 2. So, that means collecting the coefficient of x to the power n plus 2 from the both sides in order to get the recurrence relation; we can write recurrence relation is coming out to be C n plus 2, that is equal to 2 C n divided by n plus 2.

Now remember, these result is valid for n greater than or equal to 3 because, already we have equated the coefficient of constant term from both sides, coefficient of x from both sides upto coefficient of x to the power 4 from both sides. And these we have done only for the reason that, f x part contains term upto the order x to the power 4.

So, from x to the power 5 and onwards; we can write the general recurrence formula. So, from the left we are getting the coefficient of x to the power n plus 2 is simply C n plus 2. And on the right hand side; coefficient of x to the power n plus 2 is actually 2 C n divided by n plus 2.

Now, this result is valid for n greater than equal to 3. So, clearly from C 3 equal to 0 and C 4 equal to 0; you can find C 5, C 6; which is equal to C 7 and onwards, all these coefficients are identically equal to 0; C 5, C 6, C 7 are all this quantity equal to 0.

Already we have C 3, C 4 this is equal to 0. So, although we have assume an infinite series as a solution of the given problem, but we landed at a solution that is given by 1 plus x plus x square, there is no other terms involving higher powers of x. So, this is clearly a polynomial, which is a solution for the given problem.

And I hope you have experienced these type of situations will appear it in case of ordinary differential equations also. And this volterra integral equations is most of the time, we obtain it by converting the linear ordinary differential equations that is initial value problems converted to volterra integral equations. And therefore, in case of ordinary differential equation we have the experience that sometimes we are trying to find a solution, which are assume to be an infinite series, but ultimately solutions are comes out to be a polynomial.

So, just for a quick recapitulation what we have done today. So, first of all we have defined what Laplace transforms for a function with exponential order is. And then, we have considered the convolution of two functions; f 1 x and f x 2 x. And Laplace transform of convolution of two functions; f 1 and f 2 is nothing but, the product of the Laplace transform of these two functions and these Laplace transform method can be solved sorry can be used to solve the volterra integral equation of first kind, whenever this kernel K x comma s is a special type that is it is a function of x minus s only.

And in the illustrative example, we have seen in one case; we have used x minus s as the kernel. So, therefore, k x was equal to x. Secondly, K x comma s is cosine x minus s. So, that means small k x is nothing but, cosine x. And third problem that we have considered that, kernel k x minus s is actually e to the power x minus s.

So, for those problems we can apply these Laplace transform method to solve it and we need the formula for inverse Laplace transform from any table of Laplace transform; we can get this result to find out the ultimate solution of the given problem. And then, we have considered the series solution means, where we are assuming solution into the form y x equal to n runnings from 0 to infinity C n x to the power n; after substituting this expression into the integral equation; we are having some recurrence relation equating the general form of the x to the power n term, most of the time we have used the coefficient of x to power n plus 2 from the both sides.

And of course, you have to be careful about the Taylor series expansion for f x, fortunately the examples that I considered here those are having finite number of terms in the Taylor series expansion for f x. And in the last example we have observed that, these solution results in a polynomial rather than an in infinite series. So, thank you for your attention.