Calculus of Variations and Integral Equations
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Module No. # 01
Lecture No. # 24

Welcome viewers, to the lecture series on Integral Equation under NPTEL courses. This is the 4th lecture in the series. You can recall in the last lecture that is in the lecture 3 we have considered the formulation of volterra integral equations of second kind, starting from second and higher order linear ordinary differential equations subjected to initial conditions.

And you can recall that, we have discussed about a method such that, a second order differential equation with specified initial conditions for y and y dot at the initial point x equal to a, we have converted the questions of the form d 2 y d x 2 plus p x into d y d x plus q x into y equal to r x into a integral equation. And then for high order under ordinary differential equation, we have deal with the method of assuming n th derivative of y with respect to x as u and then we converted the given ordinary differential equation into an integral equation of volterra type where unknown function was u.

And then with illustrative example, we have attempted to understand the difference between solutions of two types of integral equation, either obtain directly from the differential equation by integrating it twice or solution of the integral equation obtained by assuming secondary derivative equal to u and the problem considered was a second order ordinary differential equation and we have discussed about the solution.

Now, today in this lecture we are going to discuss about solution method for volterra integral equation of second type.

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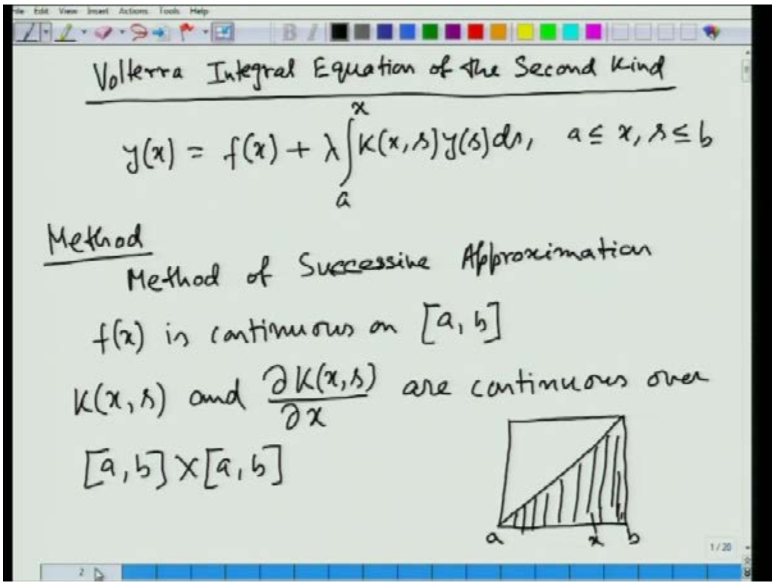

So, the equations we are going to consider volterra integral equation of the second kind and of course, these equation we are going to consider is non homogeneous, so the basic structure is given by y x is equal to f x plus lambda integral a to x k of x comma s y s d s where a less than equal to x comma s less than equal to b, this is the given integral equation. And the method by which we are going to solve this kind of equation is known as method of successive approximation method of successive approximation.

Now, immediately one question come up that, what will be the condition satisfied by f and k x comma s such that we can apply this method or by using this method, we can solve the given volterra integrity equation, answer to the question is that f x is continuous f x is continuous on the closed interval a comma b and the kernel k x comma s and it is fast partial derivative with respect to x, both of them are continuous over the rectangular domain a comma b cross a comma b.

Now, actually we are considering that k x comma s and partial derivative of k with respect to x continuous over the rectangle of length, the both sides b minus a actually square b minus a, but practically we need the continuity condition of k x comma s and partial derivative of k with respect to x over a triangle; because if we try to understand this is our a square, over which the given functions k x s and partial derivative of k x s have continuous.

So, integral is over s from a to x integral over x is from a to x, now the point is that, if we considered this rectangle, so this is the point a, so we are taking about somewhere at x and x can go up to this point that is the x equal to b, this is the point x equal to b, so integral this is the line s equal to x, so ultimately we need the condition over this particular in region, this is the region.

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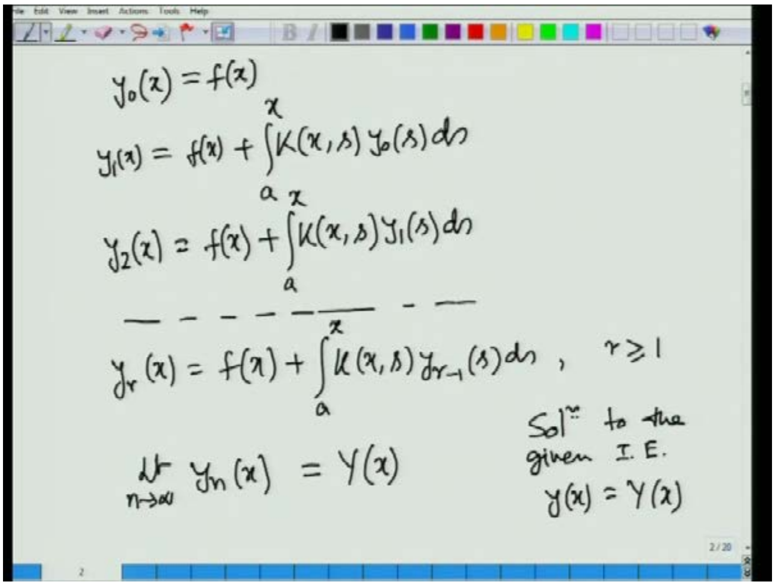

Now, what is the method of solution, that is the we are going to talk about successive approximation, it says first of all we can take a 0 the order approximation denoted by y 0 x for the given particular problem and once we substitute these 0 th order approximation into the given integral equation on the right hand side; that means, when we are substituting it into the format, f x plus integral a to x k of x comma s y 0 s d s evaluating this integral and then adding with f x we can calculate y 1 x.

Now, first question is, what will be the assumption for y 0 x? First of all we start our discussion with the assumption that y 0 x is equal to f x of course, you can keep in mind that instead of f x any continuous function like f x equal to 0 or f x equal to 1 or f x equal to x they can sort the purpose.

Now, our target is using 0 th order approximation for y which is denoted by here as y 0 x, we can calculate y 1 x, because k x comma s is known this is a continuous function; we are assume y 0 x equal to f x which is again a continuous function. So, therefore, integral a to x k x comma s y 0 s d s these part is integral and after integration f x plus this integral produces other continuous function, which is defined over the interval a comma b where x ranging from a to b and using these fast approximation y 1.

Similarly, we can calculate y 2 x this is equal to using the result f x plus a to x k of x comma s y 1 s d s, so that means, we are using a recursive formula that is y k x is equal to f x plus integral a to x k x comma s y it would be better to write r instead of k here, otherwise it would be little bit confusing with the kernel r x and this r is greater than equal to 1.

And our target is to show that, these y r x as r tends to infinity convert if converges to a uniformly continuous function than that particular function will be the solution of the given integral equation, so that means, if limit n tends to infinity, y n x exist and if the sequence of function converges to a function say y x, then solution to the given integral equation I E stands for integral equation here is y x is equal to capital Y X.

So, we can repeat it again as a 0 th order approximation we can assume y 0 x equal to a f x using y 0 x we can calculate y 1 x using the expression for y 1 x, we can calculate y 2 x and so on. And general recursive formula is y r x is equal to f x plus integral a to x k x comma s y r minus 1 s is d s where r greater than equal to 1 and if this limit function as limit n tends to infinity y n x converges to y x, then y x is the solution to the given integral equation.

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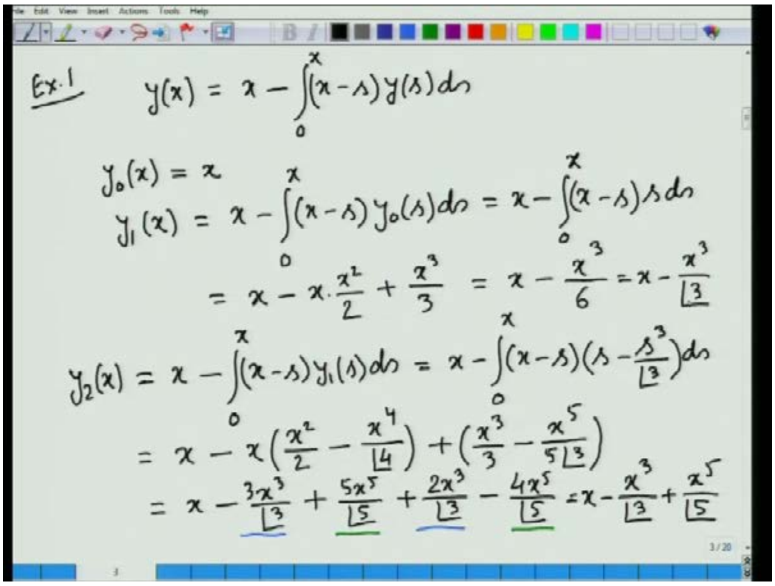

So, first of all we consider few examples to understand this technique, we consider the integral equation y x this is equal to x minus integral 0 to x x minus s y s d s. So, clearly this is an integral equation of the volterra integral equation of the second kind and we have to solve this equation by using the method of successive approximation.

So here f x is equal to x, so therefore, as per method just we have discussed y 0 x is equal to x according to the formula y 1 x is equal to x minus integral 0 to x minus s y 0 s d s and this is equal to x minus integral 0 to x minus s d s and this is equal to x minus x into x square by 2 plus x cube divided by 3 and from here, we can calculate this is equal to x minus x cube by 6.

And in order to make a correspondence between 3 and 6, just for trial we can write it here x minus x cube by factorial 3, in the next step will be able to verify whether our these assumption that denominator of x cube will be factorial 3 or not that can be easily verified. Once we calculate y 2 x, so y 2 x is equal to according to the formula x minus integral 0 to x x minus s y 1 s d s and this is equal to x minus integral 0 to x x minus s multiplied with s minus s cube by factorial 3 d s, so this is equal to x minus. So, first of all this part we have to integrate with respect to s and then we have to substitute the lower limits and upper limits in both the integrals lower limit will contribute only 0.

So, from the first term, that is x into integral of these will give us x into x square by 2 minus x to the power 4 divided by factorial 4 here, then this minus combined with this minus give us plus and here integrant is x square and s to the power 4 by 3, so we will be having x cube divided by 3 minus x to the power 5 divided by 5 into factorial 3.

Now, here you can check the calculation and just for your understanding I am writing all these terms in a systematic way, this is x cube by 2 we can write x cube 3 x cube by factorial 3, then plus here we can write 5 x to the power 5 by factorial 5 with the anticipation denominator will be comes out to be factorial 5 plus 2 x cube by factorial 3 and minus 4 x to the power 5 divided by factorial 5.

Now, if you combine these two terms, that is these with these one and then 5 th order terms these with these one, you will be getting this is equal to x minus x cube by factorial 3 plus x to the power 5 by factorial 5.

Now, of course, there is no need that you have to be guess that this will be factorial 3 this will be factorial 5, I am doing this only for that is in you can say, for this problem I already know the solution will comes out to be a closed form. In other words these type of manipulation is possible if and only if solution is coming out to be a closed form, in case the solution cannot be expressed in to a close form, then these type of manipulation will not work, but again writing this factorial 3 does not make any harm here, because if it is not true something else may be come out, so that can be understood from this second iteration for y 2 x.

So, now look at the expression y 0 x equal to x y 1 x equal to x minus x cube by factorial 3 y 2 x is x minus x cube by factorial 3 plus x to the power 5 by factorial 5.

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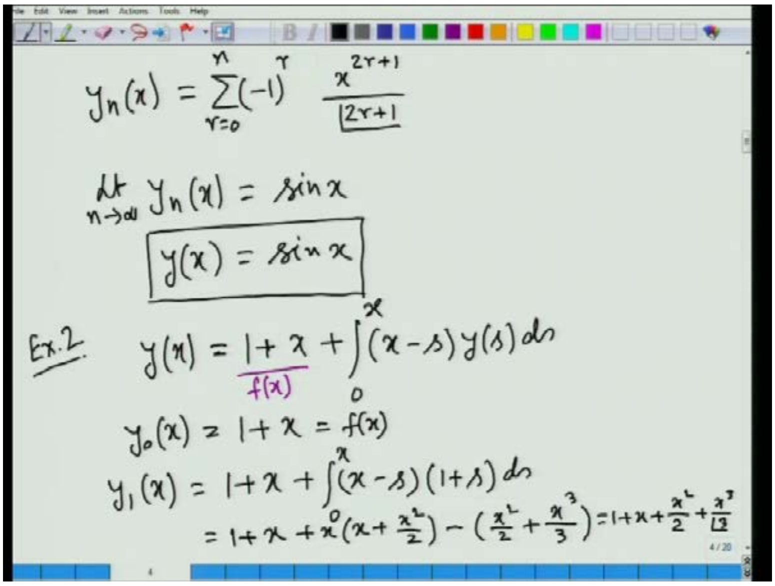

So, now, we can write the general term for this iteration, that is y n x that is equal to sigma r running from 0 to n minus 1 whole to the power r times x to the power 2 r plus 1 divided by factorial 2 r plus 1, you can easily verify that whenever r equal to 0 only so; that means, we are considering n equal to 0 substituting here you can find this is equal to x and similarly you will be having all other expressions that can be easily verified.

And finally, this limit n tends to infinity y n x this is equal to sin x, because here y n x is the maclaurin series of sin x up to n terms and therefore, this series converges uniformly to the function sin x and hence these y n x converges to sin x and the solution to the given problem y x is y x equal to sin x, this is a solution to the given integral equation.

Next we consider one more example, example two it is given by y x equal to 1 plus x plus integral 0 to x minus s y s d s this is the given integral equation, here again y 0 x is equal to 1 plus x, because here 1 plus x this particular expression is our f x, so y 0 x equal to 1 plus x that is actually f x.

Now using y 0 x we can calculate y 1 x equal to 1 plus x plus integral 0 to x minus s multiplied by 1 plus s d s and this is equal to 1 plus x plus x into x plus x square by 2 minus x square by 2 plus x cube by 3. And here after simplification it is coming out to be 1 plus x plus x square by 2 plus x cube divided by factorial 3 this is coming out to be 6 and hence we can assume that this is going to be factorial 2.

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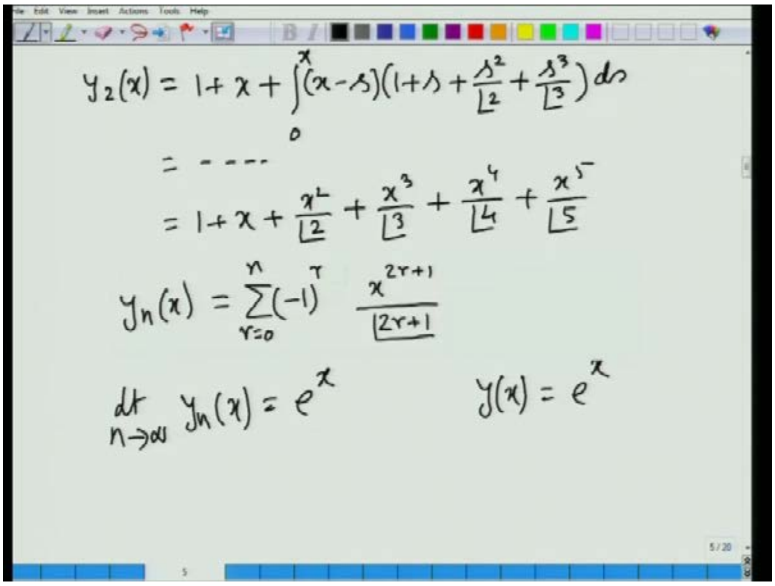

Again in a previous manner if we continue to calculate y 2 x, so y 2 x equal to 1 plus x integral 0 to x minus s multiplied by 1 plus s plus s square by factorial 2 plus s cube by factorial 3 d s if you evaluate it, then you will be having 1 plus x plus x square by factorial 2 plus x cube divided by factorial 3 plus x 2 the power 4 by factorial 4 plus x to the power 5 divided by factorial 5 this will be the y 2 x.

So, in this way from here, we can write y n x is equal to sigma r running's from 0 to n minus 1 whole to the power r x to the power 2 r plus 1 divided by factorial 2 r plus 1 and once n tends to infinity limit n tends to infinity this y n x will be equal to e to the power x and hence y x this is equal to e to the power x is the solution to the given problem.

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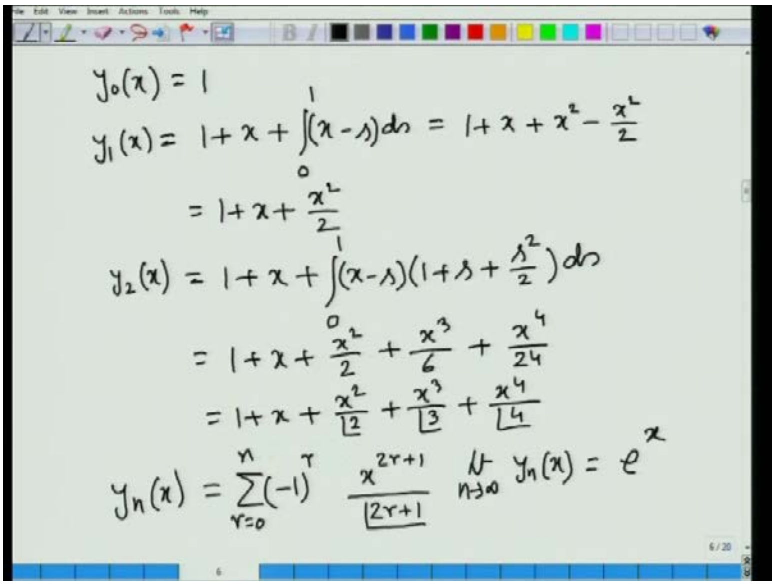

And here we like to verify that what will be the effect if we choose some other functions as y 0 x, so we are going to solve the same problem by assuming y 0 x equal to 1, so if we assume this y 0 x equal to 1, so then y 1 x will be equal to 1 plus x plus integral 0 to 1 x minus s d s this will be only the expression and here, we can find that 1 plus x plus x square minus x square by 2.

So, this is coming out to be 1 plus x plus x square by 2 and then y 2 x this is equal to 1 plus x plus integral 0 to 1 x minus s multiplied with 1 plus s plus s square by 2 d s and evaluating, we can find this will be equal to 1 plus x plus x square by 2 plus x cube by 6 plus x to the power 4 divided by 24. So, that is actually equal to 1 plus x plus x square by factorial 2 plus 2 x cube by factorial 3 plus x to the power 4 by factorial 4. So, ultimately y in x this is equal to sigma r running's from 0 to n minus 1 full to the power r x to the power 2 r plus 1 divided by factorial 2 r plus 1.

And again in this case limit n tends to infinity y n x this converges to e to the power x so; that means, we are having the same solution, whatever we have obtained using the initial guess that is 0 th order approximation, y 0 x equal to 1 plus x in case of 0 th order approximation y 0 x equal to 1 we are having the same solution. So, from here you can understand clearly that, there is no harm for considering some other function as 0 th order approximation only point you have to keep in mind, that assume function should be continuous.

Now of course, according to this kind of scheme the kernels are continuous and assuming y 0 x equal to f x is a most convenient one, because it immediately satisfies the continuity condition and of course, you will be having the solution as is in the (( )) form.

Now, we are going to see what is the proof behind the method that whether allows irrespective of the structure of the function; that means, just assuming f x and k x comma s both are continuous, whether will it be possible to prove that these type of approximation always converges uniformly to a continuous function or not.

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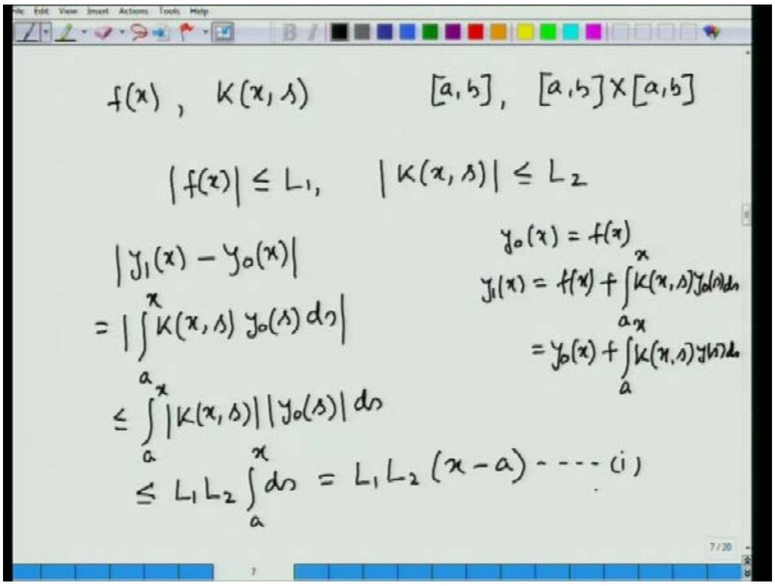

Now, for this purpose we start in this way we are assuming, f x and k x comma x as both of them are continuous over the domain a comma b and a comma b cross a comma b. So, therefore, they are bounded both of these functions are bounded, because a b are finite these are closed interval this is a closed square and we assume these bounds as L 1 and L 2 respectively; that means, f x this is less than equal to L 1 and k of x comma s this is less than equal to L 2, using the continuity condition we can found these are the bounds.

Now, our target is to prove that eta x uniformly converges to a continuous function, so for this purpose first of all we can calculate this difference modulus y 1 x minus y 0 x this modulus, now this is equal to modulus of integral a to x k x comma s y 0 s d x, this is coming in this way if you assume y 0 x is equal to f x. So, from the result y 1 x equal to f x plus integral a to x k of x comma s y 0 s d s, we can write this is equal to y 0 x plus integral a to x k of x comma s y s d s, so clearly these y 1 x minus y 0 x is equal to integral a to x k x comma s y s d s so; that means, modulus of y 1 x minus y 0 x is equal to this one.

Now, using the well known inequality you can write this is less than equal to a to x modulus of k x comma s multiplied by modulus of y 0 s d s, now this y 0 s is again our f s, so this is always less than equal to L 1 k x comma s is less than equal to L 2, so this is less than equal to L 1 multiplied by L 2 integral a to x d s, so this is equal to L 1 L 2 x minus a this is call it 1, then we calculate the difference between y 1 x and y to x.

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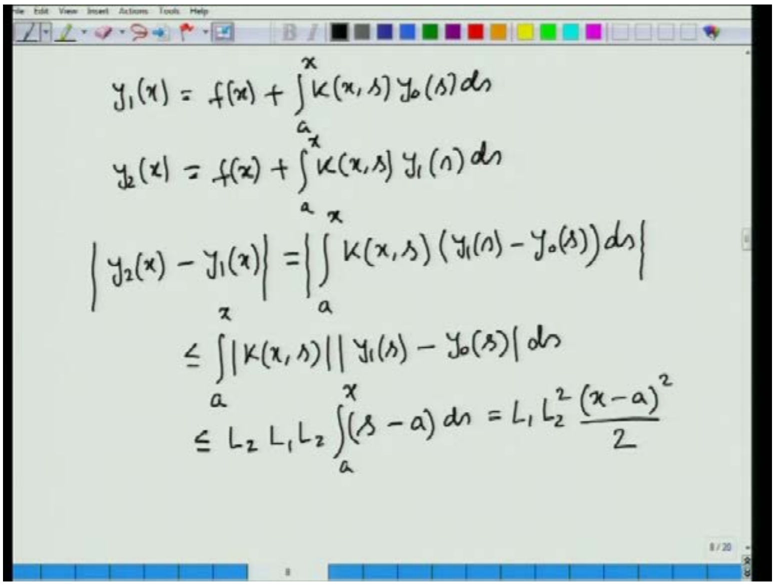

Now y 1 x this is equal to f x plus integral a to x k x comma s y 0 s d s y to x this is equal to f x plus integral a to x k of x comma s y 1 s d s, so subtracting we can find y 2 x minus y 1 x this is equal to integral a to x k of x comma s y 1 s minus y 0 s this d s; so if you take the modulus on both sides, so modulus of y 2 x minus y 1 x this is equal to modulus of integral a to x this 1.

Now, this modulus is less then equal to integral a to x modulus k x comma s this modulus multiplied by modulus y 1 s minus y 0 s d s, now be careful here, we have modulus y 1 s minus y 0 s; now this s is ranging between a to x and therefore, we can use this result that is modulus y 1 x minus y 0 x is equal to this 1, so when you replaced x by s, so in this position we will be having s.

So, first of all we can apply this is less than equal to L 2 and this part is less than equal to L 1 multiplied by L 2 integral a to x s minus a d s, now here we can write this s minus a without modulus sign, because s is greater than a and therefore, after evaluating we can find this is equal to L 1 times L 2 square times x minus a whole square divided by 2.

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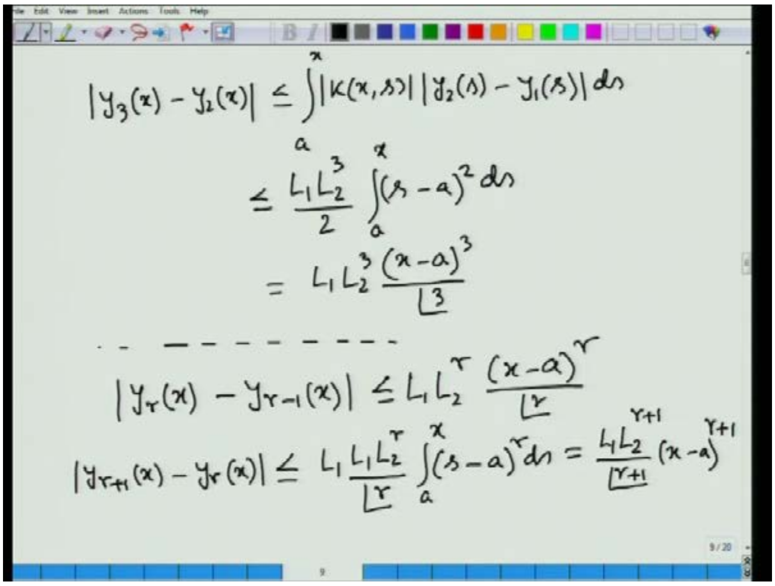

Now, in the next step you have to proceed follow the same procedure, so in the next step if we just write, so then modulus of y 3 x minus y 2 x this will be less than equal to a to x after taking modulus inside we will be having k x comma s then modulus y 2 s minus y 1 s d s. So, this is less than equal to L 1 times L 2 cube because one L 2 comes from here, L 2 square coming from here, so we will be having L 2 cube divided by 2 then, integral a to x s minus a whole square d s and this is equal to L 1 L 2 cube x minus a whole cube divided by factorial 3.

So, proceeding in this way as an induction hypothesis, we can assume that y r x minus y r minus 1 x this is less than equal to L 1 times L 2 to the power r x minus a whole to the power r divided by factorial r. And from here we can prove easily y r plus 1 x minus y r x this is less than equal to if we combine these concepts, so it will be l 1 multiplied with l 1 l 2 to the power r divided by factorial r integral a to x s minus a whole to the power r d s and this is equal to L 1 L 2 to the power r plus 1 divided by factorial r plus 1 x minus a whole to the power r plus 1. So that means, this induction hypothesis we have assume for r we proved it for r plus 1, so that means, this result is valid for all positive integral n.

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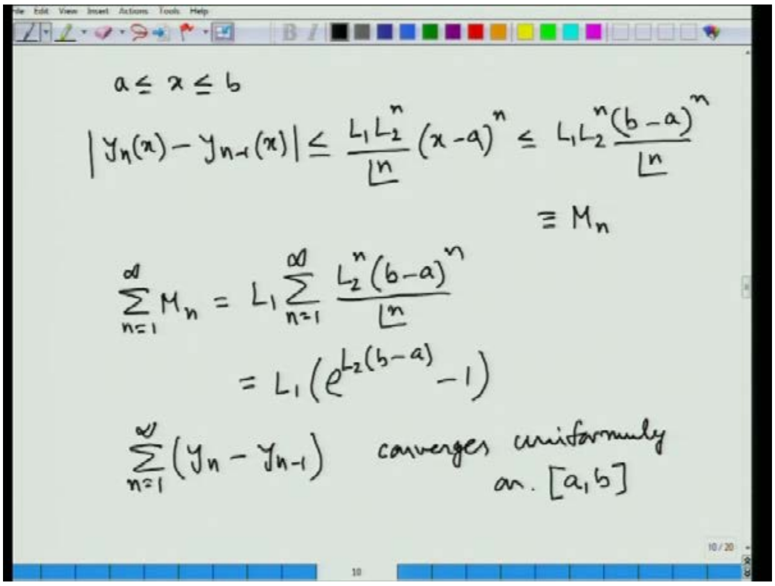

Now, we have to keep in mind that, these condition is satisfied that is x always lying between a and b, so once this x is less than equal to b, so ultimately we can write for n that is y n x minus y n minus 1 x, we have proved this will be less than equal to L 1 L 2 to the power n divided by factorial n multiplied by x minus a whole to the power n, now as x less than equal to b, so this expression will be less than equal to L 1 times L 2 to the power n b minus a to the power n by factorial n and we denote these by M n.

And therefore, we can write sigma n equal to 1 to infinity M n this is equal to L 1 times summation n running's from 1 to infinity L 2 to the power n b minus a whole to the power n divided by factorial n and this is equal to L 1 times e to the power L 2 times b minus a minus 1. And hence the conditions for y stresses M test these are satisfied and therefore, the series summation n running's from 1 to infinity y n minus y n minus 1 x this converges uniformly, this converges uniformly on the interval a comma b, that means, x belongs to this interval.

(Refer slide time: 27:21) Now, if we go back to the previous slides then we can see that y 1 x is calculated from this result, that is f x plus this 1, now f x is continuous as f x is continuous, so after integration we can find y 1 x is continuous once y 1 x is continuous from the definition of y 2 x 1 should be calculating y 2 x then y 2 x is equal to this (Refer slide time: 30:45) 1 f x is continuous k x comma s this is also continuous y 1 x is continuous, so y 2 x is continuous. So that means, y 0 y 1 y 2 these are continues and in this way the n the iteration y n x is also continuous and therefore, the sequence of functions they are continuous functions and ultimately we are able to prove that n tends to n running's from 1 to infinity y n x minus y n minus 1 x, they converges uniformly over the interval a comma b.

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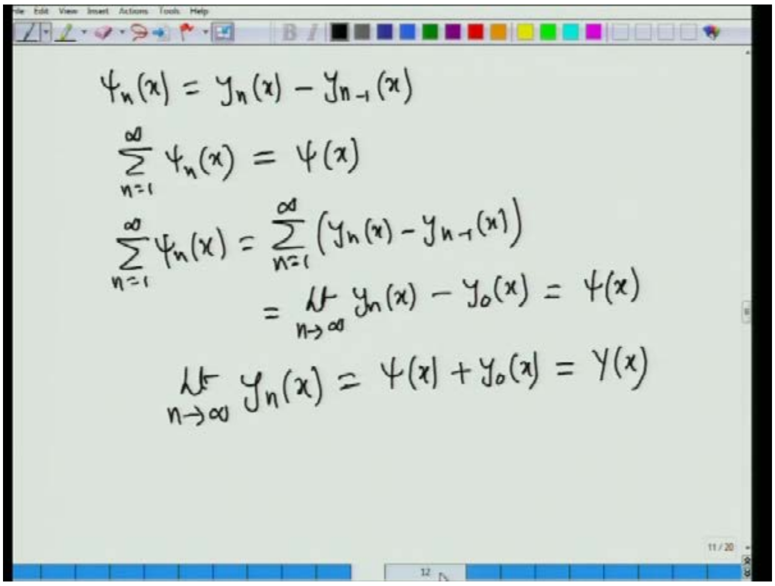

Now, for a notational convenience if we define psi and x is equal to y n x minus y n minus 1 x, then the series convergent n running's from 1 to infinity psi n x and let us assume these converges to psi x. Now if we look at this particular series, that summation n running's from 1 to infinity psi n x this is equal to summation in running's from 1 to infinity y n x minus y n minus 1 x this is equal to nothing, but limit n tends to infinity y n x minus y 0 x, because in the summation we will be having y 1 minus y 0 plus y 2 minus y 1 plus y 3 minus y 2 and so on.

So that means, y 1 y 2 y 3 all will cancels with each other, we are only left with y 0 x and the last term that is y n x as n tends to infinity, now as n tends to infinity summation psi n x is convergent these converges to psi x and therefore, limit n tends infinity y n x these converges to psi x plus y 0 x as per our previous notation we have defined it as y x.

So, this is the proof of the result that these approximate, iterates are actually converging to a uniformly continuous functions, because each iterates is continuous and the series summation n running's from 1 to infinity psi n x minus psi n minus 1 x, this is uniformly convergent over the interval a comma b and denoting the sum function psi n x and the sum function psi n x equal to psi x, we have established that limit n tends to infinity y n x they converges to this one.

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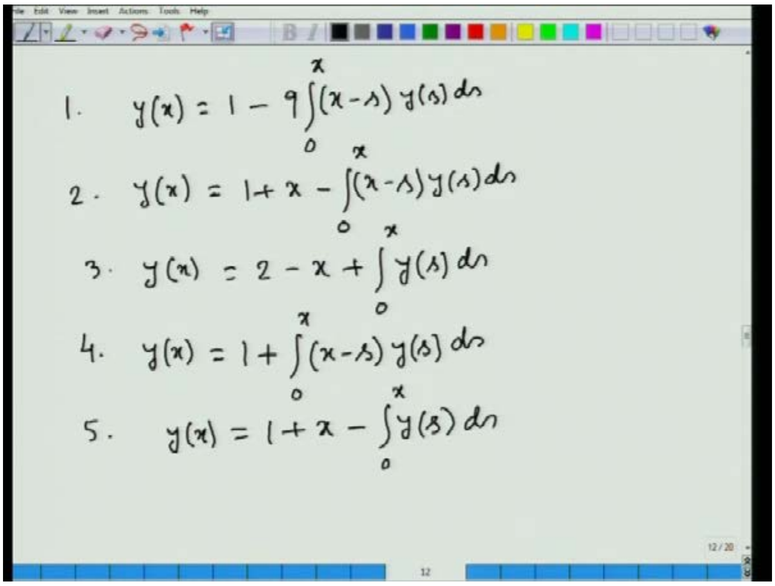

Now, I just write here some problems for your practice, that is problem number 1, y x is equal to 1 minus 9 integral 0 to x minus s y s d s, this is the first problem. Number 2 y x equal to 1 plus x minus integral 0 to x minus s y s d s, number 3 y x equal to 2 minus x plus integral 0 to x y s d s, number 4 y x equal to 1 plus integral 0 to x minus s y s d s and problem number 5 y x is equal to 1 plus x minus integral 0 to x y s d s.

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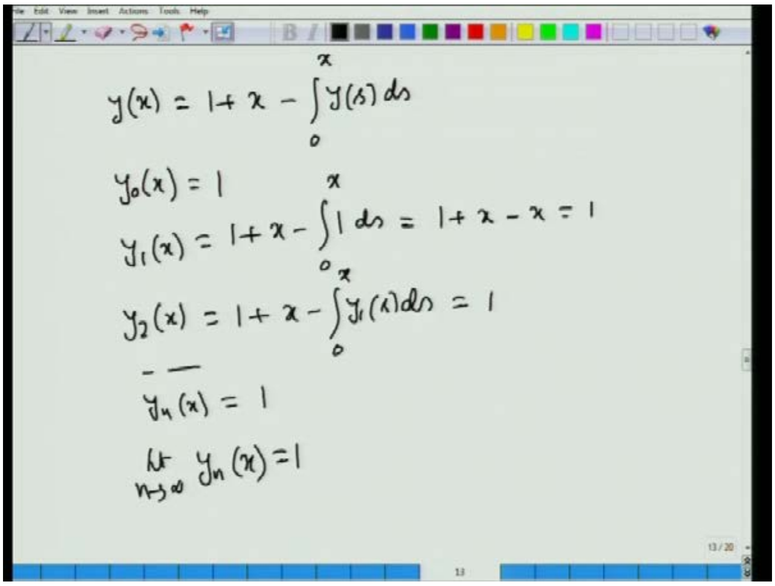

Now, before coming to the end of this lecture, I like to draw your attention towards the last problem this problem is very much interesting, in the proof you have seen that it is very easy to use the continuity of a f x to prove the result of uniform convergence of the iterates; sometimes other assumption gives us easily the decides solution, apart from the assumption y 0 x equal to f x.

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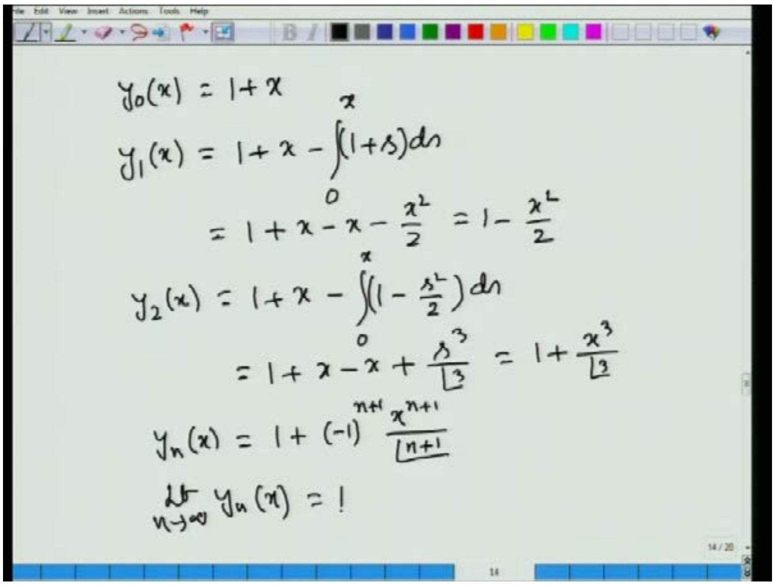

And last one is a such type of example, if we just try to solve this problem y x is equal to 1 plus x minus integral 0 to x y s d s with the assumption y 0 x this is equal to 1 you can see that y 1 x is coming out to the 1 plus x minus integral 0 to x 1 d s, so that means, 1 plus x minus x is equal to 1. So, similarly y 2 x is equal to 1 plus x minus 0 2 x y 1 s d s, so this is exactly equal to 1, so clearly you can see in this way y n x this is equal to 1 and hence limit n tends to infinity y n x this is equal to 1 (Refer slide time: 46:34).

Now, if you just solve this problem with the assumption y 0 x equal to f x; that means, 1 plus x then at every step, you will be having some additional term that is y 1 x this is equal to 1 plus x minus integral 0 to x 1 plus s d s, because this 1 plus s is your y 0 s and after integration it will give you 1 plus x minus x minus x square by 2, so this is equal to 1 minus x square by 2.

Similarly, y 2 x this will be 1 plus x minus integral 0 to x 1 minus s square by 2 this d s, so this will be 1 plus x minus x plus x cube by factorial 3, so this is equal to 1 plus x cube by factorial 3. So, in this case, in that the n th term y n x you will be having an additional term 1 plus minus 1 whole to the power n plus 1 x to the power n plus 1 by factorial n plus 1 of course, for any real x this term will goes to 0. And ultimately you will be having limit n tends to infinity y n x this is equal to 1 which is the same solution as we have obtained by assuming y 0 x.

So, sometimes that clever choice for y 0 x gives a little bit easier way to get the solution, but of course, you do not need to bother about whether will be choosing a f x or 1 plus x. Ultimately, what you are may be your choice if f x is continuous and k x comma s and it is partially derivate both of them are continuous over the hmm interval a comma b the and the square of length b minus a that is a comma b cross a cross b a comma b, then you will be having these iterates converges to a continuous function and those converges is the uniform convergence and ultimately you will be having the solution.

So, today I can stop at this point, in the next lectures we will be looking at some other different methods to solve these kinds of volterra integral equation and after discussing all those techniques for solving volterra integral equation. We will be making a comparative study of the method what are the advantages for using different types of method, but what are the, in between mathematical techniques involved to the problem which will may guide us, what method will give us the right solution very quickly, so; that means, efficiency of the method is very much important.