Calculus of Variations and Integral Equation

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Lecture No. # 23

Welcome viewers to the third lecture, for the lecture series on Integral Equation under NPTEL. In the second lecture, we have discussed the Leibnitz rule, that is let us use to convert the integral equation to ordinary differential equation. And we solved the ordinary differential equations, and verified that obtained solutions also satisfies the given integral equation. Next, we have considered the detailed proof of generalized replacement lemma.

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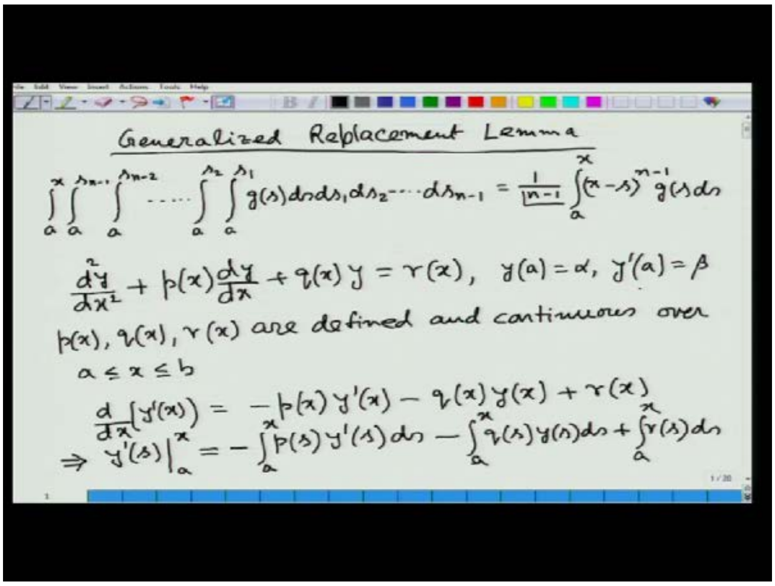

This generalized replacement lemma, we today again try to recall. Generalized replacement lemma, in terms of mathematical notations, this is integral from a to x, a to s n minus 1, integral a to s n minus 2, a to s 2, integral a to s 1, g s d s d s 1 d s 2 up to d s n minus 1. This multiple integral is equal to 1 by factorial n minus 1 integral a 2 x x minus s whole to the power n minus 1 g s d s. This was the generalized replacement lemma.

Now, we use this generalized replacement lemma in order to convert second and higher order ordinary differential equations into volterra integral equation of the second kind. At the first two lectures, we have seen some examples that how second or higher order linear ordinary differential equations can be converted into an integral equation. Now, in this lecture we start with a general second order linear ordinary differential equation which is an initial value problem.

So, first we consider the ordinary differential equation that is given by d to y d x 2 plus p x d y d x plus q x y this is equal to r x. This is the given differential equation subjected to the initial condition y a equal to alpha, y dot a, this is equal to beta. Now, in this equation p x q x and r x these 3 functions are known functions. These 3 functions p x, q x, r x, they are actually defined and continuous over, a less than equal to x and less than equal to b.

Now, our target is to convert this ordinary differential equation into an integral equation and in the process of construction of these equivalent integral equation from these ordinary differential equation, you can see that these two given initial conditions will be absorbed into the different steps of the formation of the integral equation. So, first of all we can rewrite these given ordinary differential equation into the form, that is d d x of y dot x this is equal to minus p x y dashed x minus q x y x plus r x as per the given conditions on p q r this right hand side is an integral function.

So, we are actually intended to integrate both sides from the limit a to x and seems x is already involved into the equation and of course, x is the independent variable and we are going to integrate from 0 to x. So, what we can do we can rewrite these expressions where x is replaced by s. So, these d x will comes out to be d s, and then we integrate from the range s equal to a to and 2s equal to x. So, after integration it will comes into this particular form that y dot s limit from a to x, this is equal to minus this is equal to minus integral a 2 x p s y dashed x d s minus integral a to x q s y s d s plus integral a 2 x r s ds.

Now on the left hand side it will be y dot x minus y dot a and y dot a is already given this is beta. So, left hand side will be y dot x minus beta, and we can shift these beta onto the right hand side in order to get the result that d y d x, which stands for y dot x, this is equal to beta, minus integral a to x p s y dot s d s minus integral a to x q s y s d s plus integral a to x r s d s.

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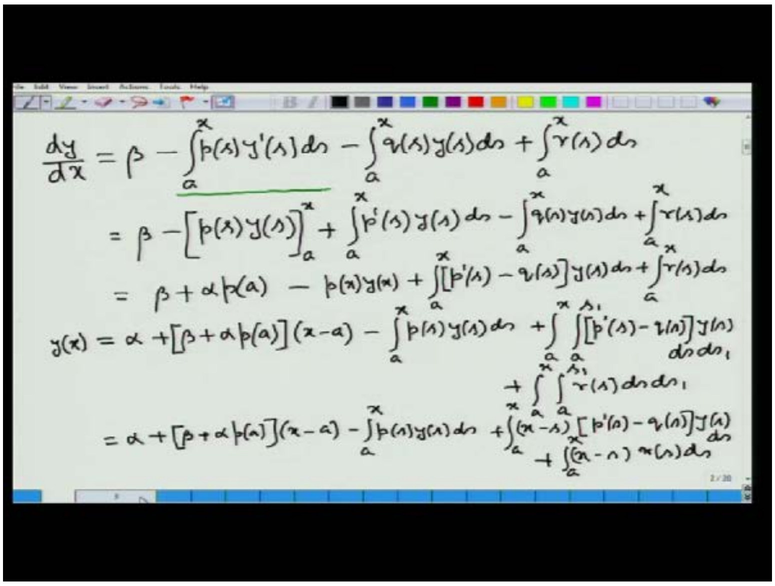

Now, we are just going to integrate these integral using the formula for integration by parts. If we integrate these integral using the formula for integration by parts, then will be having beta, minus p s y s. Actually we were considering these y dot s, as p and p as u. Limit from a to x, then it will be plus integral a to x p dashed s y s d s plus integral sorry this will be minus a to x q s y s d s plus integral a to x r s d s. And once you substitutes this lower limit this is a constant and you can recall y at s equal to a, it is given p s is a known function. So, after substituting s equal to a, this will also be a known quantity. So, will be having this constant time that is beta plus alpha p a, this quantity minus p x y x plus integral a to x p dashed s, minus q s, this multiplied by y s d s, plus integral a to x r s d s. So, this is actually expression for d y d x equal to these one.

So, now, before proceeding further, I just want to draw your attention. If you look at this mathematical expression and this is also an equation. Equation in terms of the unknown y, now in this equation derivative of y is involved here on the left hand side, again the unknown function y appears under the integral sign. So, the question is what type of equation is it. It is actually known as integral differential equation, where in the equation derivative of y is involved again integral of y is also involved. After few lectures, we will be discussing little bit about this into differential equations, this is just for your information at this moment.

Now, again if we integrate this result that is d y d x equal to given by this one, then after integration from a to x within this range will find y x minus y, y equal to alpha, we can transfer this alpha 1 to the right hand side to get alpha plus beta plus alpha p a, it is multiplied with x minus a, minus integral a to x p s y s d s plus integral a to s, sorry this will be a to x, a to x a to s p dot s minus q s y s d s d s 1, this plus integral a to x integral a to s 1 r s d s d s 1.

Now, in last two integrals, we have to apply the generalized replacement formula. After applying generalized replacement formula, we will be having this alpha plus beta plus alpha times p a x minus a, this will remains unaltered integral a to x p s y s d s, this will be plus integral a to x x minus s p dot s minus q s y s d s plus last integral will be integral a to x x minus s r s d s.

Now, you can see first two terms, that is alpha plus beta plus alpha into p minus a, into x minus a, this is known quantity, r s is a given function. So, you can find out a to x x minus s r s d s, and third and fourth term involving y s, these actually involves the unknown quantity y.

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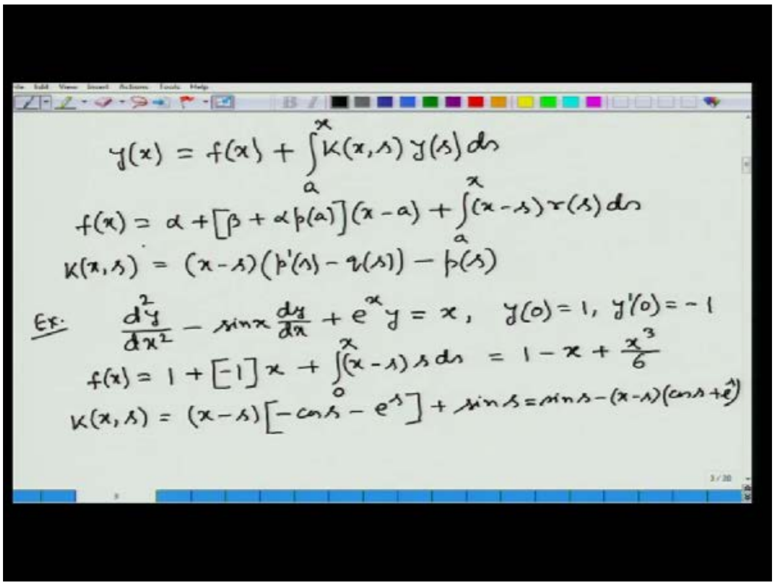

So therefore, we can write this expression into the form y x equal to f x plu integral a to x k x comma s y s d s, where this f x is equal to this expression that is alpha plus beta plus alpha p a into x minus a plus the last integral involving r; that is integral a to x, x minus s r s d s. And from the previous light, you can see that this entire expression p dot s minus q s multiplied by x minus s, and minus p s these actually give you the targeted kernel of the integral equation that is k of x comma s. And hence the kernel of the integral equation k of x comma x, that is given by x minus s multiplied with b dot s minus q s minus p s.

So, this is our target integral equation that we have obtained starting from the given general form of second order ordinary differential equation which is an initial value problem.

We can just have a look at one example; that means, we had to construct the integral equation corresponding to this differential equation d 2 y d x 2, minus sign x d y d x, plus e to the power x y is equal to x, where y 0 is given as 1 y dash 0, this is given as minus 1. So, if we compared with our general format for the second order ordinary differential equation then you can see p x is minus sin x known function, q x equal to e to the power x I is equal to x alpha is equal to 1, and beta this is equal to minus 1.

So, quickly if we substitute this expression into this formula, then we can find f x, this is equal to 1 plus beta is minus 1; now, p a p x is minus sin x, a is 0, so that means, p a is 0. So, no contribution is coming from alpha p a multiplied by x only, because a is 0 plus integral a to x here a is 0. So, lower limit will be 0 to x, x minus s, r x is x. So, r s will be s d s. And after evaluating this integral, you can find 1 minus x, first term x s d s, after integration will give you, x cube by 2 and minus s square at d s from 0 to x, will give you minus x cube by 3.

So, after simplification this will be equal to x cube by 6 and similarly, if you calculate the kernel of this integral equation. So, this will be x minus s, multiplied with p dot s, p is minus sin s, so, that means, it will be minus cos s, then q is to the power x, so, that means, minus e to the power s minus p s. So, it will be plus sin s. So, this is equal to sin s minus x minus s, multiplied the cos s, plus e to the power s.

Now, here I substitute these expressions into this easily available formula written here, but in terms of the practical problem and you have to attend, then you have proceed step by step but the way out will be these 1. Of course, this strategy can be applied to higher order ordinary differential equations also, and here I just consider another example on higher order linear differential equation, ordinary differential equation where initial condition is given. But you to keep in mind that formulation of a general result that what will be the resulting voltera integral equation from a given n th order ordinary differential equation with a little bit difficult problem.

And we can attack that problem using the strategy what we have adopted earlier, if recall from the first lecture that equations integral equations was constructed from the differential equation, where given equation was of the order second order differential equation, we have started with d 2 y, d x 2 equal to phi x. And from there we have calculated d y d x and y in terms of phi and substituted into the original equation in order to get the resulting integral equation. That point I will come later after this example.

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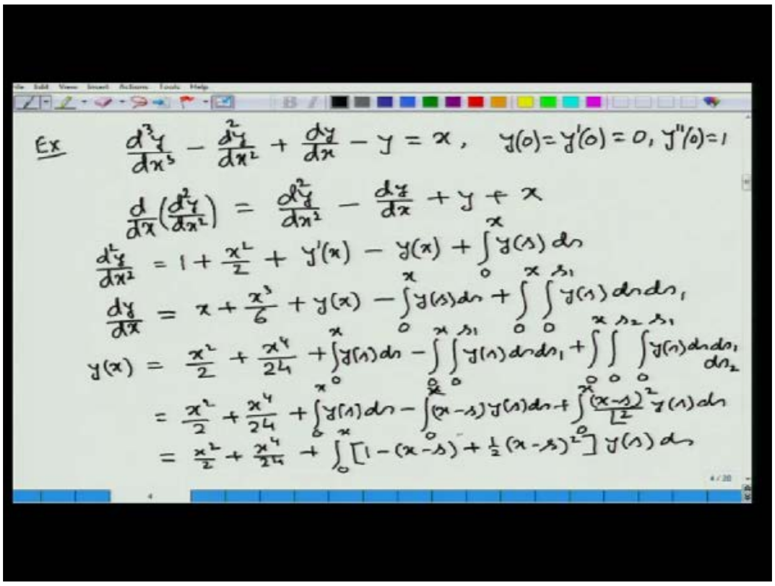

Now, here and in the example is d 3 y d x 3 minus d 2 y d x 2 plus d y d x, minus y, this is equal to x, where given initial conditions are, y 0 equal to y dot 0, this is a equal to 0,and y double dot is equal to 1. So, if we proceed a similar manner, that first of all from the give any question we can write d d x of d 2 y d x 2, that is equal to integral sorry it will be equal to d 2 y d x 2, minus d y d x, plus y, plus x.

Now, we have to integrate these from 0 to x. So, after integration you can find d 2 y, d x 2 minus d 2 y d x 2 at x equal to 0 this is 1. So, you can transferred this 1 on the right hand side, and after that we can write integral of these term that is x from the range 0 to x. So, it will be, x square by 2, plus integral of this expression that is d 2 d x 2. So, it will results in y dot x, minus 0, because y dot 0 is equal to 0, then minus y x plus 0, because y 0 is given here exactly equal to 0, plus integral 0 to x, y s d s.

Again you can write the left hand side in terms of d t x of d x and again after integration you can find from here d y d x, this is equal to no constant term will be adjusted either on the left or right, because d y d at x equal to 0. So, this is equal to x plus, x cube by 6, plus integral of y dot x, that is equal to y x minus 0, minus integral 0 to x, y s d s and this will be change to 0 to x 0 to s 1 y s d s d s 1. Again integrating from 0 to x, you can find y x, this is equal to x square by 2 plus x to the power 4 by 24 plus integral 0 to x y s d s minus integral 0 to x integral 0 to s 1 y s d s d s 1 plus integral 0 to x integral 0 to s 2 integral 0 to s 1 y s d s d s 1 d s 2.

Now, in these 2 last integrals, we had to apply generalized replacement formula. So, after applying that formula we can find this will be equal to x square by 2 plus x to the power 4 by 24 plus integral 0 to x y s d s minus integral 0 to x x minus s y s d s plus integral 0 to x x minus s whole square by factorial 2 y s d s.

So, ultimately this is equal to the integral equation x square by 2, plus x to the power 4 by 24, plus integral 0 to x 1 minus x minus s plus half of x minus s whole square these multiplied with y s d s. So, this is actually our resulting integral equation. So, this part x square by 2 plus x to the power 4 by 24, this is equal to your f x and 1 minus x minus s plus half of s minus 2 whole square, this is actually the kernel of the integral equation.

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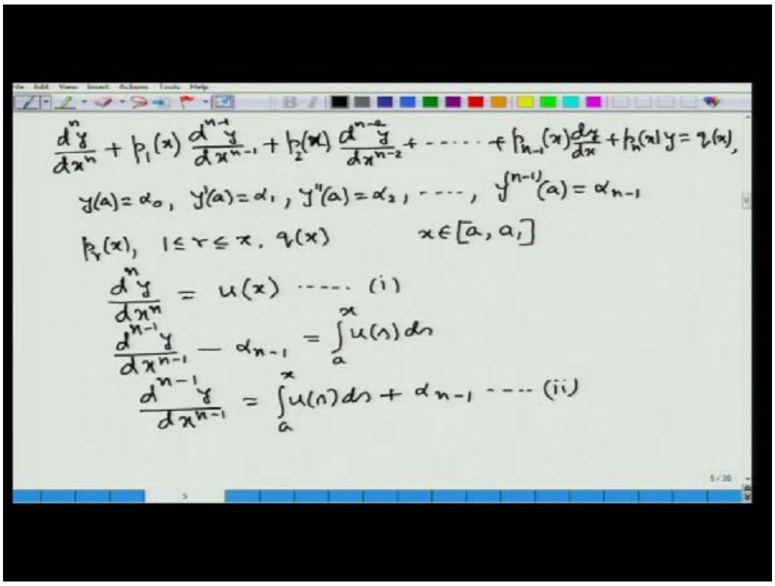

Next, we consider the formulation of integral equation starting from a given n th order ordinary differential equation with n initial conditions those are also prescribed. Now, we consider this equation that d n y d x n plus p 1 x d n minus 1 y d x n minus 1 plus p 2 x d n minus 2 y d x n minus 2, proceeding in this way, this will be up to p, n minus 1 x, d y d x, plus p n x y; this is this is equal to q x. And given initial conditions are y a, equal to alpha 0 y dot a, equal to alpha 1 y double dot a, that is equal to alpha 2, in this way y n minus 1 a, that is equal to alpha n minus 1.

And all these functions that is p r x where 1 less than equal to r less than equal to x and q x, they are defined and continuous over some closed interval of x, starting from the left at a and on the right, ending let say a 1. Now, our target is convert these general linear n th order ordinary differential equation, where in initial conditions are prescribed, convert it to a volterra integral equation, and during the process of this transformation you have to be careful about the notation at every steps.

Let us start with the assumption in this case, we have to assume d n y d x n this is equal to u x. Now, this equation possesses a solution. That means, they exist at y which is continuously n times differentiable, satisfies this equation which is a solution of this equation and therefore, n th derivative of y with respect to s is equal to u x, that is also continuous. So, we can integrate it from the limit a to x in order to get the result d n minus 1 y, d x n minus 1, minus these derivatives evaluated at a, that means, y n minus 1 a, this is equal to alpha n minus 1, and this is equal to integral a to x, u s, d s, this 1.

Next we can transfer this quantity onto the right hand side to get the result d n minus 1 y, d x n minus 1, that is equal to integral a to x u s d s plus alpha n minus 1, we call this expression as 1 this as 2. Now, we are going to integrate this second result, that is d n y by d x n minus 1 equal to this expression from a to x.

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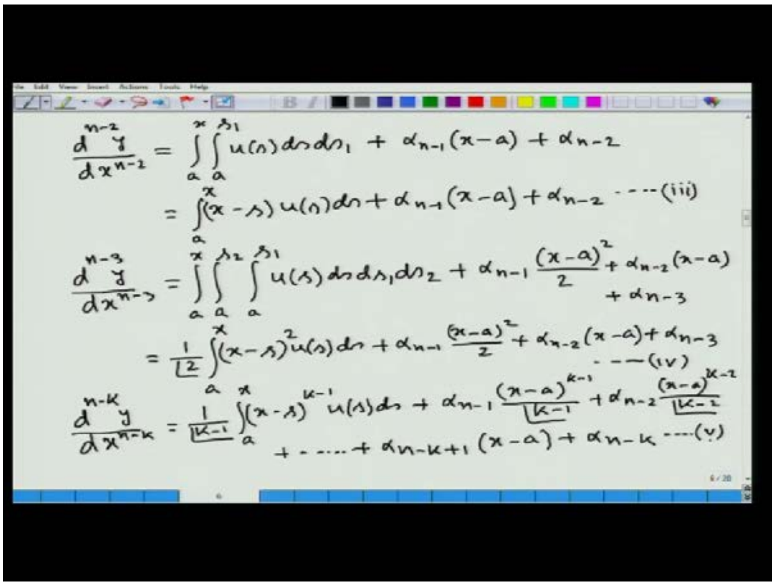

Once we integrate then we can find d n minus 2 y, d x n minus 2, equal to the constant evaluated at x equal to 0, that is alpha n minus 2, can be transformed onto the right hand side and then we will be having integral a to x, integral a to s 1, u s d s d s 1, plus alpha n minus 1, x minus a, plus alpha n minus 2. Remember this alpha n minus 2, coming from the left hand side and applying the formula, we can find, this is equal to integral a to x, x minus s, u s, d s, plus, alpha n minus 1, x minus a, plus, alpha n minus 2, call it 2, sorry it will be 3, because last question was 2.

Now, we integrate not the exactly expression d n minus 2 y, d x n minus 2, equal to this 1, rather we are integrating the first expression that is d n minus 2 y by d x n minus 2 equal to double integral this 1, such that after introducing the integral sign we can apply the general replacement formula. So, using that technique we can find d n minus 3 y, d x n minus 3, this will be equal to, integral from a to x, integral a to s 2, integral a to s 1, u s d s d s 1 d s 2.

Now, this term that is alpha n minus 1, x minus a, after integration from a to x, will produce, x minus a whole square, by factorial 2, here we will find alpha n minus 2, x minus a, plus, alpha n minus 3, that actually come from the left hand side. So, ultimately this is equal to, 1 by factorial 2, integral a to x, x minus s, this whole square, u s, d s, plus, alpha n minus 1, x minus a, whole square by 2 plus alpha n minus 2, x minus a, plus alpha n minus 3, call it 4.

Now, you just have a look at the analogy of this order of the derivative and suffixes appearing here. In case of n minus 2, when order of the derivative is n minus 2, so, constant is going up to n minus 2, alpha n minus 1 alpha minus 2. When it is n minus 3 th order derivative is involved then alpha minus 1 alpha n minus 2, up to alpha n minus 3, and in this case, x minus s whole to the power 1; in this case x minus whole square, so that means, at the k th step, we can find this result that, d n minus k y, d x, n minus k, that will be equal to…

Now just try to understand when 3 is here that is n minus 3. So, we have 1 by factorial 2 and x minus s whole square, when this is n minus 2, so 2 is here. So, this is only 1 by factorial, we can assume this expression is 1 by factorial 1 into x minus s whole to the power 1. So, in case n minus k, we will be having 1 by factorial, k minus 1, integral a to x, x minus s, whole to the power k minus 1 u s d x.

And then we will be having alpha n minus 1, x minus a, whole to the power k minus 1, here it will be factorial k minus 1 plus alpha n minus 2, x minus a, whole to the power k minus 2, by factorial k minus 2, plus dot dot. Last but one term will be plus, alpha n minus k plus 1, x minus a and last one will be alpha n minus k.

So, here it was alpha n minus 3, when we have the n minus 3 th order derivative of y on the left hand side, here it was alpha n minus 2, last constant term, where left hand side we have the derivative of n minus 2 th order. So, in case of n minus k, last constant will be alpha n minus k, we call this expression as 5.

So, proceeding in this way we can arrive at the second order, first order as well as only y, from here by taking k equal to n minus 2, k equal n minus 1, and finally, k equal to n, but here I am writing the result only for d y d x and y only.

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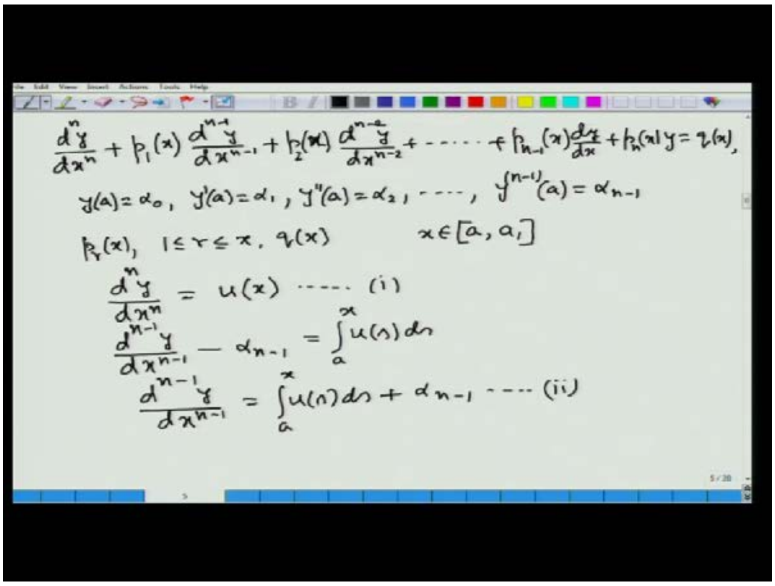

So, d y d x, this will be equal to, 1 by, factorial n minus 2, integral a to x, x minus s, whole to the power n minus 2, u s d s, plus, alpha n minus 1, x minus a, whole to the power n minus 2, by factorial n minus 2, plus, alpha n minus 2, x minus a, whole to the power n minus 3, by factorial, n minus 3, plus dot dot.

Proceeding in this way, last but one term will be, alpha 2, x minus a, plus, alpha 1 and again integrating this expression you can find y x is equal to, 1 by factorial, n minus 1, integral a to x, x minus s, whole to the power n minus 1, u s d s, plus, alpha n minus 1, x minus a, whole to the power n minus ,1 by factorial, n minus 1, plus alpha n minus 2, x minus a, whole to the power n minus 2, by factorial, n minus 2, plus dot dot, plus alpha 2, x minus a, whole square by 2, plus alpha 1, x minus a, plus alpha 0. This alpha 0 is coming from the left hand side because y x minus y a, y a is given to be alpha 0. So, this alpha 0 will come to the right hand side.

So, last equation we numbered it as 5. So, this will be 6 and this is 7. So, we have all the expressions starting from y, d y d x, d 2 y d x 2, up to d n y d x n, in terms of u. So, now, we are going to substitute all these expressions into the original given ordinary differential equation this one and after transferring n minus 1 th derivative of y up to y term onto the right hand side.

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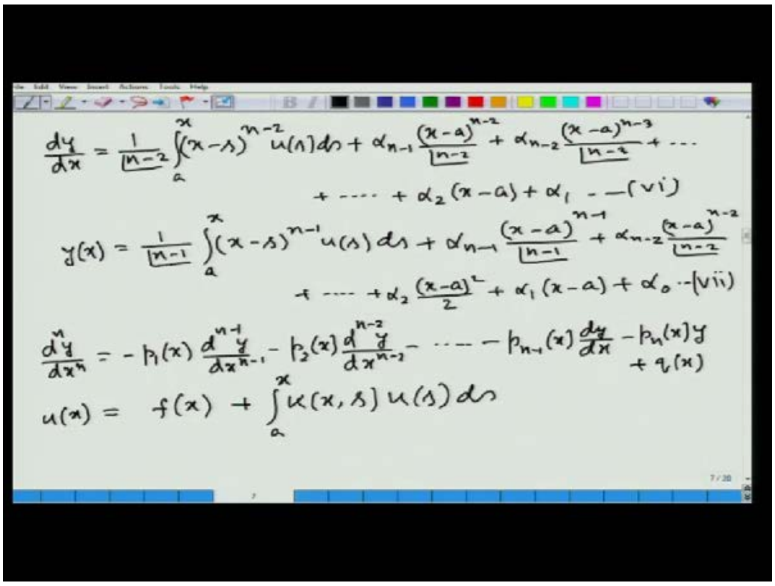

So, that means, we are rewriting the equation into this form. d n y, d x n, this is equal to minus p 1 x, d n minus 1 y, d x n minus 1, minus p 2 x, d n minus 2 y, by d x, n minus 2, minus dot dot, in this way ultimately p n minus 1, x, d y d x, minus p n x, y, plus q x. So, d n y d x n, this is equal to u x state for what. Now first of all we try to collect the terms which does not involve integral of use.

So, try to understand this point. First of all you will be having this term q x, then this entire expression in 7, that is alpha n minus 1, x minus a, whole to the power n minus 1, by factorial, n minus 1, up to alpha 0, this will be multiplied by p n x, then in expression 6, except the integral, entire expression will be multiplied with p n minus 1, proceeding in this way, finally, this this alpha n minus 1, it will be multiplied with p n minus 1 x.

So, what we can do, we can just claim that after collecting this term, we will be having an expression f x, plus integral a to x, k x comma s, u s d s. Now the task is we have to write what is f x and what is k x comma s. Now, just recall from here, first of all this alpha n minus 1, this alpha n minus 1 it will be multiplied with p n minus 1. So, which gives u ultimately,

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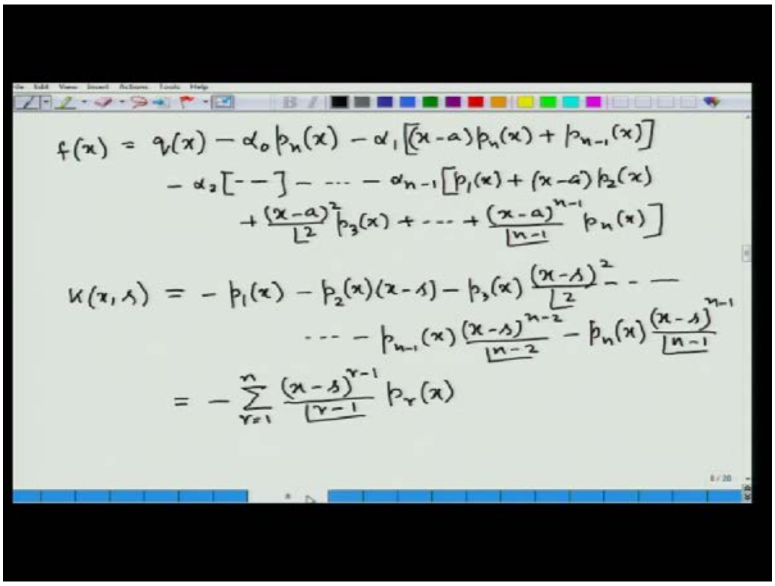

This f x equal to, q x minus just see alpha 0 will appear only with y x. So, first of all we can write the coefficient of alpha 0 now alpha 0 will be multiplied the p n x. So, therefore, it will be minus alpha 0, p n x. Then alpha 1 will comes from only expression of y x and d y d x. So, alpha 1, this x minus a, it will be multiplied with p n and alpha 1 will be multiplied with p n minus 1. So, therefore it will be minus alpha 1, then x minus a, multiplied with p n x, plus p, n minus 1 x.

Then u can write the coefficient of alpha 2, it will be 1 expression, proceeding in this way last term will be alpha n minus 1 and now just see what is the expressions will be involved from, the first 1, this alpha n minus 1, will be multiplied with p 1 x. Then from the second 1 alpha n minus 1 this x minus a will be multiplied with p 2 x, then x minus a whole square by factorial 2 this will be multiplied with p 3 x.

So, proceeding in this way we will be having this p 1 x, plus x minus a, p 2 x, plus x minus a, whole square by, factorial 2, p 3 x, plus dot dot, last term will be, x minus a, whole to the power n minus 1, by factorial n minus 1, p n x. This is actually expression for f x. And just recall q, p n, p n minus 1, up to p n x, all these quantities are known quantities, as well as the given quantities to this is completely a known function.

Now, what will be the structure the kernel, k x comma s, is actually collection of the integrant involved with y sorry u a stars. So, from the first 1 you'll be having just minus p 1 s. Then from the second 1, p 2 s multiplied with x minus s, then p 3 s multiplied with this one. So, proceeding in this way ultimately we will be having this expression that is minus p 1 x, minus p 2 x, it will be multiplied with x minus s, minus p 3 x, x minus a whole square, by factorial 2, minus dot dot, last but one term will be, minus p n minus 1 x, x minus s, whole to the power n minus 2, by factorial n minus 2, minus p n x, multiplied with x minus s, whole to the power n minus 1, by factorial n minus 1.

And we can write this entire expression into a compact form, using the summation notation, that is minus sigma r runnings from 1 to n, x minus s, whole to the power, r minus 1, divided by, factorial r minus 1, multiplied with p r x. So, if we start substituting r equal to 1, 2, 3, up to n. So, you will be having this entire expression. So that means, a general n th order linear differential equation can be converted into volterra integral equation of second kind by this method.

Now, the most crucial question is that, in one method, we have obtained the differential equation, in terms of the unknown function y, that is the first approach, and in the second approach we obtained the integral equation, in terms of the unknown function u, where u is the n th derivative of y. So, what is the relation between these 3, that is the given ordinary differential equation, integral equation in terms of the unknown variable involved with the ordinary differential equation and integral equation in terms of the unknown function, which we are introducing in terms of the formula d n y d x n equal to u x.

Answer to this question is that, if I x is a solution of the given ordinary differential equation, then u can verify, phi x will be solution of the integral equation what we have written in terms of y and n th derivative of this phi x, will be solution of the integral equation what we have written in terms of u.

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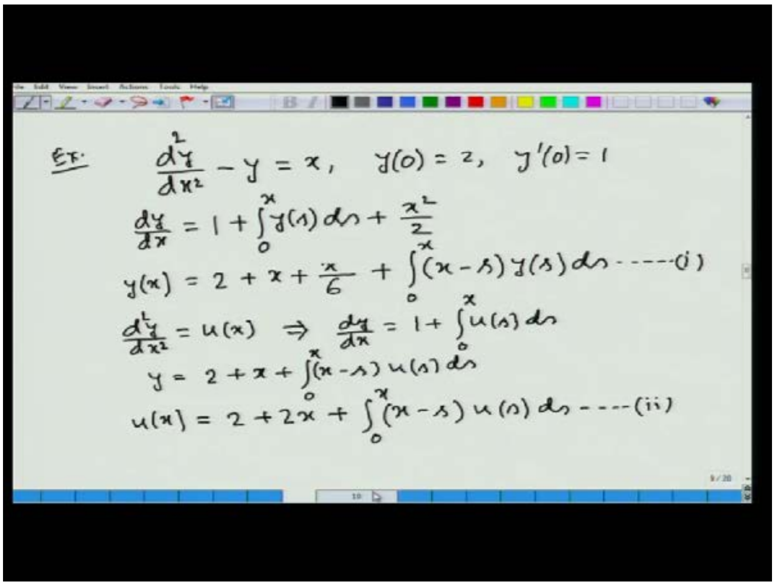

So, just have a quick look at this example, then u can clearly understand this idea. Consider this equation d 2 y, d x 2 minus , this is equal to x with the given conditions y 0 equal to 2 and y dot 0, this is equal to 1.

So, using the first approach integrating this equation, u can write, d y d x, this is equal to 1, plus integral 0 to x, y s, d s, plus, x square by 2, this 1 is coming from y dot 0 and further integrating, you can find y x, this is equal to 2, plus x, plus, x cube by six, plus it will be actually integral 0 to x, then integral 0 to s 1, y s, d s, d s 1 and after using the replacement formula you'll be having 0 to x, x minus s, y s, d s.

So, this is integral equation in terms of unknown function y. But if u use this notation d 2 y, d x 2 equal to u x, then from here, integrating from 0 to x, we can find d y d x, this is equal to 1 plus, integral 0 to x, u s d s and again by integrating this expression, we can find y equal to, 2 plus, x plus, again using the replacement formula, it will be x minus s, u s d s.

So, y d 2 y d x 2 equal to u x y is this 1. So, clearly from the given equation you can understand the required integral equation is u x equal to, y plus this one. So that means, 2 plus, 2 x plus, integral 0 to x, x minus s, u s d s. So, call it as 1 and call it as 2. So, this is the integral equation in terms of the unknown function y and this is the integral equation in terms of the u x where u is the second derivative of y.

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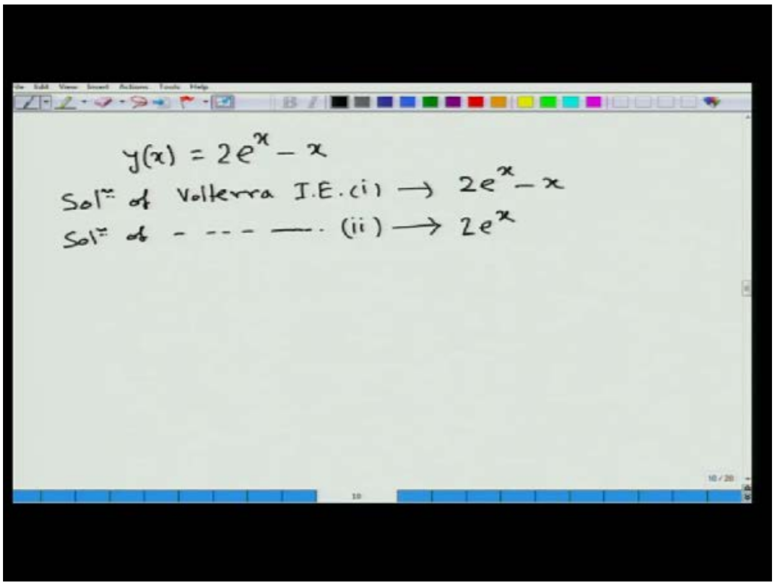

And for the given ordinary differential equation, if u solve it then u can find, solution will be y x equal to, 2 e to the power x, minus x. And therefore, solution of volterra integral equation, in short we can write this I E 1 is 2 e to the power x, minus x and solution of the volterra integral equation 2, that is in terms of u what we have written that will be the second derivative of this function, that is 2 e to the power x. You can verify these are the solutions of these 2 problems.

So, this lecture we can stop at here. Just for a quick recapitulation that we have started with a second order ordinary differential equation, initial value problem where p x, q x, r x is given, then using the generalized replacement formula and integrating the given equation both side twice, we obtained the general format for the integral equation in terms of y. This is a volterra integral equation of second kind where this f x given by this formula and kernel is given by this one.

Then we have considered one example where p x is minus sin x, q x is e to the power x and right hand side function is x, and this technique can be adopted for third or fourth order equation, but of course, it is not possible to extend this idea, that, it will be little bit difficult to arrange these terms appearing after integration to write down the volterra integral equation in terms of for the n th order ordinary differential equation. For those case we have to assume d n y d x n equal to u x, and then using the successive steps you can find n minus 1 th derivative of y in terms of u, n minus 2 th derivative of y in terms of u and in general n minus k derivative of y in terms of u; finally, you will be arriving at this integral equation.

And then I have explained here the relation between the 2 methods; that means, solution of the original differential equation will be solution of the integral equation, if it is written in terms of the unknown function involved with the differential equation, and it is nth derivative, in this case we are considering second order differential equation. So, second order derivative of solution of the given equation will be solution of the volterra integral equation, written in terms of u. So, I stop for this lecture, at this point. Thank you.