Calculus of Variations and Integral Equation

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Lecture No. # 22

Welcome viewers. This is second lecture for lecture series of NPTEL on integral equation. Before going to discuss about today's topic, I like to recapitulate quickly what we have discussed in the first lecture. We have started with the formation as well as formulation of integral equations, I have given 3 examples where initial value problems of ordinary differential equations, and boundary problem of ordinary differential equation is converted into integral equation. And then we have considered a physical problem whose description in terms of mathematical tools also leads to an integral equation which is known as Abel's integral equation. You can recall, we have considered mainly two types of linear integral equation, once is Fredholm integral equations.

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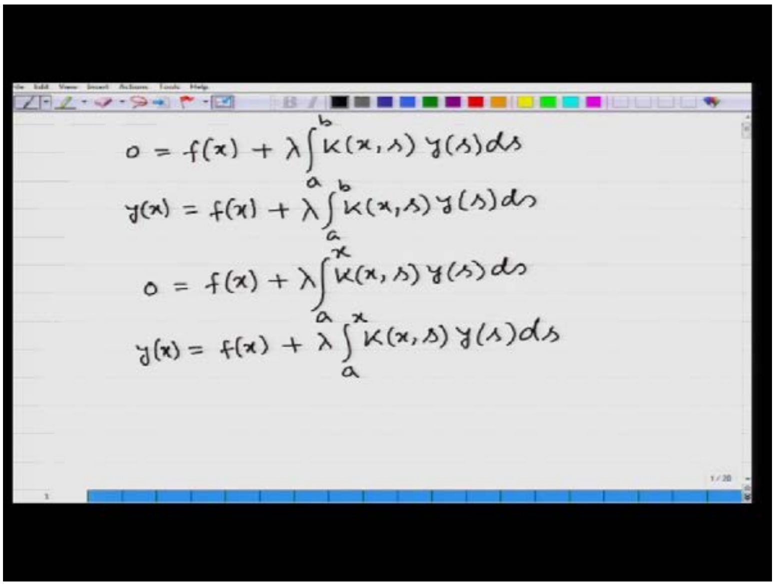

Fredholm integral equations are of two kinds. First kind of Fredholm integral equations is of the form, 0 equal to f x plus lambda integral a to b K (x, s) y(s) ds, and Fredholm integral equation of second kind is y(x) is equal to f x plus lambda integral a to b K (x, s) y(s) ds. This was first equation was of the first kind and second one is of the second kind, where range of integrations are 2 finite real numbers. Another type of the equations I have introduced yesterday, that is volterra type integral equations, if you have 0 equal to f x plus lambda times, integral a to x K (x, s) y s ds. This is a volterra integral equation of the first kind and volterra integral equation of the second kind is given by y(x) equal to f x plus lambda integral a to x K (x, s) y s ds.

So, last two are volterra integral equations and first of two are Fredholm integral equation and also we have discussed about the singular integral equations, and in One example we have considered where a given function was shown that it is a solution of the singular integral equation. Now, before proceeding further I just like to mention two main text books for this lecture series.

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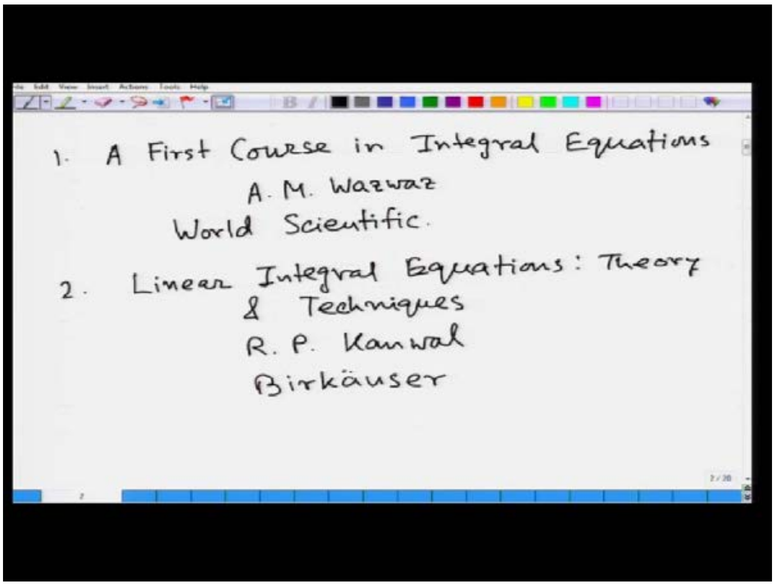

First one is a first course in Integral Equations by A M Wazwaz. This is a book from world scientific this is first, and another book that is Linear Integral Equations linear integral equations theory and techniques by R P Kanwal publisher this there are also several other books, but for preliminary level personally I like this two book very much and most of the lectures, and ideas without within the series is based upon these two books.

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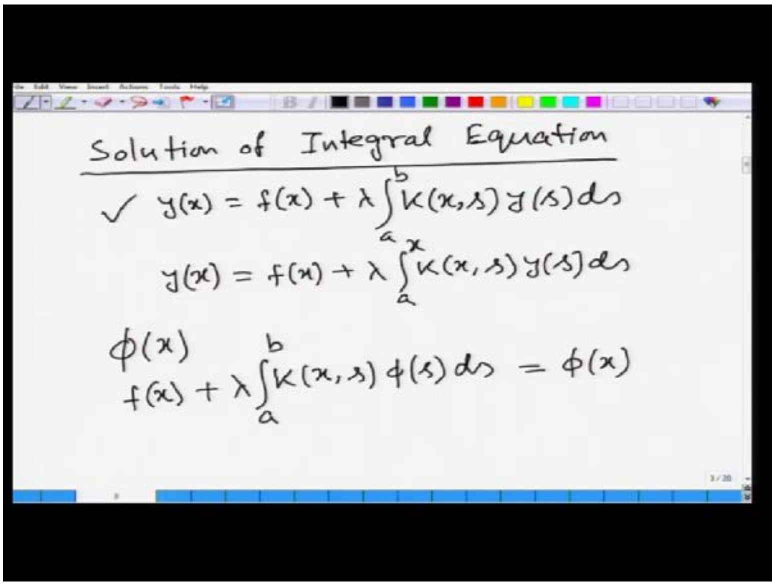

Now, today we start with the concept of Solution of Integral Equation solution of integral equation. Suppose we are considering either a Fredholm equation of the form y(x) equal to f x plus lambda integral a to b K (x, s) y s ds or we are talking about volterra integral equation y(x) equal to f x plus lambda integral a to x, K (x, s) y(s) ds. Of course, we may consider second first kind of equations in this case both the equations I have written for the second time also instead of second kind in as first kind of equations as well as singular integral equations.

Now, if we able to find out a function phi x such that when this phi x substituted into either these 2 equations, either of these 2 equations, and if it happens that right hand side after integration will be equal to the left hand side, then we say phi x is a solution of this integral equation.

So, that means, if we just consider this first example once we substitute it on to the right hand side then it will be f x plus lambda times integral a to b, K (x, s), then we are actually looking for unknown function y(x) and phi x is a possible candidate for the solution of the targeted problem. So, therefore replacing y by phi, if these expression after substitution and integration is comes out to be phi x, then we can say phi x is a solution of this integral equation. Now, this is just formal definition. First we consider few examples which will be actually verification of functions phi x, those are going to be solutions for the given integral equations.

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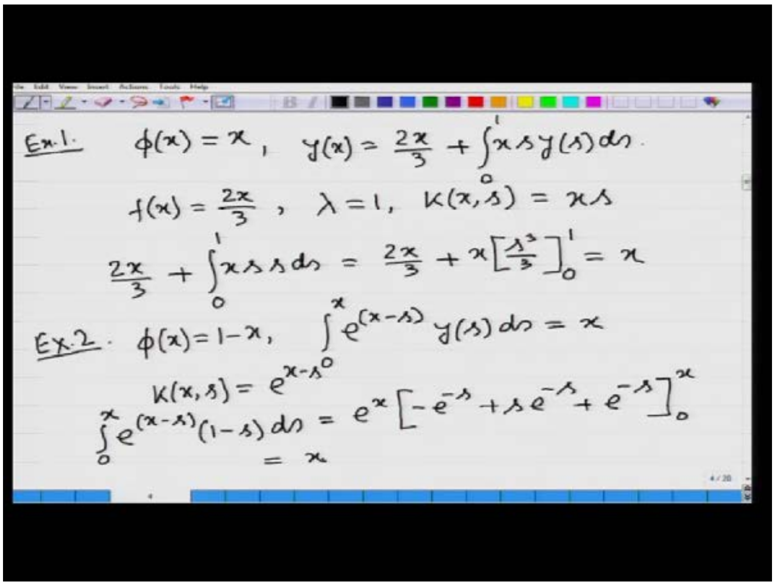

So, we consider example 1. This first example we are interested to check whether this function phi x equal to x is going to be a solution of the integral equation y x, is equal to 2 x by 3, plus integral 0 to 1 x s y s ds. If you look at this integral equation then you can understand this is a Fredholm integral equation of second kind, and here if you compared with the standard form then immediately you can verify f x is equal to 2 x by 3, lambda this is equal to 1 and kernel of the integral equation x (x, s), that is x into s.

Now, we have to verify whether this function phi x equal to x is a solution of this integral equation or not. So, that means, we have to substitute y(s) equal to s, because phi x is equal to x into this integral and we have to verify what will be the outcome. So, after substitution we will get 2 x by 3 plus integral 0 to 1, x s this is this part is the kernel and for y(s) if we substitute this phi. So, another is ds. So, this is equal to 2 x by 3 plus x and s cube by 3 with limit 0 to 1, and after substituting this limit you can verify this is coming out to be x. So, that is exactly equal to phi x and hence y(x) equal to x is a solution to this problem. Again just recall this is a Fredholm integral equation of second kind.

Now, we consider a volterra integral the equation, example 2. Here we are interested to verify whether phi x equal to 1 minus x is a solution to this integral equation 0 to x, e to the power x minus s y s ds equal to x. If you have a close look at this equation, then you can see y(x) does not appear explicitly into the equation outside of the integral sign, and here range of integration is 0 to x. So, therefore, this is a volterra integral equation of first kind. This given equation is a volterra integral equation of the first kind. If you compare this equation with a standard form then you can find f x equal to minus x, lambda equal to 1 and kernel K (x, s), this is equal to e to the power x minus s this is actually kernel for the given problem.

If we substitute this function phi into the integral then you can verify integral 0 to x, e to the power x minus s, 1 minus s, ds and this will be equal to e to the power x, if you integrate this integral then it will be minus e to the power minus s, this is coming out to be well, e to the power minus s will be multiplied with 1 and then you have another integral that is s into the power minus s.

So, using the formula of integral by parts you can find this is equal to plus s e to the power minus s plus e to the power minus s, and limit from 0 to x. If you substitute this limit and after simplification, you can find this is exactly equal to x and hence phi x equal to x is a solution of this integral equation.

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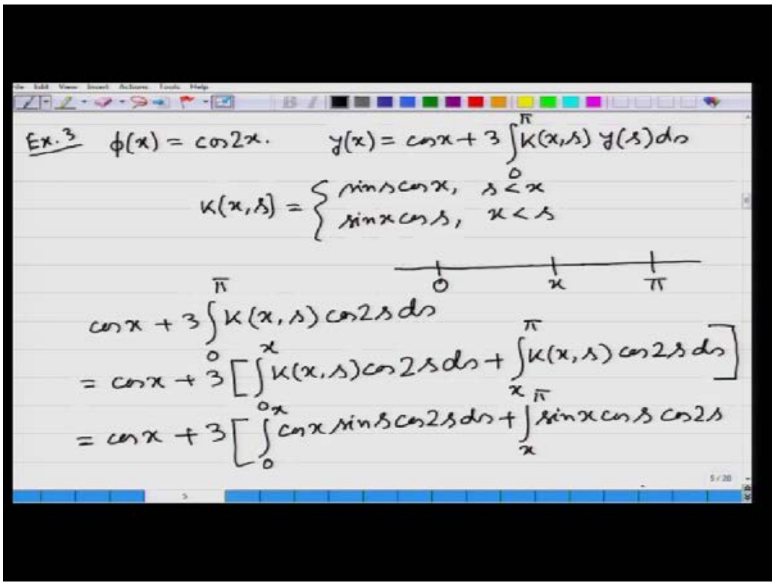

Next we consider another example. This example is little bit interesting in the sense in terms of the kernel involved with the problem. We have to verify whether phi x equal to cosine 2 x, this is a solution of the integral equation given by y(x) equal to cos x plus 3 integral 0 to phi K (x, s) y s ds, where this particular kernel is given by K (x, s), this is equal to given by sin s cosine x when s less than x and sin x cosine s, if x less than x. So, this is actually the given kernel and again here range of integration is finite that is 0 to pi. So, this is a Fredholm integral equation of second kind.

Now, we try to verify whether this function satisfies this equation or not. Before proceeding further just try to understand on this real line our range of integration is 0 to pi and here this kernel is defined in this way, it is equal to sin s cosine x whenever s less than x and this is equal to sin x cosine s whenever x is less than s. So, in order to incorporate this function under the integral sign and part from the integration we introduce the point x in between 0 and pi, and we divide this integral 0 to pi into 2 integrals 1 ranging from 0 to x another one ranging from x to pi and this calculation is little bit tedious, but still we can try to verify this.

So, we start with this integral cos x plus 3 integral 0 to pi, cosine sorry K (x, s) y(s) is cosine 2 s ds this is equal to cosine x plus 3 1 integral is 0 to x K(x, s) cosine 2 s ds, plus second integral x to pi cosine k s cosine 2 s ds.

Now, we can use the definition of cosine k x here s is ranging from 0 to x in the first integral and in the second integral range of integration is x to pi. So, after substitution of this expression we will be having cosine x plus 3 integral 0 to x, whenever x is greater than s; that means, s less than x. So, this will be cosine x sin s cosine 2 s ds plus integral x to pi, sin x cosine s cosine 2 s ds this one.

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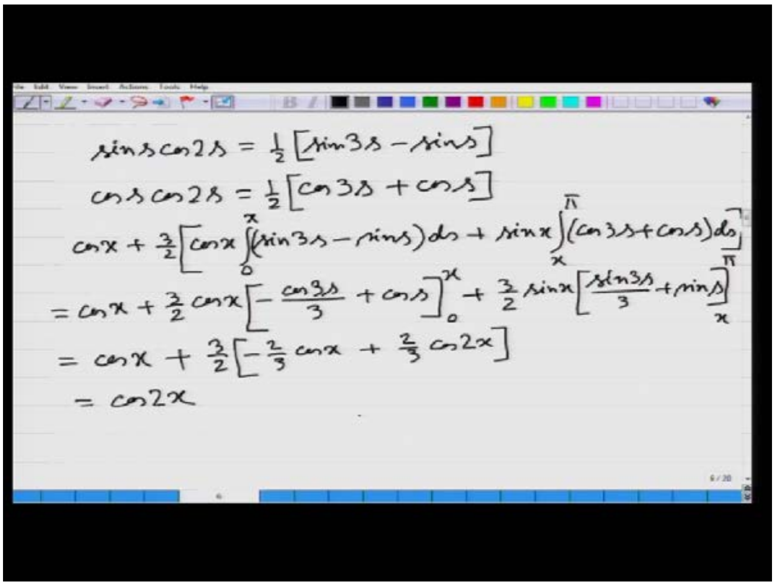

In order to solve this problem we have to use this formula that is sin s cosine 2 s, this is equal to half of sin 3 s minus sin s, this is actually coming under this first integral integrant is sin s cosine 2 s and for the second integrant you can find cosine s cosine 2 s. So, using the formula optigonometry, you can write cosine s cosine 2 s, this is equal to half of cosine 3 s plus cosine s.

So, after substituting this 2 results into the integral, it becomes cosine x plus 3 by 2, we can take cosine x outside the integral then it will be 0 to x sin 3 s minus sin s ds plus sin x multiplied with the integral from x to pi then cosine 3 s plus cosine s ds, this...

So, this will be called to after integration cosine x plus 3 by 2 cosine x into, minus cosine 3 s by 3 plus cosine s this limit will be 0 to x plus 3 by 2, sin x multiplied with sin 3 s divided by 3 plus sin s this limit will be from x to pi. After substituting this limit at this upper limit it will be minus cosine 3 x by 3 cosine x at the lower limit this is minus 1 third, this is plus 1 third. Similarly here at the upper limit both the quantity exactly equal to 0, because sin 3 pi is 0 sin pi equal to 0 and at the lower limit and after substitution the limit, and after simplification you can find this result is coming out to be cosine x plus 3 by 2 multiplied with minus 2 third cosine x plus 2 third cosine 2 x, and this is equal to cosine 2 x.

So, now you can see that we have started from this integral equation that is on the right hand side of the integral equation. We have used the definition of the kernel which is defined into 2 parts, 1 part is valid for s less than x and other part that is valid for s greater than x. We have substituted these 2 expressions into the integral sign and then you have used this formula for trigonometric functions, and after integration and simplification we can find this is equal to cosine 2 x. So, this cosine 2 x is exactly equal to y(x) and hence the function phi x equal to cosine 2 x is the solution of this integral equation.

Now, we just why just like to make some remark. First of all, all those examples which we are considered, their solutions are comes out as a closed type functions are either x or 1 minus x cosine 2 x. So, all the solutions appears in the closed form. So, the question is whether in all cases we will be having solution in the closed form or not. Answer is it depends completely upon the problem in some cases, if you are fortunate then you will be having the solutions into the closed form. That means, solutions can be expressed in terms of either polynomials or trigonometric functions, logarithmic functions, exponential functions or a combination of all these functions, and in some cases we will be having solutions which are functions of x, but we are unable to find out any particular closed form of the function which can represent the solution.

And another important point I like to remark here in case of ordinary differential equation, most of the time we are concerned with the existence and uniqueness of the solution. In the first lecture, you have observed that integral equation have been constructed from the ordinary differential equations those are either initial value problem or boundary value problem. For all those differential equations the concept of existence and uniqueness of the solutions are very much important.

But here we let the question of existence and uniqueness of solution of the integral equation for the further studies, now I give 2 illustrative example in one case, I am not deriving the solution at this moment, but where you can verify the given series is a solution of the integral equation, but that cannot be put into the closed form.

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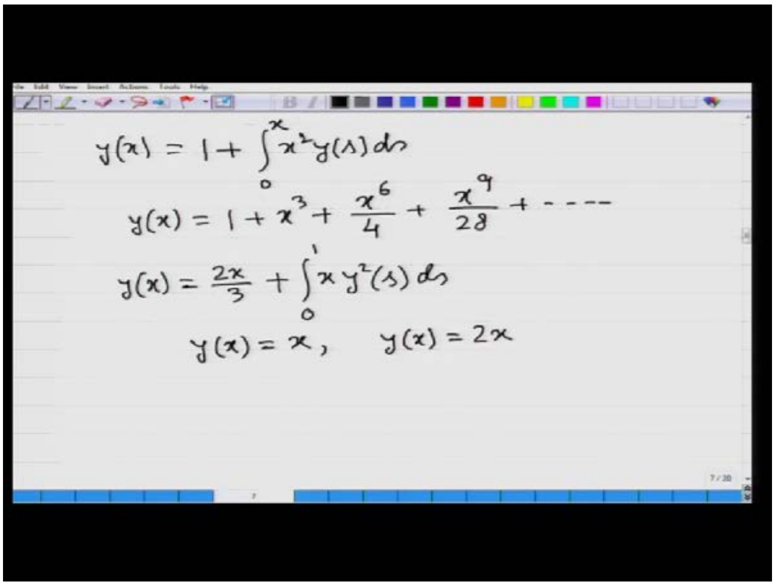

Example is we consider this integral equation y(x) is equal to 1 plus, integral 0 to x x square y s ds this is a volterra integral equation of the second kind. For these problem you can verify that y(x) equal to 1 plus x cube plus x to the power 6 by 4 plus x to the power 9 by 2 8 plus dot dot; these series is a solution of this integral equation and for these series we are unable to find out any closed form of functions such that y(x) will be equal to that closed function apart from this series of presentation, but you can verify this is a solution of these integral equation.

And next we consider a non-linear integral equation given by y(x) equal to 2 x by 3 plus integral 0 to 1 x y square s ds. This is an non-linear integral equation, interestingly you can verify these equation possesses 2 solutions; one is y(x) equal to x and another y(x) is equal to 2 x.

So, for this non-linear integral equations the question of existence uniqueness is more difficult, and this is a nice example from where you can verify that this non-linear integral equation, possesses 2 solutions - one is x and another is 2 x, so that means solutions of this equation is not unique.

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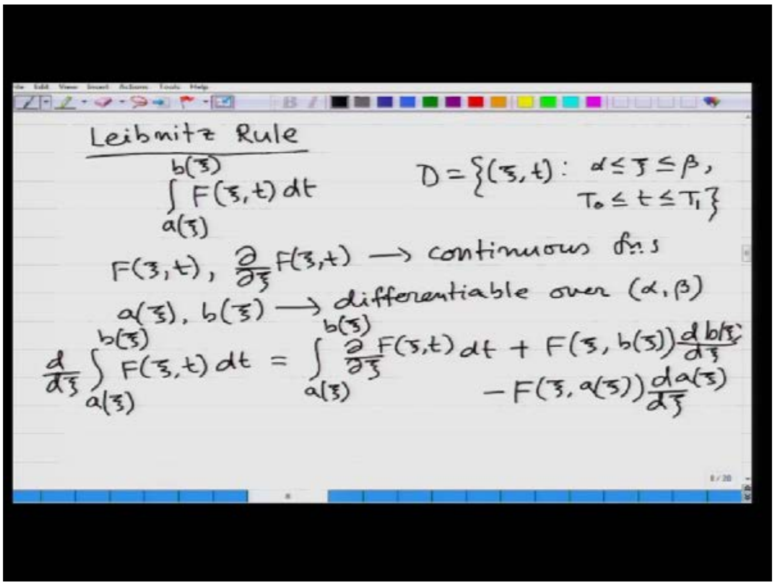

Next before proceeding further I like to remind you one important formula for the calculus that is known as Leibnitz Rule. This Leibnitz Rule is required for forthcoming discussions on integral equation. As well as today, within this lecture, I will give you some preliminary idea that how you can find out solution of an integral equation by differentiating the integral equation.

Of course, you have to keep in mind this is not the only possible way to solve this equations, but in some cases it will be possible to differentiate the integral equation to obtain some ordinary differential equations associated with the given problem, and you can easily solve that ordinary differential equation and ultimately we will able to verify solution of the differential equation obtain from the integral equation also satisfies the given integral equation. And hence this is 1 way by which you can find out solution of some integral equations. There are several other methods to solve Fredholm integral equation as well as volterra integral equation and singular integral equation those discussions will come in next lectures.

Now, this Leibnitz rule is related with the differentiation of this function, that is integral a psi to b psi f (psi, t) d t. And were considered here a rectangular domain d this is collection of the point (psi, t), such that alpha less than equal to psi less than equal to beta and t 0 less than equal to t, less than equal to the t 1.

This is the domain and we assume that f (psi, t) and del del psi of f (psi, t); these are continuous functions these are continuous functions and a psi comma b psi they are differentiable differentiable over the open interval alpha comma beta. If these conditions are satisfied then we can differentiate this function which is actually result of integration which is a function of psi with respect to psi, formula is given by d t psi of integral a psi to b psi f (psi, t) d t, this is equal to…

First of all we have to integrate the partial derivative of f with respect to psi, that is del del psi f (psi, t) d t plus f (psi, b psi), that means b psi substituted in place of t multiplied with d (b psi, d psi) minus f (psi, a psi) multiplied with derivative of a psi with respect to psi. This is the Leibnitz rule.

You can find proof of this result in any standard book on calculus, and will be using this result to convert volterra integral equations to ordinary differential equation. And we are intended to verify that solution of those ordinary differential equations actually solution of the given integral equation.

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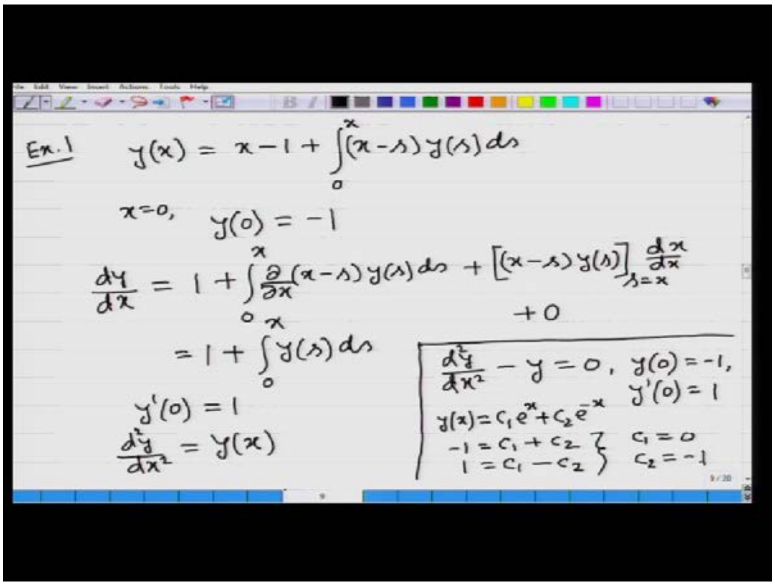

For these purpose first we consider 1 example that given integral equation is y(x) is equal to x minus 1 plus integral 0 to x, x minus s y s ds. Our attention is to convert this equation into an ordinary differential equation which will, which is going to be initial problem and by solving the obtain ordinary differential equation we can verify solution of ordinary differential equation is a solution of these integral equation. So, as we are going to solve ordinary differential equation, of course we need the initial conditions.

From here if you take x equal to 0, then you can find y 0, this is equal to minus 1, these x identically equal to 0 and substituting here this integral from 0 to 0 . So, this is also equal to 0, and therefore y(0) is equal to minus 1. Now, differentiating the given equation with respect to x this will be dy by dx, x minus 1 will results in 1 and in order to differentiate these quantity with respect to x, we have to use the Leibnitz rule.

So, according to the Leibnitz rule, this will be integral 0 to x del del x of x minus s y(s) ds is the first part then plus x minus s y(s); these expression we have to evaluate at s is equal to x with dt x of x plus another term is 0, because derivative of 0 is going to be 0. Of course, if you substitute x is equal to s here, then you can find this is also identical equal to 0. And del del x of minus s this is 1. So, ultimately we can find dy dx equal to 1 plus, integral 0 to x, y(s) ds, this is the first derivative. And again from here if you substitute x equal to 0 here, then you can find immediately y dot 0 this is equal to 1.

Now, again if you differentiate this result that is dy dx is equal to 1 plus integral 0 to x, y(s) ds then you can find d 2 y dx 2, this is equal to y x, in order to get this y(x) again we have to apply Leibnitz rule to obtain derivative of this right hand side is equal to y x. So, therefore, the given integral equation is now converted into an ordinary differential equation that is the d 2 y, dx 2 minus y, this is equal to 0 with initial conditions that is y 0, this is equal to minus 1 and y dot 0 this is equal to 1.

And quickly if we just solve this equation, then you know the general solution of this equation is given by y(x) is equal to c 1 e to the power x plus c 2 e to the power minus x, using first initial condition that is y(0) equal to minus 1. We find minus 1 equal to c 1 plus c 2 and using the second initial condition that is y dot 0 equal to 1, you can find 1 equal to c 1 minus c 2. If you solve these 2 constants, then you can find c 1 equal to 0 and c 2 this is equal to minus 1.

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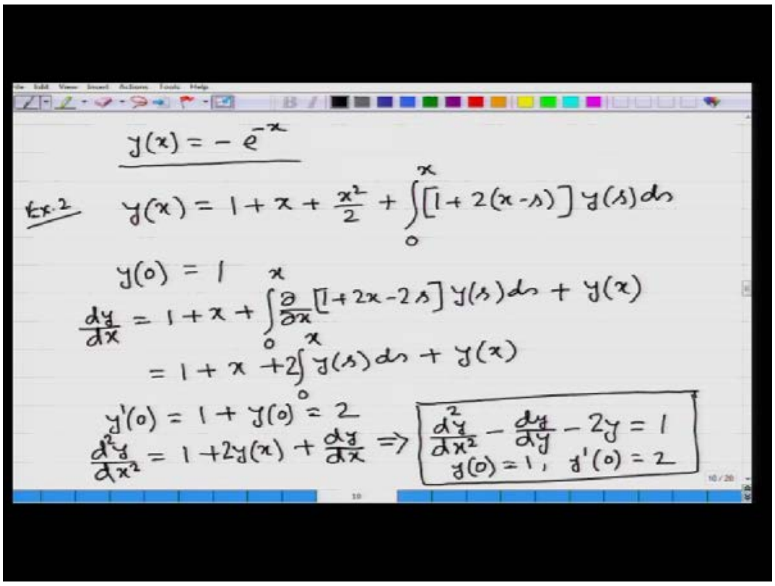

And hence solution of the given second order ordinary differential equations which is an initial value problem is y(x) is equal to minus e to the power minus x, and you can verify yourself these y(x) equal to minus e to the power minus x is a solution of these given integral equation, I left this problem for your practice problem.

Next we consider another example. Next example is y x, this is equal to 1 plus x plus x square by 2 plus integral 0 to x 1 plus 2 into x minus s y(s) ds. In these case once you proceed you can see that resulting equation will be a second order ordinary differential equation, but it will be not a straight forward equation like d 2 y dx equal to y final differential equation will also include first order derivative dot. So, in order to obtain the desired differential equation starting from this integral equation again we had to take help of the Leibnitz rule.

First of all you can verify that y(0) this is equal to 1, this y(0) is equal to 1, because these 2 terms are identical equal to 0, then x equal to 0 this integral is 0. So, therefore y(0) equal to 1, then dy dx this is equal to 1 plus x plus integral 0 to x del del x of 1 plus 2 x minus 2 s y(s) ds, this is coming from the first term of the Leibnitz formula. Then for the second term once you substitute here s equal to x. So, this term will be equal to 0. So, ultimately we are left with only 1 term that is y(x) and after differentiation these term will produce 2. So, ultimately it results in 1 plus x plus integral 0 to x, 2 will come out here y(s) ds plus y x. And you should be very much careful for finding out initial condition for the first derivative; you have to use the initial condition that is y(0) here.

So, y dot 0 this is equal to 1 plus y 0. So, this is equal to 2. And if you differentiate this result dy dx equal to 1 plus x plus 2 integral 0 to x, y(s) ds plus y(x) once again with respect to x, then you will be having this result that is the d 2 y dx 2 this is equal to 1 plus these integral after using Leibnitz rule as above, you find this will be equal to 2 y(x) and derivative of y is dy d x. So, from here you will be having the second order differential equation that is the d 2 y dx 2 minus dy dx minus 2 y, this is equal to 1 this is our target differential equation, along with 2 initial conditions that is y(0) this is equal to 1 and y dot 0 this is equal 2.

So, this is actually the second order ordinary differential equation which is an initial problem associated with the integral equation this one or you can say corresponding to this integral equation this, and once you solve this equation using this initial condition then unique solution of this equation is a solution of the given integral equation. I am not going to find out this result whether I can give you some practice problem at this moment for this topic, in all these cases you can try to solve the integral equation by converting the given integral equation into ordinary differential equation. And finding out the solution of the ordinary differential equation obtained from the given integral equation, you can verify those solutions are actually satisfying the given integral equation.

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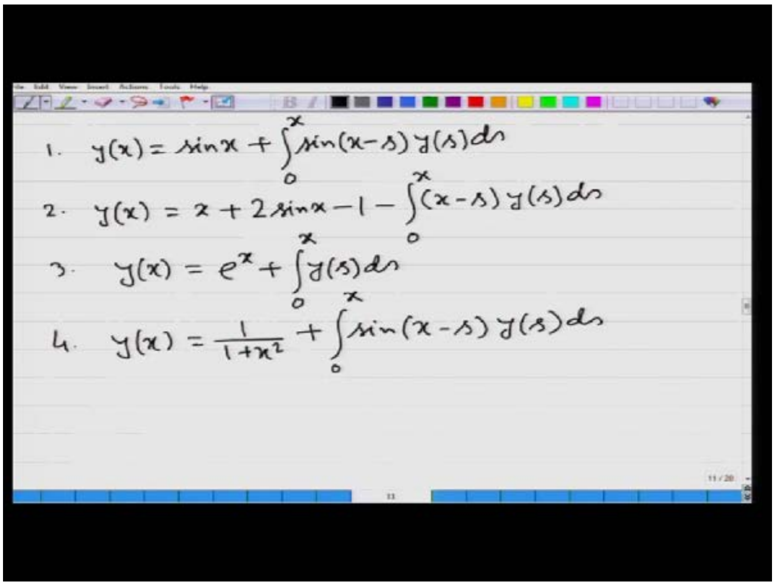

So, first problem is y(x) is equal to sin x plus integral 0 to x sin of, x minus s, y(s) ds, second problem y(x) equal to x plus, 2 sin x minus 1, minus integral 0 to x, x minus s y(s) ds. Third problem y(x) equal to e to the power x plus integral 0 to x y(s) ds, and fourth one y(x) is equal to 1 by 1 plus x square plus integral 0 to x, sin of, x minus s, y(s) ds. So, all this equation can be converted to ordinary differential equation with prescribed initial conditions that conditions can be obtained from these equations, and its derivative, and once you able to find the solution of those corresponding differential equations that will actually satisfy these integral equations.

And of course, you take a note here that all these equations are actually volterra integral equations, range of integral is 0 to x in all these exercises, and also if we just have a look at the last 2 examples that I considered here, those are also volterra integral equation. So, these volterra integral equations sometimes can be converted into ordinary differential equations and by solving those ordinary differential equations, you can find the solutions of the integral equations.

Next we consider one important lemma that is very much important for our conversion of initial value problem, and boundary value problem to integral equation when we will be considering in a general format.

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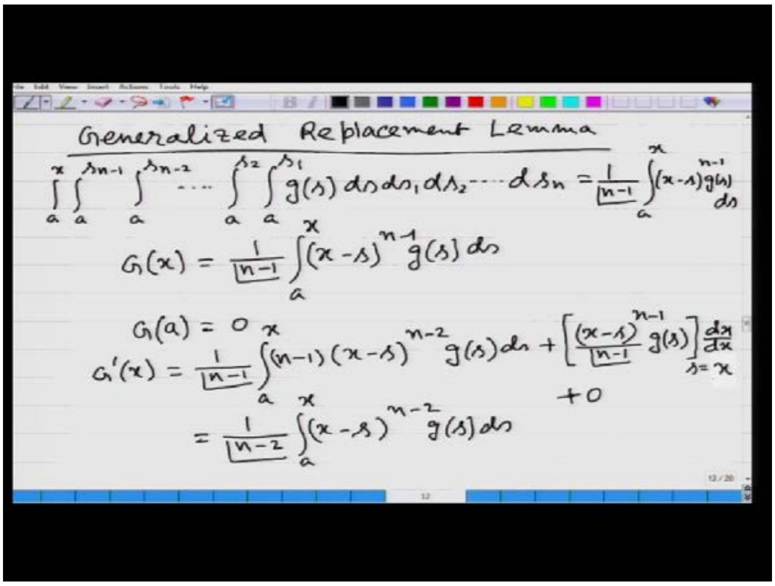

In the first lecture, we have considered some preliminary examples, but now we have to formalize all those results for general form of the ordinary differential question. This lemma is known as generalized replacement lemma generalized replacement lemma. This lemma says that integral a to x integral a to s n minus 1 integral a to s n minus 2; in this way a 2 s 2 a 2 s 1 g s ds d s 1 ds 2 up to ds n. This is equal to 1 by factorial n minus 1, integral a to x x minus s whole to the power n minus 1 g(s) d x.

This is actually the formula; that means, a collection of n integrals can be converted into a single integral, and you can recall a miniature version of this formula we have used in the first lecture were a double integral is converted into a single integral. And where I have mentioned that it can be done easily by interchanging the order of integration, but this is actually generalization of that particular result; and of course, you can verify those result is coming directly from here in case of n equal to 2.

So, in order to prove this result, we take G x equal to 1 by factorial n minus 1, integral a to x x minus s to the power n minus 1 g(s) ds. At a later stage it will be required that the value of G a, you can take note of it, that G a is identically equal to 0. If we apply Leibnitz rule on this G x equal to 1 by factorial in minus 1 integral a to x, this expression, then you can find G dot x, this is equal to, 1 by factorial in n minus 1, partial derivative of x minus, s to the power n minus 1 with respect to x will results in n minus 1 x minus s, to the power n minus 2, g s ds plus we will be having x minus s to the power n minus 1 by factorial n minus 1 g s.

This expression we have to evaluate at s equal to x with dx dx and another term will be equal to 0, because lower limit of the integral is a constant; and of course, this is also equal to 0. And this is equal to 0 leads us to the result 1 by factorial n minus 2 integral a to x, x minus s to the power n minus 2 g(s) d x. This is the G dot x.

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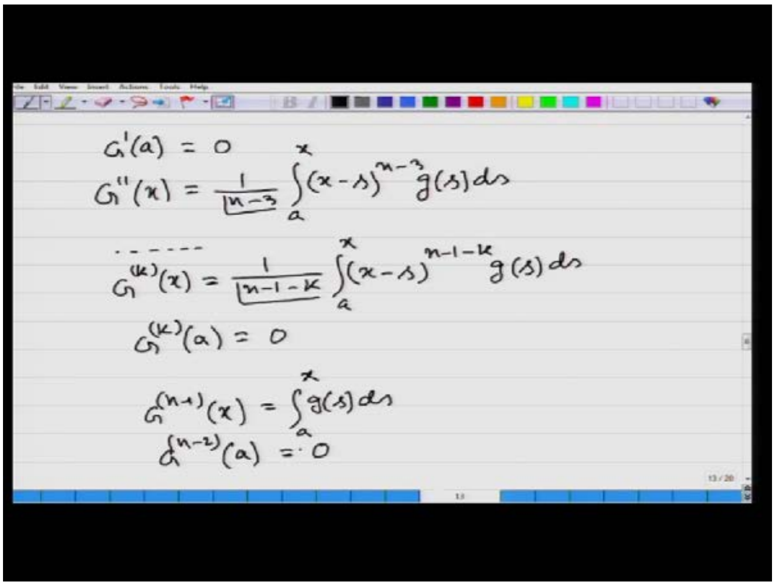

Similarly if you calculate G dot a from here this is equal to 0 and G double dot x, this will be equal to 1 by factorial n minus 3 integral a to x x minus s to the power n minus 3, g s ds. Proceeding in this way you can find general formula for G k x this will be 1 by factorial n minus 1 minus k integral a to x x minus s to the power n minus 1 minus k g s ds. And of course, these G k a is equal to 0. So using this result, if we proceed up to n minus 1 at step then finally, we will be having G n minus 1 x, this is equal to integral a to x, g s ds.

Now, from here if we try to recover G x by integration you will be arriving at the desired result, that is the generalized replacement lemma. And of course, here from the previous step you can recall that g n minus 2, a this is also equal to 0.

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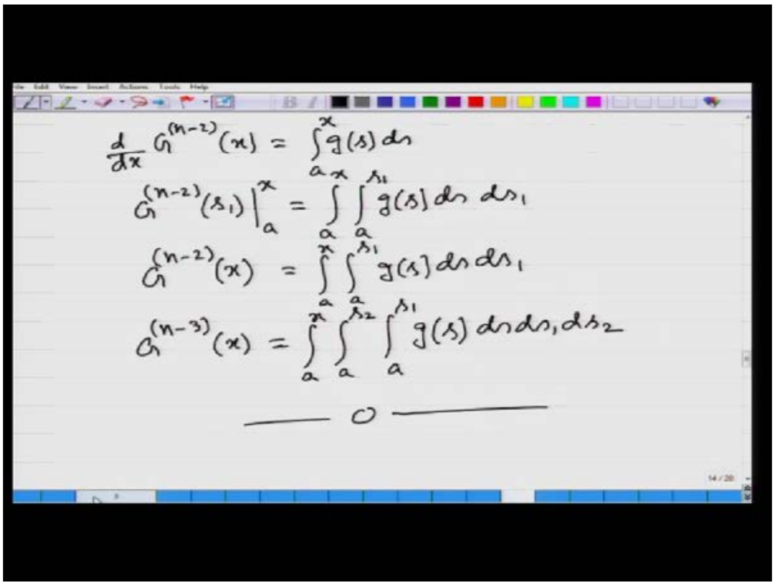

So, if we apply this result we can find that d dx of G n minus 2, x this is equal to integral a to x, g s ds, if we integrate both sides from a to x, then will be having G n minus 2, this step is very important, s 1 from limit a to x, as your range of integration is a to x. So, we are replacing this independent variable x by s 1, and once it is x is replaced by s 1 within this differential equation. So, on the right hand side x will be replaced by s 1 and therefore, it will be a s 1 g s ds and integral a to x with respect to the ds 1.

Now, substituting s 1 equal to x this will be G, n minus 2, x at lower limit this is 0 and this is equal to integral a to x, integral a to s 1 g s ds d s 1. If you proceed in a similar way at the next step, you will be having G, n minus 3, x is equal to in this case you have to integrate this expression, and before integration you have to replace this x by s 2 and range of integration will be a to x. So, we will be having a to x integral a to s 2 integral a to s 1 g(s) d s ds 1 ds 2.

So, if you proceed in this way after n minus 1 step, starting from here, you'll be arriving at the generalized replacement formula. This lecture I stop here and before ending I just quickly recapitulate what we have done today. First of all we have considered the solution of the integral equation then with illustrate examples, we have seen that phi x equal to x is a solution of this Fredholm integral equation, here phi x equal to 1 minus s is a solution of the volterra integral equation.

This is another example were kernel is not a single function whether it is defined over the 2 interval, in 1 case is less than x, and in other case is greater than x and then we have verified this is the solution. And here is 1 example, where solution cannot be obtain into the closed form and this is a non-linear integral equation, where we have 2 solutions, that we solutions of this problem is not unique.

And this is Leibnitz rule it will be required for our further discussions, this is the application of Leibnitz rule, using this rule we can convert these integral equation to an ordinary differential equation with prescribed initial conditions. And these initial conditions once imposed on the general solution of the differential equation gives you the solution of the differential equation, and of course you can verify these solution y(x) the equal to minus e to the power minus x is a solution of the given integral equation that is this one.

And then these are some exercises for you, and finally we approved the generalized replacement lemma, these lemma will be required for the next lecture where we can see, how general differential equation, which is either initial value problem or boundary value problem can be converted into integral equation. And integral equation corresponding to initial value problem will be volterra integral equation, and integral equation corresponding to the boundary value problem will be the Fredholm integral equations. So, thank you for your attention.