Calculus of Variations and Integral Equation

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Welcome viewers to NPTEL lectures on Integral Equations. This is going to be the first lecture on Integral equation. Now, Integral Equation is a very useful tool in mathematics from your mathematical point of view as well as from applied mathematical point of view. And there are different areas where these types of Integral equations are used in a wide sense.

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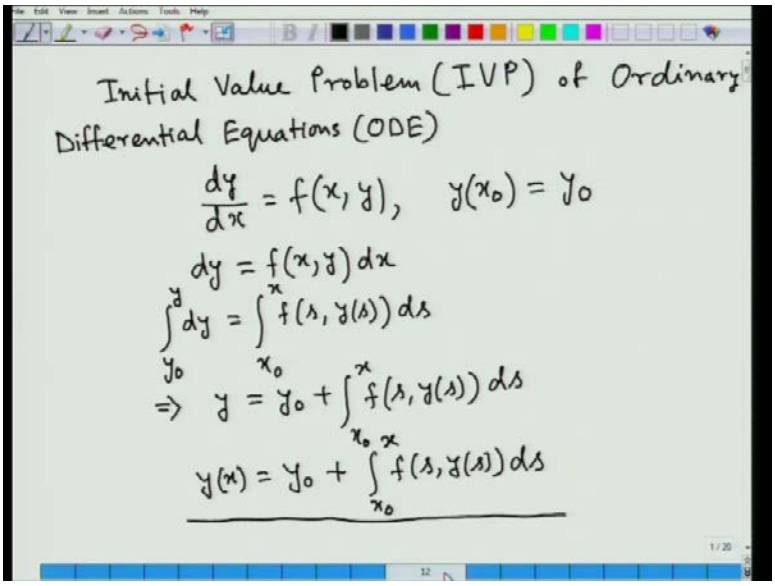

Now, first we start with a our known formulation that is ordinary differential equation from where you can see how Integral Equation comes into the picture. For these purpose we consider, initial value problem initial value problem, which is usually denoted by I V P of Ordinary Differential Equations, this is in general denoted by ODE. And throughout this Lecture series I will use these I V P and ODE for Initial value problem, and ordinary differential equations.

So, first of all we consider a general first order differential equations equal to f(x) y subject to the Initial condition y(x) 0. This is equal to y 0. For these kind of equations we know how to solve this problem. Now, our intention is to convert this Initial value problem to an Integral equation. So, in order to solve it, we can write dy this is equal to f(x) y d x. Now, if we integrate between y equal to y 0 2 y and x equal to x 0 2 K x then you can find y 0 to y d y, this is a equal to Integral x 0 to x, f (s, y s) ds from where we find y, this is equal to y 0 plus Integral x 0 to x f (s, y s) ds.

Now, this y(x) equal to y 0 plus Integral x 0 to x f s y s ds, this is actually our Integral Equation because the unknown function y here appeared under the Integral sign. Now, if y equal to y(x) is a solution of this equation that is the Initial value problem then second function y(x) will be a solution of this Integral equation. Now, if we just look at this equation. So, in this case we have to find out y(x) such that y(x) satisfies these equation.

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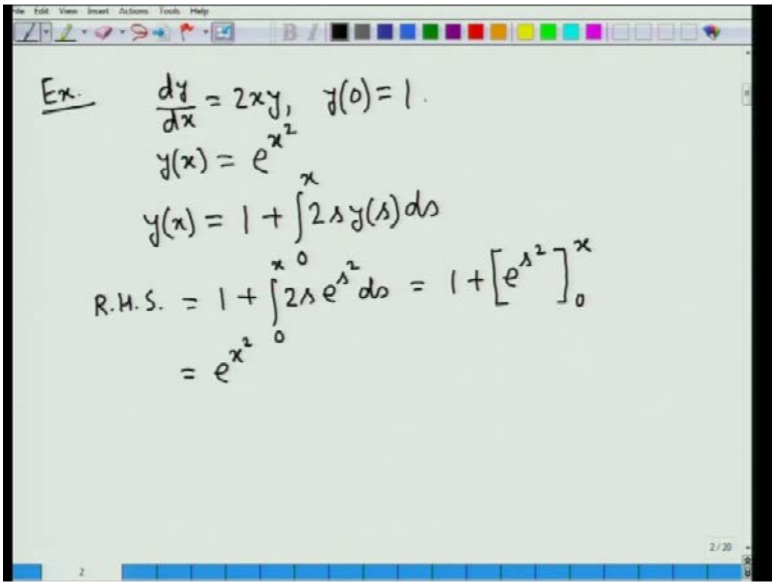

Next we will consider a particular example to verify either a solution of the Initial values problem is going to satisfy the corresponding integral equation or not. For these purpose we choose the equation dy d x is equal to 2 x y, subject it to the Initial condition y 0 this is equal to 1.

So, this is our given o d and of the season Initial problem and you can easily verify solution to this problem is given by y equal to e to the power x square. If we recall the last equation what we have constructed, once dy d x equal to f(x) y subject to y(x) 0 equal to y 0., so, this is our Integral equation. So, corresponding to these given differential equation our integral equation will be y(x) this is equal to 1, that is y 0 plus Integral 0 to x, 2 s y s ds.

Now, here we intended to verify if whether this y(x) equal to e to the power x square satisfy this or not. So, for these purpose, we start with right hand side. So, this is equal to 1 plus Integral 0 to x, 2 s now y(s) is e to the power s square ds, and if we integrate it then we find 1 plus e to the power s square, Integral from limit from 0 to x. So, this is coming out to be e to the power x square which is equal to left hand side. So, that means, the solution of this Initial value problem is also a solution to the corresponding Integral equation. Now, it is a first it was a first order equation.

Next we consider a second order ordinary differential equation, again this is the second order Initial value problem.

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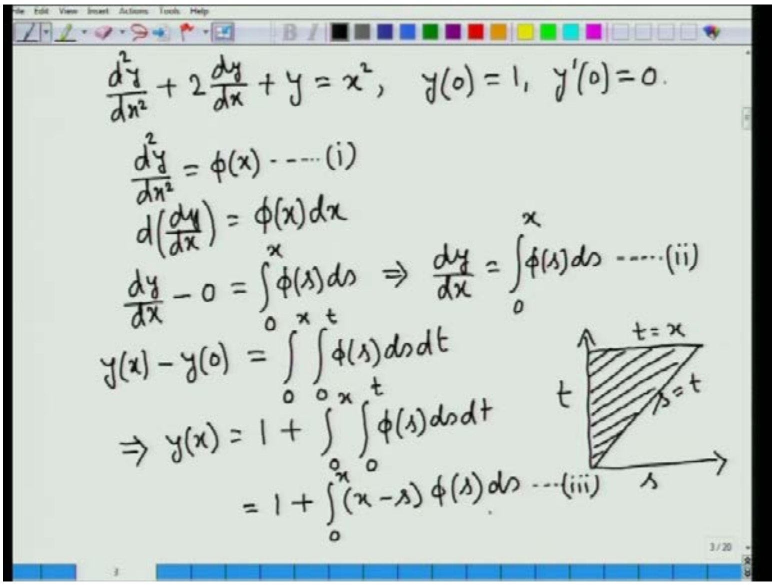

We consider this equation d to y d x 2 plus 2 dy d x plus y, equal to x square with given Initial condition y 0 equal to 1 and y dash 0, this is equal to 0. Now, at a later stage of this Lecture series I will show you directly from here after integration, how can I find out the Integral equation. Now, as it is a starting point here we just consider second derivative of y with respect to x we define it as a new function, phi x such that after integrating this expression twice, that means you are going to evaluate dy d x and y in terms of phi and its Integral such that the entire equation can be converted into an Integral equation.

So, this equation we can rewrite as d of dy d x this is equal to phi x d x. Now, where Initial condition is given at x equal to 0, so if we integrate from 0 to x. So, then you can find dy d x minus 0, this 0 is coming due to the given condition y dot 0 equal to 0 this is a equal to Integral 0 to x phi s ds. So, this actually implies dy d x is coming out to be Integral 0 to x phi s ds, call it 2.

Again if we integrate this result after transferring d x onto the right and again from the limit 0 to x, then you can find y(x) minus y 0. This is equal to… Now here the range of integration is going to be x. So, we change this limit to t and we get 0 to x Integral 0 to t phi s ds d t and these y 0 is equal to 1 So, this implies y(x) this is equal to 1 plus Integral 0 to x Integral 0 to t phi s ds d t.

Now, if you look at the range of the integration in this direction we can take s in a practical direction we can take t the first limit was on ds and s is ranging from 0 to t. So, that means, we have a straight line whose equation is s equal to t. So, ds had is from 0 to t and then this limit t hiding from 0 up to the line that is t equal to x.

So, these area is actually our domain over which we are performing our integration now if we interchange the order of the Integral, then after interchanging the order of Integral we can find this is going to be 1 plus Integral 0 to x, x minus s phi s ds. So, interchanging the variable of integration we will be getting this equal to y.

So, we have assume d 2 y d x 2 equal to phi x then from there we have derived using the given Initial condition y dot 0 equal to 0 dot dy d x equal to this 1 and finally, using y 0 equal to 1 we have obtained y(x) equal to these 1.

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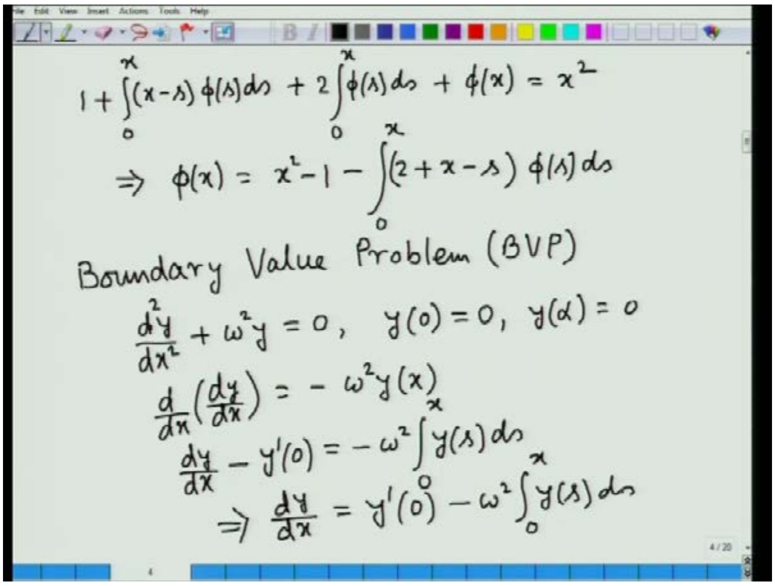

Now, if we substitute all these expressions into the given equation then you will be will be having 1 plus Integral 0 to x, x minus s phi s ds this is actually coming out for y then plus 2 Integral 0 to x phi s ds plus phi x, this is equal to x square and from here we can write that phi x, this is equal to x square minus 1 minus Integral 0 to x 2 plus x minus s multiplied with phi s ds.

So, in this case you can see phi is our unknown and these phi x comes under Integral sign. So, this is an integral equation corresponding to the given second order Initial value problem. Now, if y(x) is a solution to the given second order Initial value problem then second derivative of y, that is y double dot x equal to phi x will be the solution of these Integral equation. Now, so far we have considered to Initial value problem and in both the cases resulting equation comes out to be an Integral involving 0 to x within this range.

Next we consider a boundary value problem. This boundary value problem is denoted by BVP consider the second order differential equation d 2 y d x 2 plus omega square y, this is equal to 0 subject to the given boundary conditions y 0 equal to 0, and y alpha this is equal to 0 . So, that means, we are considering this problem over the interval 0 alpha that is x ranging from 0 to alpha.

Now, in the previous example, in order to convert this type of second order differential equation to an Integral equation we have chosen d 2 y d x 2 equal to phi x. Now here without introducing that type of function directly we can make an attempt to convert this equation to an integral equation. At a later stage of this Lecture series I will describe in which cases we have to consider what type of technique to convert the differential equation into Integral equation, but that is not a very serious question, because most of the time will be considering solution of the Integral equation.

Now, this equation can be rewritten as the d 2 x of dy d x this is well known strategy is equal to minus omega square y x. Now, if we integrate this equation from 0 to x then you can find dy d x minus these derivative of y evaluated at y equal to 0, that is y dot 0 this is equal to minus omega square Integral 0 to x y(s) d s

Now, in this boundary value problem the values of y on the left hand and on the right hand are given we do not have any information for y dot zero. But at a later stage using this boundary condition we can find out y dot 0 in terms of y. So, from here we can write dy d x this is equal to y dot 0 minus omega square Integral 0 to x y(s) ds.

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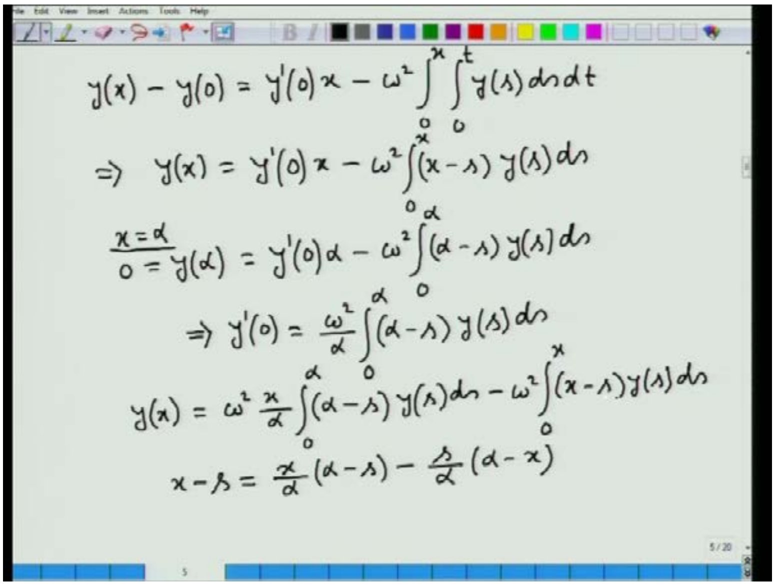

If we again integrate the previous expression then we can find y(x) minus y 0, this will be equal to y dot 0 multiplied by x minus omega square Integral 0 to x, Integral 0 to t y(s) ds d t. Now, we can recall that given condition for a y 0 was 0. So, from here we can write y(x) is equal to y dot 0 x minus omega square, and again if we apply the same technique as we have adopted earlier, from here you can find Integral 0 to x, x minus s y(s) ds still this y dot 0 is appeared in here.

Now, we can make an attempt to find out this y dot 0, if we substitute x equal to alpha from this result there you can find 0 this is equal to y alpha actually substituting x equal to alpha you will be having y alpha here and y alpha is given to be 0. So, 0 equal to y dot 0 alpha minus omega square, Integral 0 to alpha, alpha minus s y(s) ds. So, from here we find y dot 0 that is equal to omega square, divided by alpha, Integral 0 to alpha, alpha minus s y(s) ds. So, if we substitute this expression here then we find y x, this is equal to omega square times x by alpha, Integral 0 to alpha, alpha minus s y(s) ds, this part coming for y dot 0 then minus omega square Integral 0 to x, x minus s y(s) ds.

Now, we have to proceed further in order to put this integral equation into more compact form. In order to put into more compact form we can use these identity x minus s this is equal to x by alpha times alpha minus s minus s by alpha times alpha minus x, this is an identity and we are going to replace these x minus s with the help of this result.

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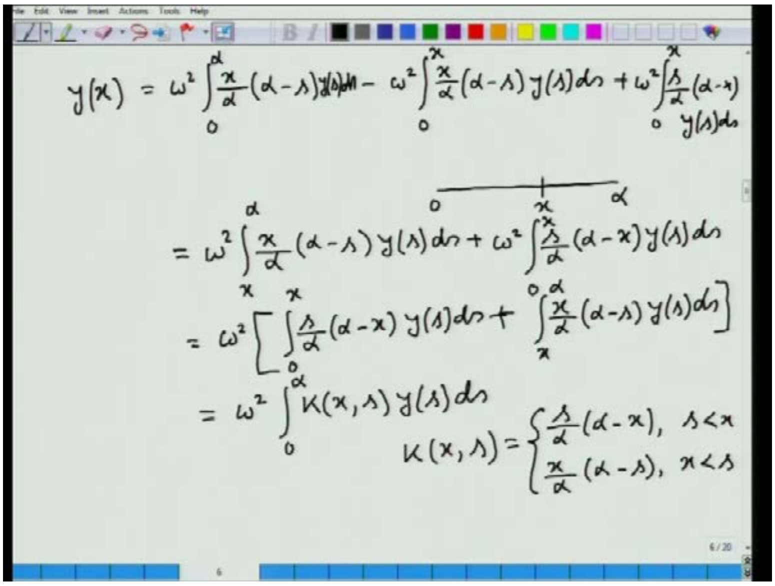

So, if we replace this x minus s by this identity then we find y(s) equal to omega square times Integral 0 to alpha x by alpha alpha minus s ds minus omega square Integral 0 to x, x by alpha times alpha minus s y(s) ds plus omega square, Integral 0 to x, s by alpha times alpha minus x y(s) ds here I forgot to write y s, this will be y s ds.

Now, here range of integration is 0 to alpha. So, we can be derive this range of integration 0 to alpha by introducing a point here that is 0 to x and x to alpha. So, if we derive these into 0 to x and x to alpha then 0 to x Integral will cancels with this e 1, and then we are left with this is equal to omega square Integral x to alpha x by alpha times alpha minus s y(s) ds plus omega square Integral 0 to x, s by alpha times alpha minus x y s ds.

So, this is equal to omega square, bracket 0 to x, s by alpha, times alpha minus x y s ds plus Integral x to alpha x by alpha times alpha minus s y s ds and this can be written into a compact form omega square times Integral 0 to alpha K (x, s) y(s) ds where this K (x, s) is defined in this way. This is x by alpha, times alpha minus x whenever s less than x and this is equal to x by alpha, times alpha minus s for x less than s. So, that means, y(x) equal to omega square Integral 0 to alpha K (x, s) y ds; this is the integral equation corresponding to the given second order boundary value problem.

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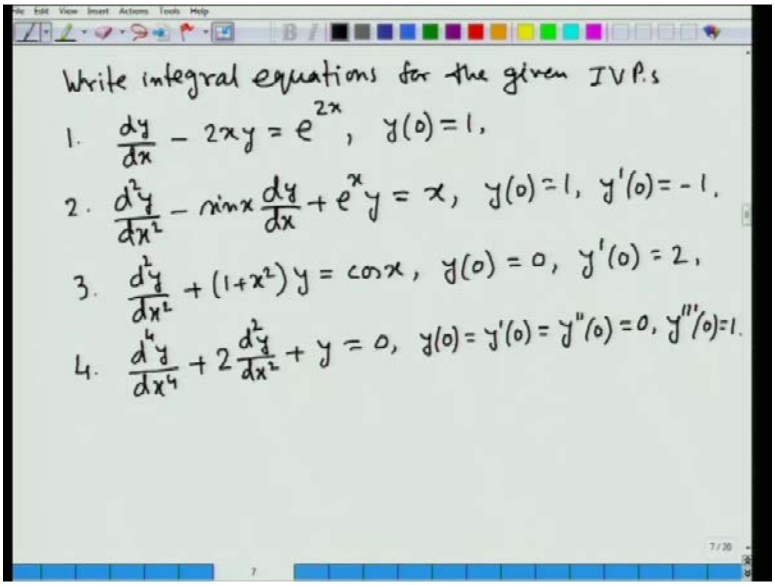

Now here I give just some exercises that you can try to convert this problem to Integral equations. So, the problem is write Integral equations for the given Initial value problems number 1 dy d x minus 2 x y equal to e to the power 2 x with given Initial condition y 0 equal to 1 for number 2 d 2 y d x 2 minus sin x dy d x plus e to the power x into y this is equal to x with Initial condition y 0 equal to 1, and y dot 0 this is equal to minus 1. Problem number 3 d 2 y d x 2 plus 1 plus x square y this is equal to cosine x with given Initial condition y 0 equal to 0 y dot 0 this is equal to 2. And problem number 4 d 4 y by d x 4, plus 2 d 2 y d x 2 plus y, this is equal to 0 with the given Initial conditions y 0 equal to y dot 0 equal to y double dot equal to 0, this is equal to 0, and y triple dot 0 this is equal to 1.

So, this was all examples based upon the ordinary differential equation such that ordinary differential equations can be converted into Integral equations. Now, I give a physical example regarding the origin of Integral equation.

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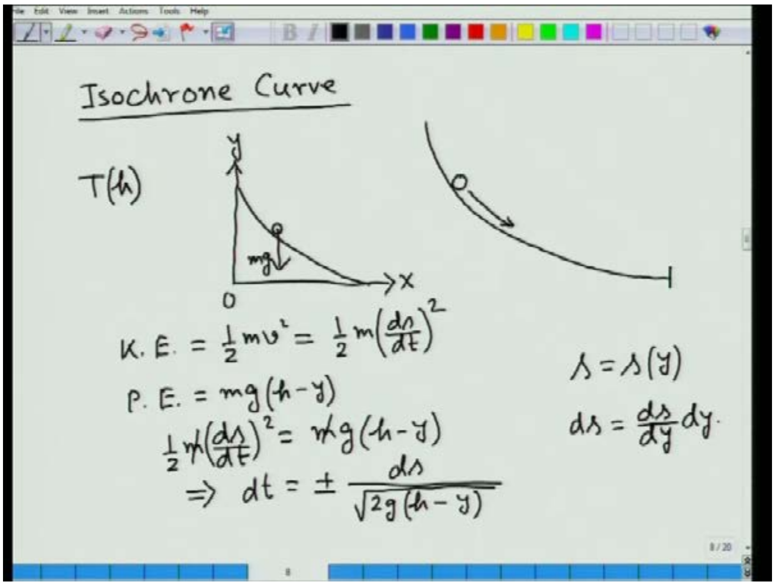

This integral equation is actually known as Abel's Integral Equation and this problem is related with Isochrone curve. This Isochrone curve problem is something like that you have a curve like this, a particle starts from this point it is coming down along the curve under gravitational force only, and this wire is a frictionless wire and suppose this is the point up to which this particle can come down.

So, the problem is that this Isochrones curve is the curve for which the time taken by an object sliding without friction under uniform gravity to its lowest point, this point is the lowest point is independent of its starting point. So, main problem is we have to find out the shape of the curve such that if we start from either at this position or at this position or at this position all the particles will arrive at the fixed point at the same time.

Now, let us denote capital T is the time taken by a particle coming to the lowest point from a particle height h. So, this is the h stands for a particle height through which the particle came down along this particular curve. Now, there is no friction if we look at the geometry and relation with the coordinates you can see, this is the x axis origin, this is y axis, curve is something like this 1 and this is the position of the particle at any time T weight m g acting vertically downwards. So, this height from the lowest point can be measured as the y coordinate of this particular position.

Suppose it started from a height h somewhere here, then at this point kinetic energy this is nothing but half of m v square, and since the particle is coming according along a smooth curve. So, this is equal to half m, ds d t whole square ds is the arc length of this particular curve at this position and potential energy, this is equal to m g multiplied by the height. So, it distance and as it is a smooth curve there is no friction; therefore, kinetic energy will be equal to potential energy as per the conservation of energy is concern.

So, from there we can write half m, ds d t this whole square is equal to m g times h minus y, this m cancels from both side and from here we can write d t this is equal to plus minus ds divided by root over, 2 g times h minus y. Now, we need 1 assumption regarding the equation of this curve without any loss of genet we can assume we can find out this arc length in terms of y coordinate of the point. And then using the chain rule we can write ds equal to ds dy times dy.

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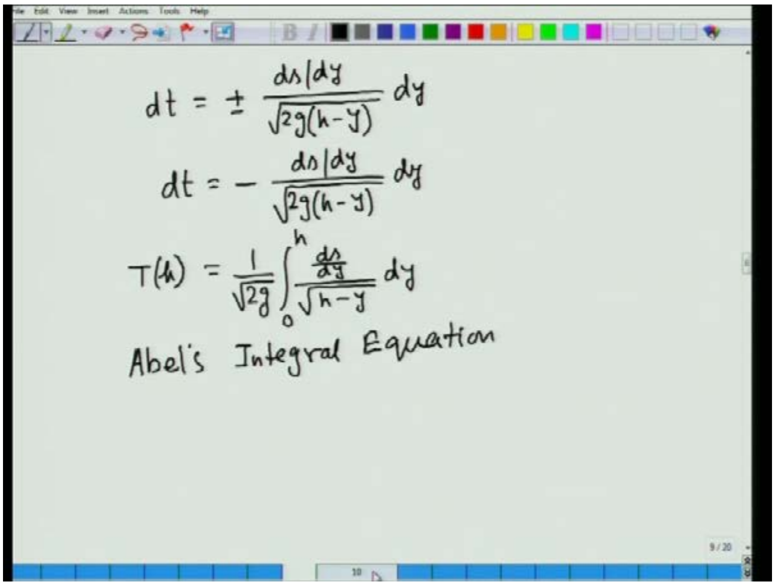

So, if we use this result here then we will be having d t this is equal to plus minus ds by dy divided by, root over 2 g times h minus y, d y. Now, if we just go back to the previous picture then you can see the remaining distances this actually height decreases with the increase of time this height y remaining height this is the decreasing function with the increase in t. And therefore, we take negative sign to get this result d t equal to minus ds by dy divided by root over 2 g into h minus y dy, and then the time required for coming to the lowest point from a height h; that is T h is given by 1 by root over 2 g integral 0 to h ds d y divided by root over h minus y dy. This equation is actually called Abel's Integral Equation. This is called Abel's Integral Equation.

So, the problem is that h t is given and we are interested to find out ds dy for which the equation of the curve can be obtained, such that a particle starts from any height it will reach at the lowest point that will be independent of the height from where it has been started. So, this is 1 physical example behind the formation of Integral equation.

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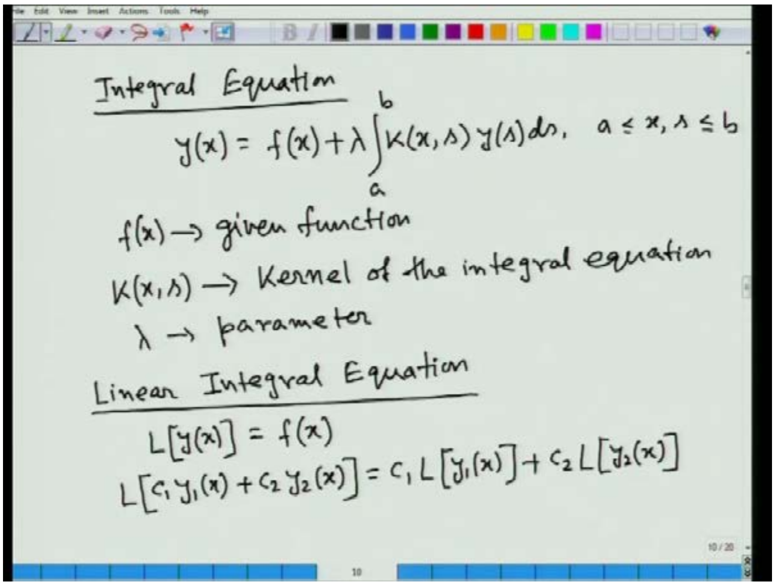

Now, we are in a position to give the general definition of an integral equation. In general integral equation is given by this particular format y(x) is equal to f(x) plus lambda times Integral a to b, K (x, s) y s ds, where a less than equal to x, s less than equal to b.

So, the problem is we had to find out y(x) such that this equation is satisfied and the unknown function y is under the Integral sign, and that is why this equation is called an integral equation, and here f(x) this is a given function K (x, s) this is actually called

Kernel of the integral equation. This is Kernel of the integral equation and lambda, this is a parameter.

Now, in some of the problem this upper limit b maybe variable x and in some cases either lower limit or upper limit or both of them will be infinite quantities, and depending upon whether both the limits are finite real numbers or upper limit is a variable x or the limits are infinite based upon this we can classify this integral equation into different classes.

Now, before going to that I want to attract your attention towards the most important point that Linear Integral equation. The integral equation what I have written here that is actually a Linear Integral Equation, because y appeared in this equation having the first power in general, we can define the linearity property in this way entire equation can be written in the form as a operator L y(x) this is equal to f x.

If we denote this y(x) minus lambda times Integral a to b K (x, s) y s ds as L of y(x), then we can easily check for this equation it is satisfies the condition L of c 1, y 1 x plus c 2 y 2 x this is equal to c 1 L of y 1 x plus c 2 L of y 2 x, this is the linearity condition where c 1 and c 2 are 2 constants. So, this is actually the standard form of Linear Integral Equation; of course, we will be curious to know what will be the format of the non-linear Integral Equation.

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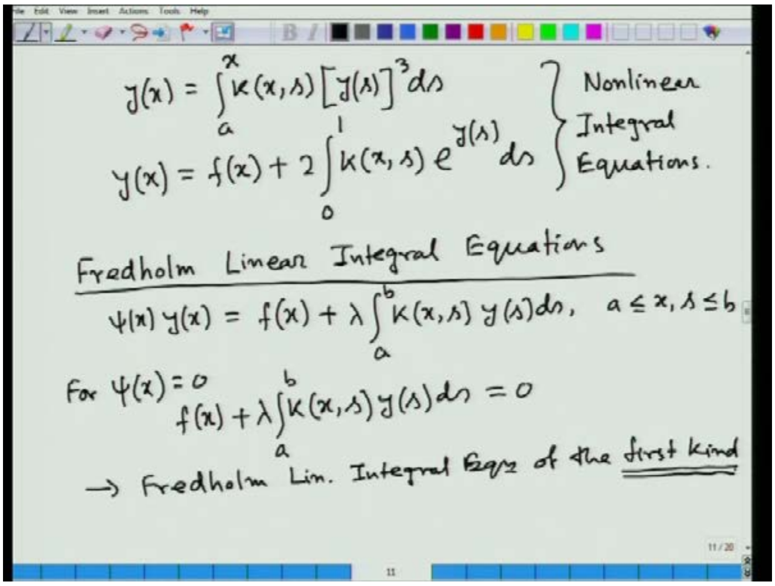

I am just giving you some examples that is y(x) is equal to Integral a to x K (x, s) y(s) full cube ds, here the unknown function y appeared as y cube which is a non-linear function of y. And hence this is a non-linear Integral equation, another example y x, this is equal to f(x) plus 2 integral 0 to 1 K (x, s), e to the power y(s) ds this is again another example of non-linear Integral Equation s; these are actually non-linear Integral equations.

Now, we come to the classification of different Linear Integral Equation, first we consider Fredholm Fredholm Linear Integral equations. (No audio from: 42:08 to 42:18) This Fredholm Linear Integral equations are written as psi x y(x) equal to f(x) plus lambda Integral a to b K (x, s) y s ds, where a less than equal to x, s less than equal to b. This psi is actually introduced here in order to classify further, this integral equation into 2 types that is Fredholm Linear Integral Equation of first kind and Fredholm Linear Integral Equation of the second kind for psi x equal to 0 then will be having this integral equation of the form f(x) plus lambda Integral a to b K (x, s) y(s) ds, this is equal to 0. This is actually Fredholm Linear Integral Equation of the first kind, this is important Fredholm Integral Equation of the first kind.

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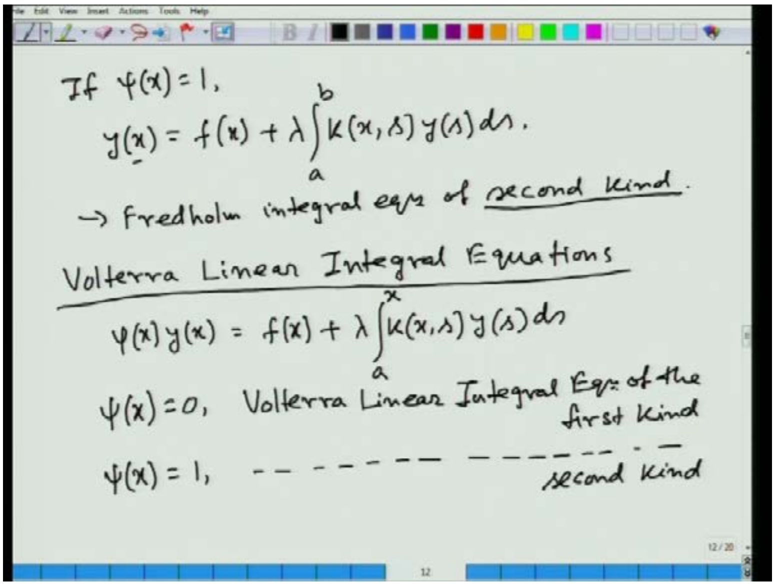

And If we take psi x equal to 1 then equation is y(x) equal to f(x) plus lambda Integral a to b, K (x, s), y(s) ds, this is actually Fredholm Integral Equation of second kind.

Next we consider Volterra Linear Integral Equations, these Volterra Linear Integral equations. In this case 1 limit is coming out to be a variable limit general format is psi x, y(x) is equal to f(x) plus lambda times Integral a to x, K (x, s), y(s) ds. So, this x comes out here appears here instead of earlier b. So, these type of equations are actually known as a Volterra Linear Integral equations I am not repeating the same thing, and just for your note that if psi x, this is equal to 0. Then we will be having Volterra Linear Integral equations of the first kind, and similarly as above if psi x equal to 1 then we will be having Volterra Integral Equation of the second kind. So, these two things are related only with the fact that whether y(x) appearing outside the Integral signs also or not. So, based upon this we can classify them into Volterra Integral Equation of fast kind or second kind and Fredholm Integral Equation of the first kind or second kind.

Now, we just look at some particular features in case of Fredholm Linear Integral Equation range of integration is finite, and we have to find out this unknown function y(x) in case of Volterra Integral Equation the range of integration is variable 1 upper limit is variable lower limit is fixed constant a in some problem this may be the reversed one thus lower one is a variable x upper one is some constant b.

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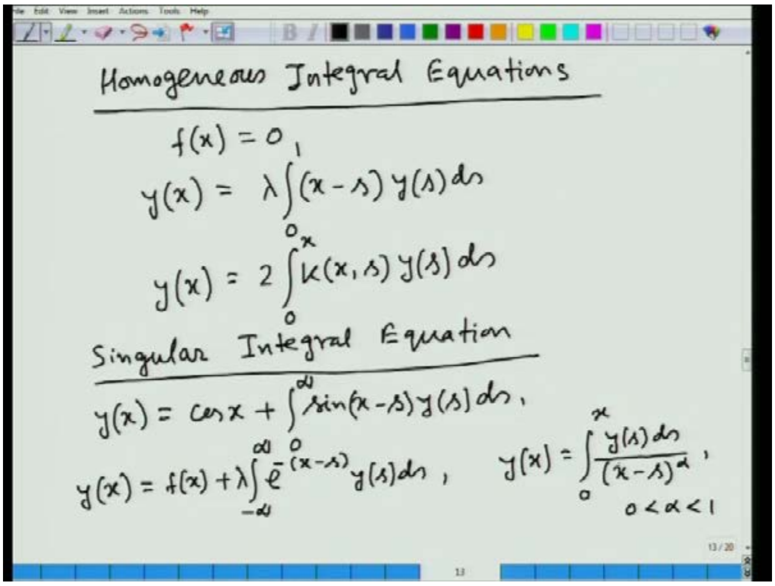

Next we comes to the Homogeneous Integral equations, this Homogeneous Integral equations these are related with the appearance of f x, if f(x) is identically equal to 0, then we can find Homogeneous Integral equations. So, Homogeneous Integral Equation of Fredholm type we consider just one example that lambda 0 to 1 x minus s y(s) ds this is an Fredholm Homogeneous Integral Equation and y(x) is equal to 2 Integral 0 to x K (x, s) y s ds this is an example of Volterra Integral Equation which is Homogeneous.

Next we considered another classification that is known as singular integral equation in case of singular integral equation either lower limit or upper limit or both the lower and upper limits are infinite quantities or integrant may have 1 or more singularities within the range of integration. If this happens then Integral equations are known as singular Integral equations we just consider few examples of singular Integral equations, y(x) this is equal to cos x plus, Integral 0 to infinity sin of x minus s, y s ds this is an example of singular integral equation.

Another one is y(x) equal to f(x) plus lambda Integral minus infinity 2 plus infinity e to the power minus x minus s y s ds this is another singular integral equation in the first one upper limit is infinite in the second one lower limit, and upper limit both of them are infinite and another integral equation y(x) is equal to Integral 0 to x, y(s) ds divided by x minus s to the power alpha where 0 less than alpha less than 1. So, at a s to x this integrant and is infinite. So, x equal to s is a singular point of this integrant and therefore, this is a singular Integral equation.

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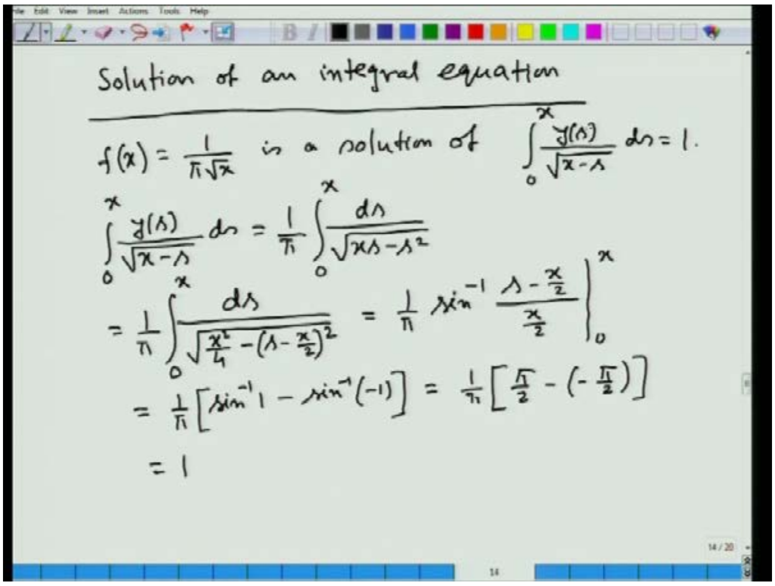

Finally for today we just consider one problem, that is solution of an integral equation. We are not going to solve any integral equation rather we can verify that a given function satisfies an Integral equation. Here we are going to verify that f(x) equal to 1 by pi root over x, this is a solution of the integral equation given by 0 to x y(s) divided by root over x minus s, ds this is equal to 1.

So, in order to verify this we start from here. 0 to x y(s) by root over x minus s ds. So, substituting for y(s) equal to 1 by pi root over s we can find this is equal to 1 by pi Integral 0 to x ds divided by root over x s minus s square this is equal to 1 by pi, Integral 0 to x, ds divided by root over x square by 4 minus s minus x by 2 this whole square this is equal to 1 by pi sin inverse s minus x by 2 whole divided by x by 2 and limit 0 to x. So, this is equal to 1 by pi after substituting the limit will be having sin inverse 1 minus sin inverse minus 1. Considering principle values of this inverse function, we find this is equal to 1 by pi pi by 2 minus minus pi by 2.

So, this is equal to 1, so that means, the function f(x) equal to 1 by pi root over x satisfies this integral equation, where y was the unknown, and this 1 by pi root over x this is the solution of this Integral equation.

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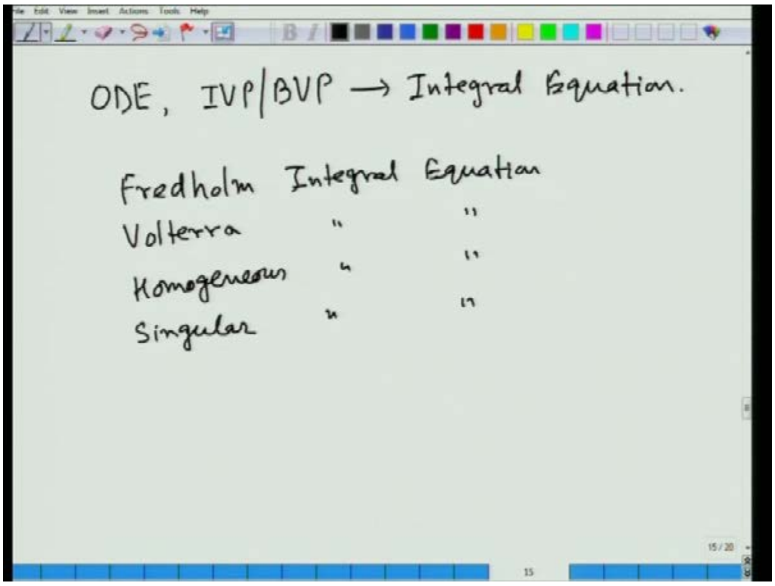

So, just in order to sum up first of all today we have seen that how ordinary differential equations, they are either Initial value problem or boundary value problems that can be converted into Integral equations. Then we have considered a physical problem behind the formation of the integral equation, and after that we have classified different types of Integral equations.

We are considered Fredholm Integral equations, these are of 2 types it is first kind and second kind, then Volterra Integral equations Volterra Integral equations and then we have had a look at Homogeneous Integral equations, and finally we had a look at the Singular Integral questions. Now, in rest of the lectures most of the time we will be considering Linear Integral equations only, and at the end I just give one Lecture on the solution of non-linear Integral equations.