Calculus of Variations and Integral Equations

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Lecture No. # 20

Welcome viewers to the NPTEL lecture series on the Calculus of variations, this is the last lecture 20 th lecture of the series. Recall that in the last lecture, we discuss sufficient conditions, Jacobi's condition, and Lysander, and Weierstrass function, where we showed that if Jacobi's condition is satisfied, and if Weierstrass function has a positive sign or negative sign throughout in a certain interval, then we get a minimum strong minimum or weak minimum depending upon, whether y prime is arbitrary or y prime is to be assumed to close to p.

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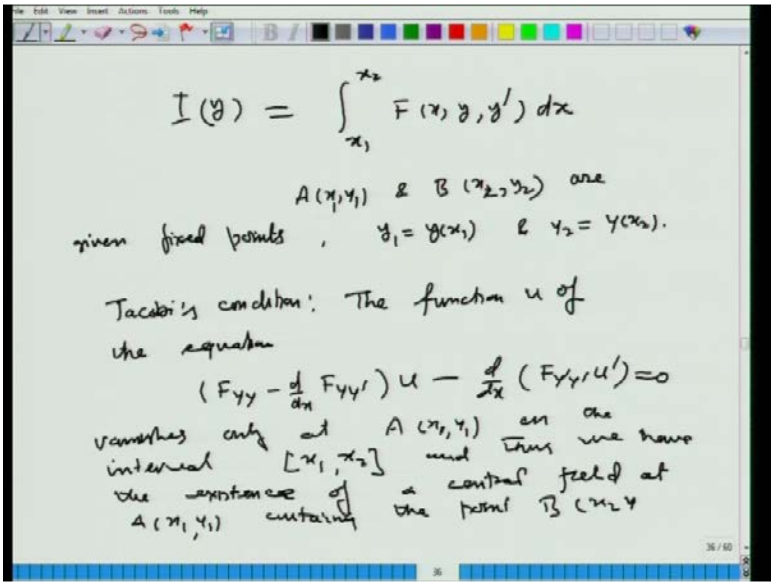

So, recall that we had the functional. So, recall that we have the functional I (y) equal to integral x 1 to x 2 F of (x, y, y prime) d x, where the boundary points are given A (x 1, y 1) and B (x 2, y 2) are given fixed points, here y 1 equal to y at x 1, and y 2 equal to y at x 2.

So, the Jacobi condition, that is the function u of the function u of the equation (F y y minus d by d x of F y y prime) time's u minus d by d x of (F y prime y prime u prime equal to 0) vanishes only at A that is (x 1, y 1) on the interval x 1, x 2; and this we have the existence of a central field at A (x 1, y 1) containing the point B (x 2, y 2). So, that is the Jacobi condition

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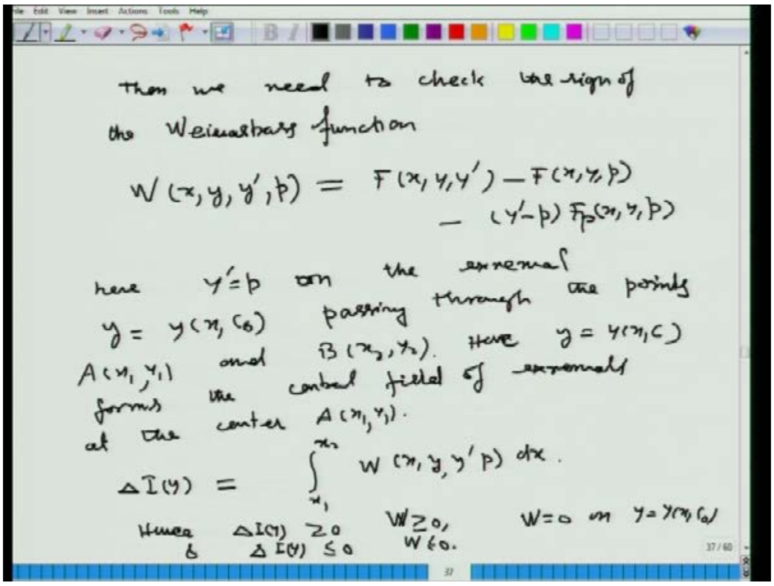

And then then we need to check the sign of the Weierstrass function; that is W, which is function of (x, y, y prime, p) equal to F (x, y, y prime) minus F (x, y, p) minus (y prime minus p) F p (x, y, p); here y prime equal to p on the exremal y equal to y (x, c 0) passing through the points A (x 1, y 1) that the central centre of the central field and B (x 2, y 2).

So, here we have the exremal y equal to y (x, c 0) which is a fixed, c 0 is fixed value of the parameters c, here y (x, c) forms the family, here y equal to y (x, c) forms the central field of exremals at the centre and for this c equal to c 0, we are on this exremal, which passes through the given points A (x 1, y 1) and B (x 2, y 2). So, we want to check whether this exremal, which is y (x, c 0) minimizes or maximizes the given functional I.

So, here we need to check that since, here we have seen that this delta I (y) as delta I (y) is integral x 1 x 2 here W (x, y, y prime p) d x. So, we see that; this delta. Thus, hence delta I (y) will be greater than equal to 0; if W is greater than equal to 0. And similarly delta I (y) be less than equal to 0; if W is less than equal to 0; and W equal to 0 on y equal to y (x, c 0), because here this y prime equal to p. So these two terms are canceled, and y prime equal to p here, this factor will be 0. So, W equal to 0, so delta I (y) equal to 0. If we are taking y equal to this is general y here, the any here this y is any curve joining these two points.

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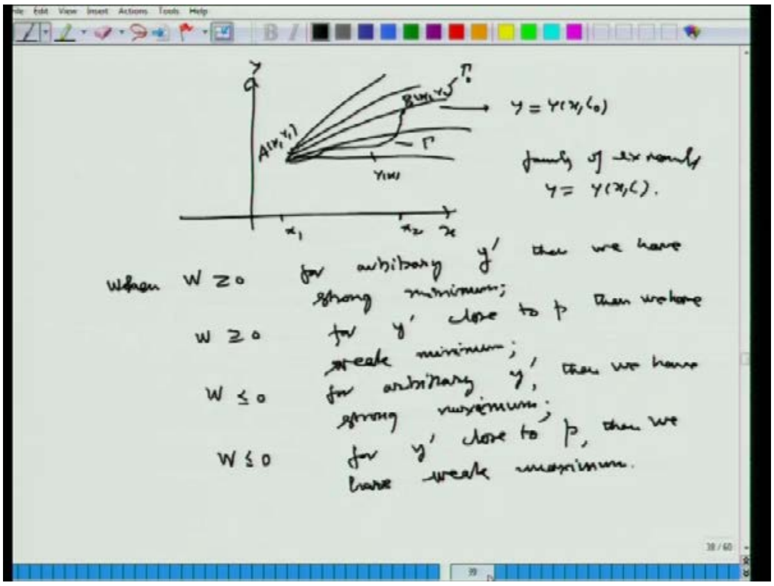

So, here situation is like this. We have central field here at A like this. So, this is A (x 1, y 1) and B is there on this (x 2, y 2). So, this is the curve is the exremal y equal to y (x, c 0) and this is the family of exremals given by y (x, c). So, a family of exremals y equal to y (x, c).

So, here c equal to c 0, we have this exremal, which passes through these given points. So, we want to see, whether this exremal is minimizing or maximizing at the given functional I; and this any y here is, this is general y x, which this we had denoted as gamma 0, and this we had denoted as general gamma. And so that is what we have here this on Lysander gamma, joining these two points A and B.

So, this is x 1 here and this point is x 2 here. So, we had seen that in various examples; that we calculated W and we see that, when W is when W is greater than equal to 0 for arbitrary y prime, then we have strong minimum. And if W is greater than equal to 0 for y prime close to p, then we have weak minimum; and if W is less than equal to 0 for arbitrary y prime, and we have strong maximum, and if W is less than equal to 0 for y prime close to p, then we have weak maximum.

Since, if we have to consider y prime close to p; that means, we are considering that this in the first order proximity, if y prime is arbitrary, then we are considering the 0 order proximity. So, as defined earlier, we get in case y prime is arbitrary, we get a strong minimum or maximum depending upon the sign of W. And in case when y prime is is to be assumed to be close to p, then we get a weak minimum or maximum depending upon the sign of W.

And here checking the sign of in various examples, we had considered here like in this case, we saw that this W expression comes out to be quite complicated, and checking the sign of this is not that easy. So, we look for a simple condition, which would guarantee as the same thing, but has been concluded earlier like this.

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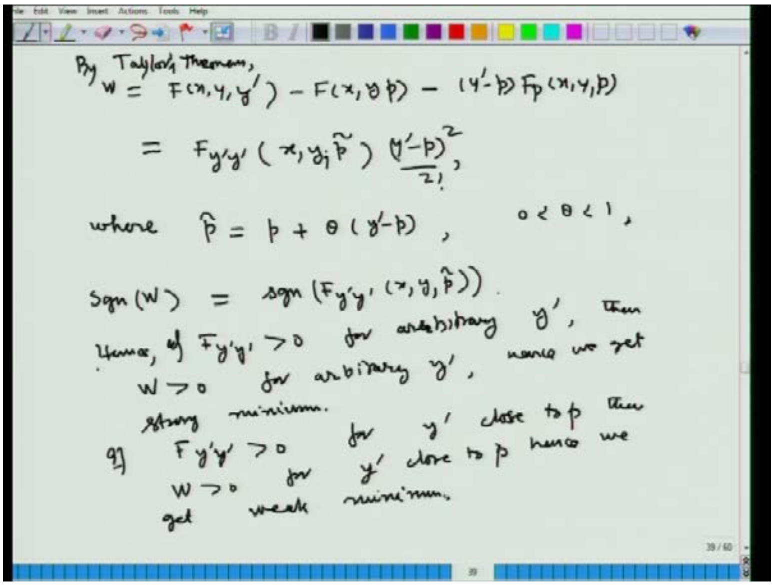

So, we just see that; if we use Taylor expansion here. So, W is F (x, y, y prime) minus F (x, y, p) minus (y prime minus p) F p (x, y, p); if we expand assuming that y prime is close to p, then by Taylor theorem. By Taylor theorem Taylor's theorem, this can be expanded like this. So, you will have F here, this F y double prime prime into (y prime minus p), because here we expand this in the neighborhood of p.

So, this term will cancel; and that first derivative term will cancel here like this; and so, we will be having thus y double prime, double prime this is to be evaluated at some intermediate point like this is (x, y, p tilde) into (y prime minus p) whole square upon factor two, where p tilde is p plus some theta times y prime minus p, where is 0 less than theta less than 1.

So, we see that the sign of W will then, if y prime is close to that p; and we see that the sign of this can be determined by the sign of y prime y prime. So, here this is since, this (y prime minus p) whole square on factor 2 is positive. So, this sign of W, then is sign of F y prime y prime (x, y, p tilde) here.

So, we see that in the neighborhood of this p prime will be close to p, in case this factor is small. Here, we see that the sign of W, if it can be seen that if sign of W is sign of a prime y prime for arbitrary y prime, then again we get here that is same thing to be determined by whether the this sign of that this is plus 1 or minus 1.

So, like that we will see that; hence F y prime y prime positive for arbitrary y prime, then this W is greater than 0 for arbitrary y prime, hence we get strong minimum. If here if this... So, if F y prime y prime greater than 0 for y prime close to p, then W is greater than 0 for y prime close to p, hence we get weak minimum.

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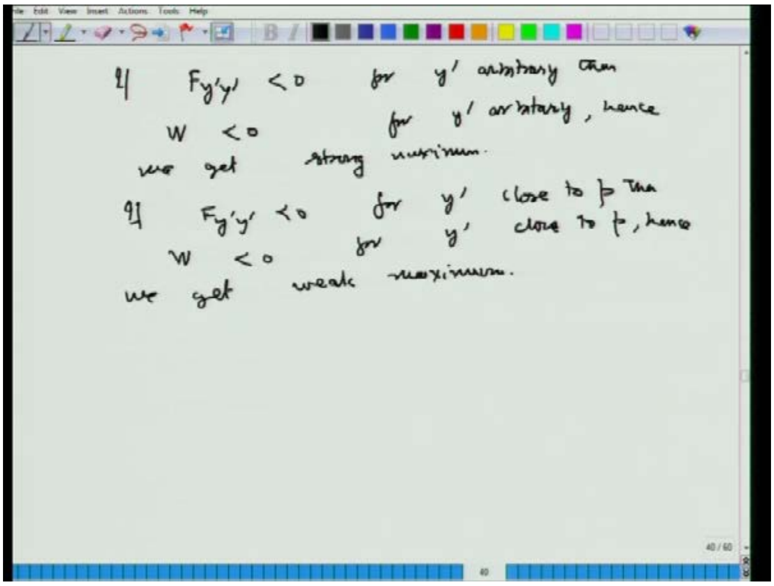

And if F y prime y prime is negative for y prime arbitrary, then W will be less than 0 for y prime arbitrary, hence we get strong maximum. And if this F y prime y prime is negative for y prime close to p, then W is negative for y prime close to p, hence we get weak maximum. So, that is what we have to check here; that F y prime y prime is whether positive or negative, which is easier to check compared to the sign of W.

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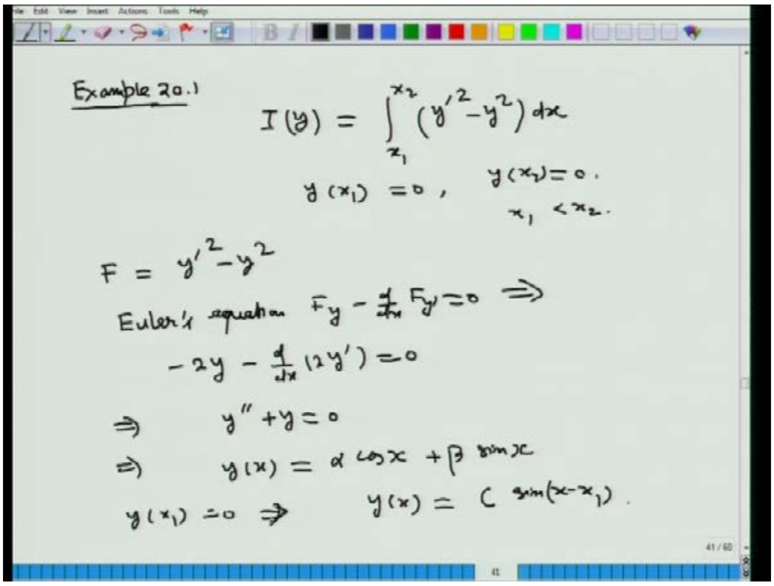

So that is, what we will do in this example. So, let us say, this is 20.1. Here, we have I (y) equal to integral x 1 to x 2 (y prime square minus y square) d x and y at x 1 equal to 0 and y at x 2 equal to 0.

Here, we have this x 1 less than x 2; obviously, and so, here F is y prime square minus y square; and Euler's equation F y minus d by d x F y prime equal to 0 implies that minus 2 y minus d by d x of 2 y prime equal to 0, this implies that y double prime plus y equal to 0; and so, solving this, we get y (x) equal to alpha cos x plus beta sin x; and y condition y at x 1 equal to 0 implies that y (x) equal to some constant c sin x minus x 1, this has already been done. So, that is what we get here.

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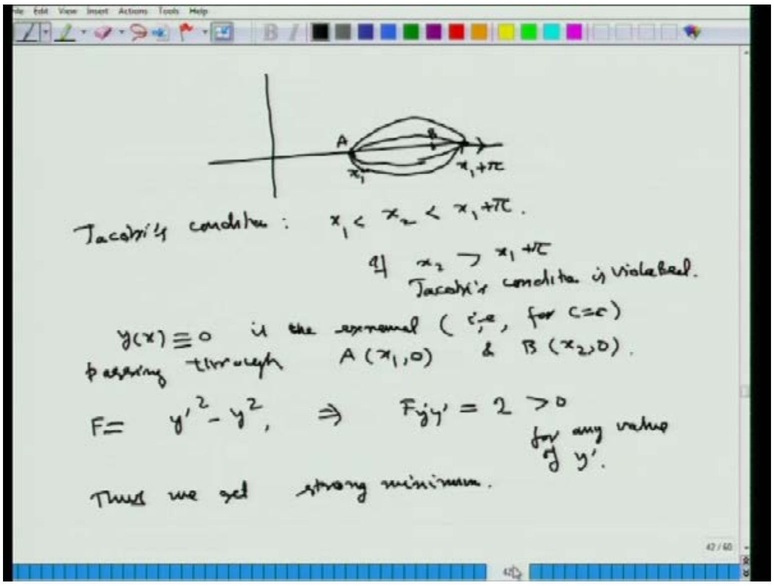

So, we have this x 1 is somewhere here and we have x 1 plus pi. So, we get these curves here like this. And we know that the Jacobi condition is that this x 2 must be… So, Jacobi condition Jacobi's condition would be fulfilled, if this x 2 which is greater than x 1, it should be less than x 1 plus pi, if x 1 plus… If x 2 is greater than if x 2 is greater than x 1 plus pi, Jacobi's condition is violated.

So, we need to have this x 2 here, this point and clearly this y (x) identically 0 is the exremal that is, for c equal to 0, we have this exremal y identically 0; passing through A that is (x 1, 0) and B that is (x 2, 0). So, this is the exremal here, y identically 0 to this x axis part of this axis, which is passing through these two points is A; and this point is B here, on the x axis.

Now, we need to check whether on this exremal, we have maximum value of I or minimum value of I. So, for that we had checked it earlier, by the sign of W. Now, we check, we have to calculate this F, here F is y prime square minus y square. So, this implies that F y prime y prime is 2, which is positive for any value of y prime. Thus, we get strong minimum in this case.

So, this I (y) here will be having minimum value, strong minimum value on the exremal y identically 0, which h is actually 0 here.

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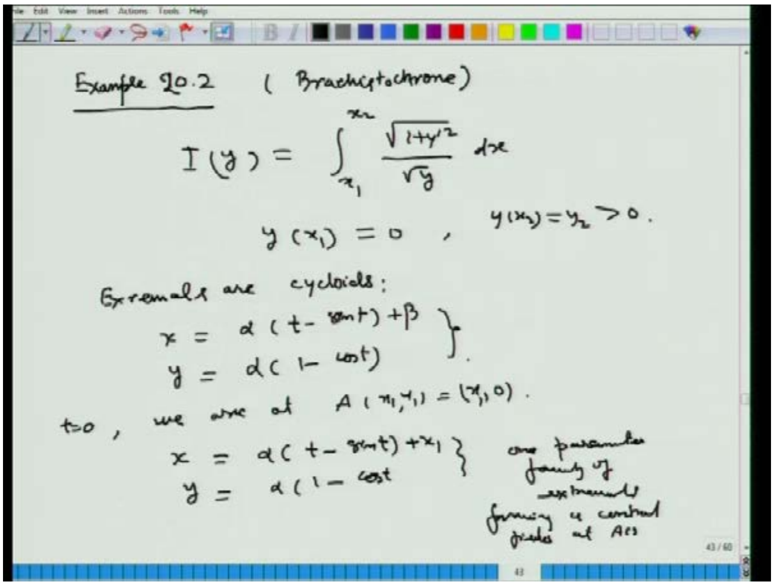

And so, now, next we will consider the example of Brachistochrone that is 20.2. So, this is Brachistochrone ,which is come in our discussions at several points, here I (y) is x 1 to x 2 root 1 plus y prime square over root y d x; and we take y at x 1 equal to 0, and y at x 2 equal to y 2, this is strictly positive here. And we know that; the exremals are cycloids given by x equal to alpha (t minus sin t) plus beta; and y equal to alpha 1 minus cos t; and here this at t equal to 0. So, t equal to 0, we have y at x 1.

So, we are at at t equal to 0; we are at we are at A that is (x 1, y 1) which is actually equal to (x 1, 0); and so, it is plus that x must be equal to alpha (t minus sin t) plus x 1 and y equal to alpha 1 minus… So, this is the one parameter family of exremals, one parameter family of exremals forming central field at A, which is (x 1, 0).

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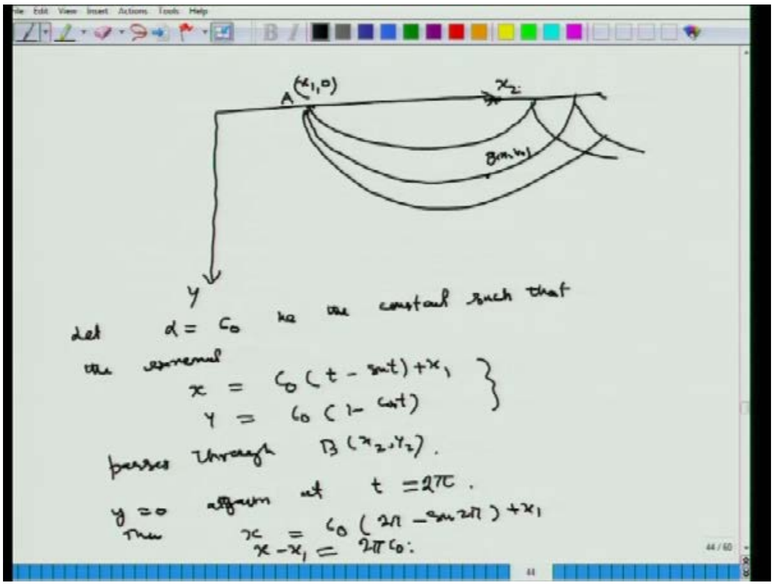

So, that is the picture here we have. So positive y will take like this and this is positive x and this is the point x 1 here, and we have the cycloids going like this, and here the then go like this, and so, here we need to see that. So, this is the family of this is the central field at this is the point A, and we need to see that B is somewhere here. So, this is x 2 here. So, B is (x 2, y 2), A is (x 1, 0) like this.

So, we see that here, let us say, let this alpha equal to c 0 be constant, such that the exremal here, x equal to c 0 to (t minus sin t) plus x 1 and y equal to c 0 (1 minus cos t) passes through the point B which is (x 2, y 2). So, we see that this c 0 should be such that it is not going to be this that exremal, where I mean, this is B point should be before where these exremals start intersecting each other.

So that means, we see that this y will be 0. So, y will be 0, y equal to 0 again at t equal to pi; and so, we see that, because here we substitute t equal to 2 pi, then this becomes 0. And so, y equal to 0 and t equal to of course, when t equal to 0 here we are at A, so there y equal to 0. So, y will be 0 again at t equal to 2 pi and so, then x will be equal to so, at 2 pi this so, c 0; t equal to (2 pi minus sin 2 pi) plus x 1. So, this is will be... So, x minus x 1 must be equal to 2 pi c 0.

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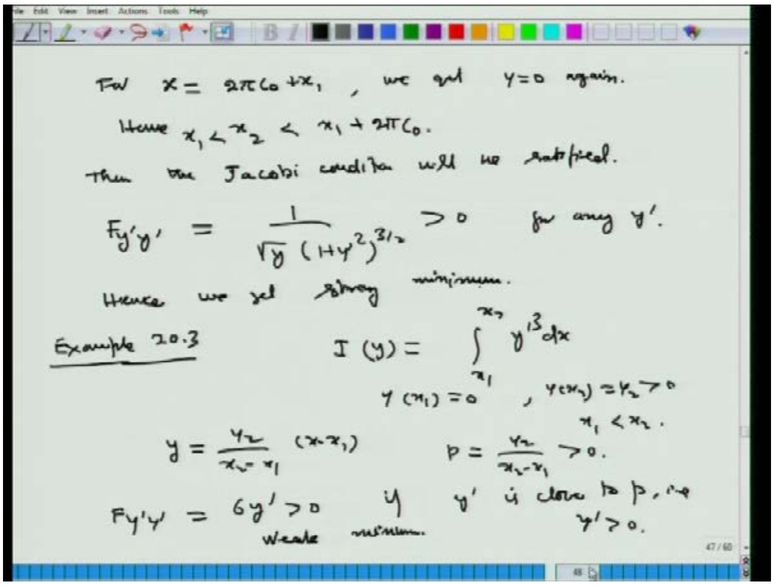

So that means, here x equal to, so x 2 must be. So, we have this x minus… So, x equal to 2 pi c 0 plus x 1; that is where we will get the value of y. So, for this, we get y equal to 0 again; hence, this x 2 must be less than. So, this x 1 less than x 2 must be less than x 1 plus 2 pi c 0. So, this is the Jacobi condition. Then the Jacobi condition will be satisfied. Thus we see that here, in this case, we have x 2 must be like this that x 1 plus 2 pi this.

Now, we check in this case F y y F y prime y prime actually comes out to be in this case 1 over root y times 1 plus y prime square 3 by 2, which is always 0 for sorry which is greater than 0 for any y prime, hence we get strong minimum in this case. In the last example here, which we have several times visited earlier, x 1 to x 2 y prime cube d x and we see that here y at x 1 equal to 0, and y at x 2 equal to y 2, which is positive here x 1 is less than x 2.

And we see that here, this exremals are y 2 over x 2 minus x 1 x minus x 1. So, here we see that; p is y 2 over x 2 minus 1, which is positive. And here, F y prime y prime comes out to be 6 y prime, which will be positive, if y prime is close to p; that is y prime is positive; and so, we get weak minimum in this case. And we cannot conclude more than this thing here, we we had earlier by directly seeing the sign of W we could say something more than this of course.

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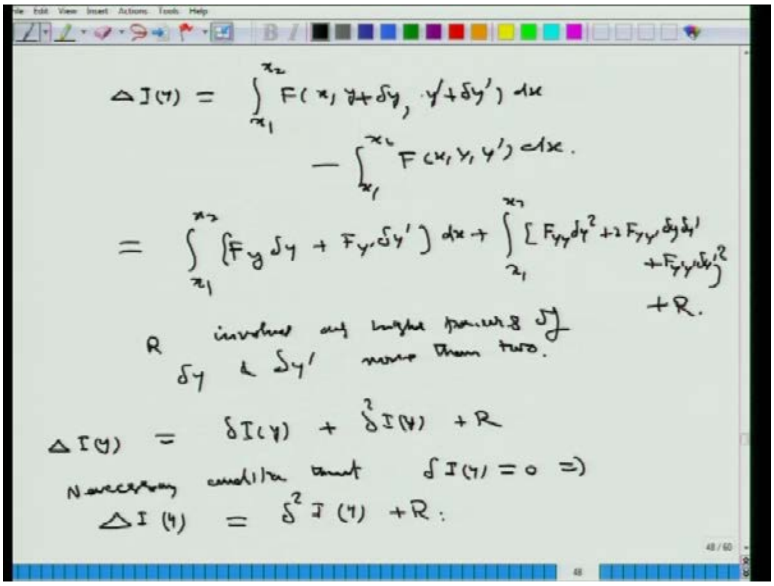

Now, let us see that; how we actually get these conditions in some other way, here let say that this delta I (y) this is x 1 to x 2 integral F of x, y y prime d x y is change to y plus delta y, and y prime plus delta y prime) d x minus x 1 x 2 F (x, y, y prime) dx.

And here this can be written at this x 1 to x 2 expanding it by Taylor series, we get F y delta y plus F y prime delta y prime d x plus we get x 1 to x 2 F y y delta y square plus 2 F y y prime delta y delta y prime plus F y prime y prime delta y prime square plus R. Which involves, R involves only higher powers of delta y and delta y prime more than 2.

So here, so that therefore, this I delta I (y) is delta the first variation here, and plus this delta is square I (y) plus R. So, this the first variation, and this the second variation, which is the second term here plus R, and the necessary condition condition that delta I (y) must be 0 and implies that delta I delta I (y) then finally, is equal to delta square I (y) plus R.

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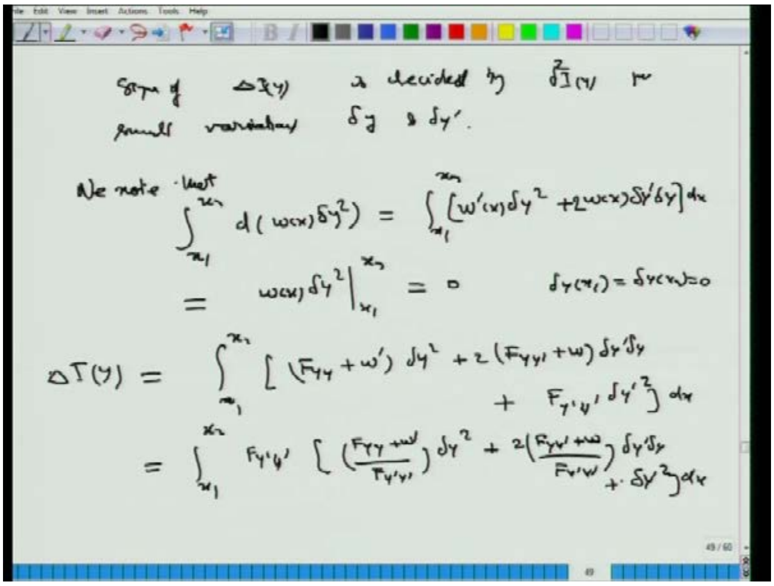

So, this sign of the sign of delta I (y) is decided by delta square I (y) for small variations delta y and delta y prime. So, we see that we need to check only this sign of this. So, what we do here, we check we note that this, integral x 1 to x 2 this d of some function let say, w (x) of into delta y square is actually, x 1 to x 2 w prime (x) delta y square plus 2 w (x) delta y delta y prime delta y dx, but on the other side this thing is w (x) delta y square evaluated at x 1 to x 2 and this is 0, because delta y is delta y at x 1 as well as delta y at x 2 is 0.

And so, this term, we can add to this. So, we have delta I (y) equal to integral x 1 to x 2, we add these things here, we get here F w prime x. So, F y the second variation here, we will add in this, we get F y y plus W prime delta y square plus 2 (F y y prime w) delta y prime delta y plus F y prime y prime delta y prime square d x. And we take this x 1 to x 2, this F y prime y prime we take out; and we see that here, we get F y y plus w prime over F y prime y prime delta y square plus 2 F y y prime plus W over F y prime y prime delta y prime delta y plus delta y prime square dx.

So, we want to make this perfect square. So, we select this gabling, such a way that this is perfect square, so that means the square of this must be this into this. So, there is one here.

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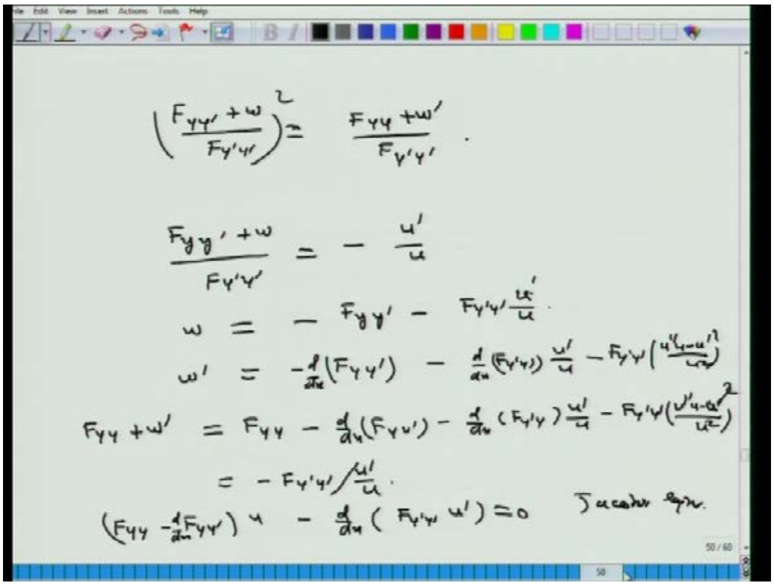

So that means, we have (F y y prime plus w over F y prime y prime), this must be square of this must be equal to F y y plus w prime over F prime F y prime y prime. So, that is, how we have to check, we have to select this w, in such a way so; that means, we take this substitutions y F y y prime plus w over F y prime y prime equal to minus u prime over u, we substitute it like this, and so, here w must be selected like this that it should be minus F y y prime minus F y prime y prime u prime over u.

And so, from here, we see that w prime must be equal to minus (F y y prime) prime, prime is here d by d x. So, we will write it as d by d x. So, there are other variables also minus d by d x of (F y prime y prime) u prime over u minus F y prime y prime, here (u double prime (u minus u prime square over u) and substituting here, we see that; we get so this adding these two F y y. So we will have a F y y plus w prime will be this F y y minus d by d x of (F y y prime) minus d by d x of (F y prime y) u prime over u minus F y prime y prime u prime u double prime minus u prime square over u square u prime square.

And so here we see that this is nothing but over this, this is actually equal to F y prime this is from here, this is equal to minus F y prime y prime u prime over u. So, we see that the last term here will cancel with this and simplifying this, we get the Jacobi equation. That is (F y y minus F y y prime d y d x) of this u minus d by d x of (F y prime y prime u prime) equal to 0; Jacobi is equation. So, which we had already got; so, if they solution of the Jacobi satisfies that only at A it vanishes and not before B, then we get that condition satisfied.

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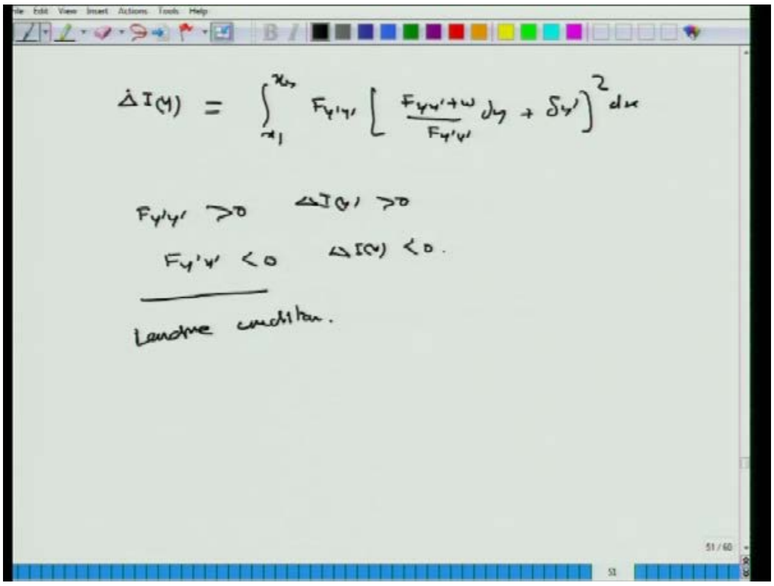

And so, next we see that here. So, using that we finally, get this delta I (y) comes out to be x 1 to x 2 F y prime y prime, which we had taken out and this becomes perfect square here F y y prime plus W over F y prime y prime plus times delta y and plus delta y prime whole is square d x.

So, clearly the sign of… So, if F y prime y prime is positive, we get delta I (y) positive, if F y prime y prime is negative and delta I (y) negative; and so this condition is this F y prime y prime positive, this call actually Lysander condition, these conditions are called Lysander condition; is an conditions.

So, we have the Jacobi condition and Lysander conditions are satisfied, you see that we get the required results that are in the case that F y prime y prime is positive, we get strong or weak a minimum provided in the case, whether y prime is closed p to be assumed or not. Similarly, F y prime y prime negative it imply the strong or weak maximum depending upon the arbitrariness of y prime or whether it is assumed to be close to p.

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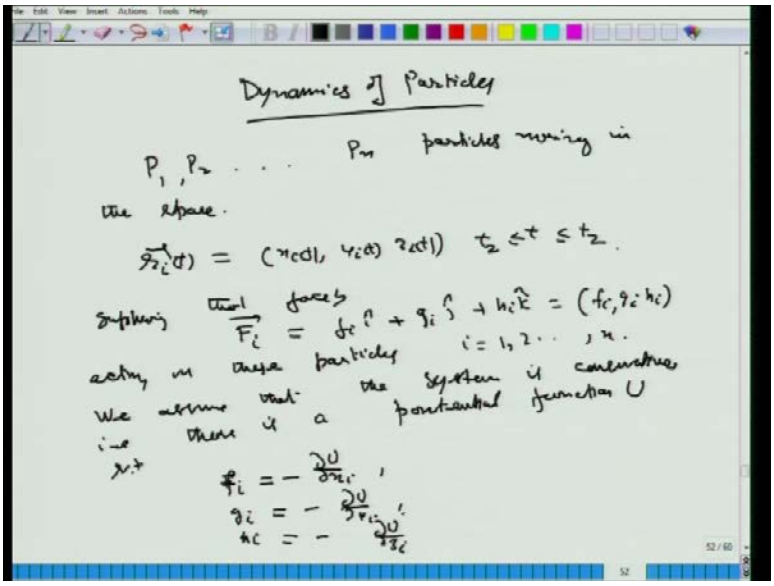

So, now we have one more application that is in the mechanics, this dynamics of particles. So, here let say, there are n points p 1, p 2, p n particles moving in this space and let say, their position vectors are r i t, which is (x i t, y i t and z i t) t line between t 1 to t 2.

And so here we say consider, supposing that that F I, F i which is F i i plus g i j plus h i k or in competent form (f i, g i, h i) acting on these particles, here i equal to 1, 2 to n. And we assume that the system is conservative that is, there is a potential function U, such that this f i equal to minus del u over del x i; and g i equal to minus del u over del y I; and h i to minus del u over del z i.

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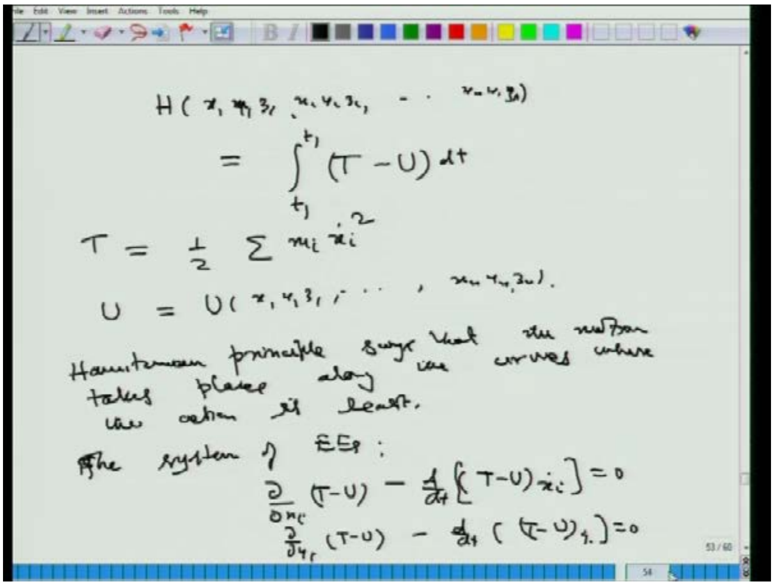

So, here we consider the functional H, which is function of now x 1, x 2, x 1, y 1, z 1, x 2, y 2, z 2 and so on x n, y n and z n, these are functions of t; and we have this t 1 to t 2, we take (T minus U) d t, where t is the kinetic energy, T is the kinetic energy given by half summation m i x i dot square; and U is the function of x 1, y 1, z 1, and so on x n, y n, z n here.

So, the Hamilton principle says that; the Hamiltonian says that the motion takes place place along the curves, where that the action is least is called principle of least action. So, we will established this thing, we have using the techniques of the calculus of variation. So, we need to see that here, we need to find the minimum of H. So that means, the Euler equation for this will be the system of Euler's equation of del over del x i of (T minus U) minus d by d t of T minus U of x i dot equal to 0. So, similarly you will have del over del y i (T minus U) minus d by d t of (T minus U) y i dot equal to 0.

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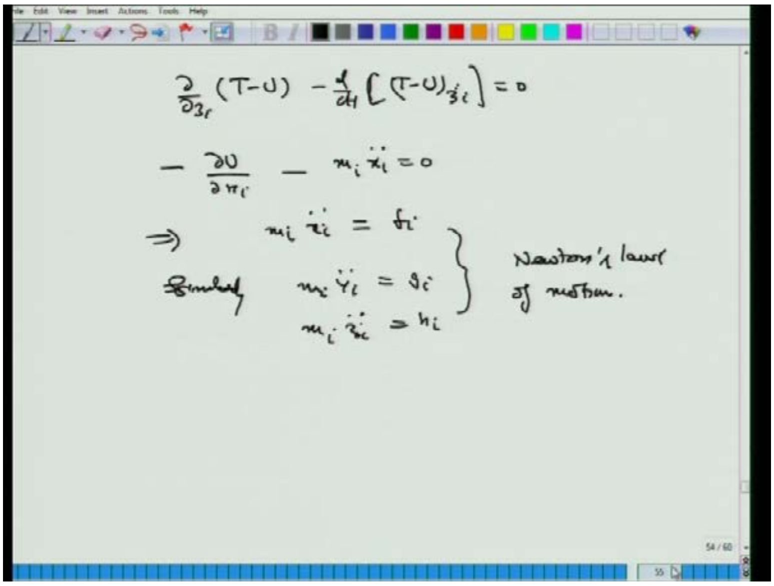

And the last one that; del by del z i of (T minus U) minus d by d t of (T minus U) z i dot partially like this. So, we get here, T is involving only here dots. So, you will have minus del u over del x i dot del x i minus here T is only involving the dots. So, you get minus m i x i double dot equal to 0. So, we get here this minus minus sign. So, here this implies that m i x i double dot equal to f i.

Similarly, similarly we get m i y i double dot equal to g I; and mi z i double dot equal to h i. So, these are the Newton's law, Newton's laws of motion. So, we see that which is so, this necessary condition here the Newton's laws, which are always holding and therefore, the action will be taking along those curves, where the action is least.

So, like that we have the applications of the calculus of variation in several fields' mechanics, and optics, and dynamics, and various other problems can be tackled using the calculus of techniques. So, thank you very much for viewing all this lectures, I hope that this will be very useful. Thank you very much.