Calculus of Variations and Integral Equation

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Lecture # 02

Hello, welcome viewers on the second lecture of NPTEL on Calculus of Variations. Let us recall what we did in the first lecture. In the first lecture, we got introduced to certain concepts on the calculus of variations.

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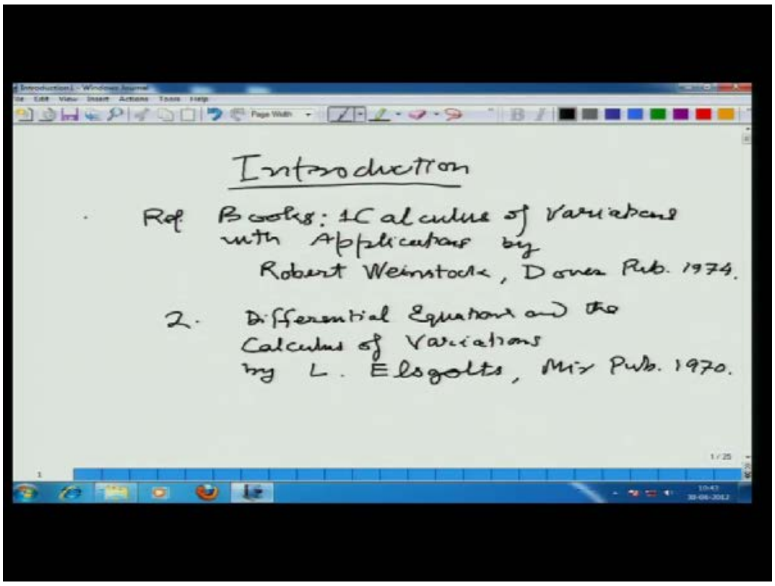

Here, I mentioned these two books, which will be followed as reference books that the Calculus of Variations with Applications by Robert Weinstock, is a Dover Publication appeared in 1974. The second book is a famous book by L. Elsgolts, Mir Publications, 1970; the title of the book is Differential Equations and the Calculus of Variations. So, these two books will be covering the material for 20 lectures, which are going to be delivered by me on the calculus of variations. So, let us recall what we did in the first lecture.

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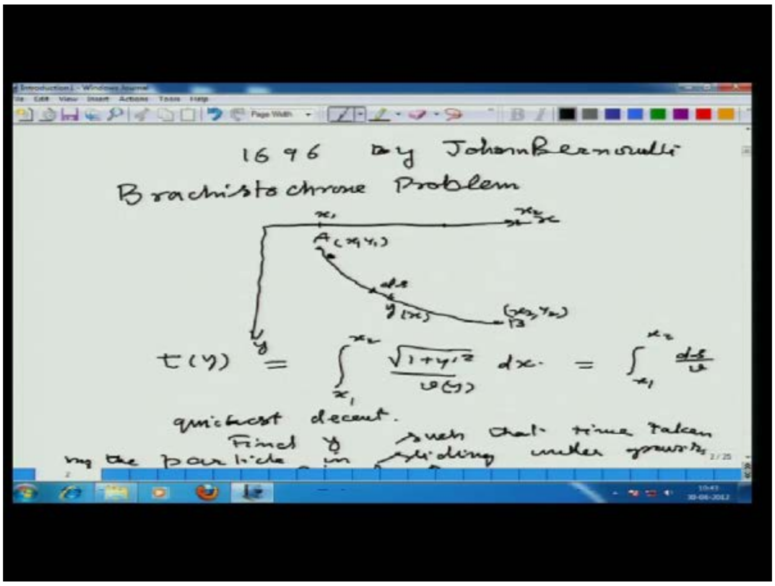

As mentioned, the study of the calculus of variations is started by the problem posed by Johnn Bernoulli. Johnn Bernoulli in 1696, he proposed a problem known as Brachistochrone problem, where he asked to find a smooth curve in a vertical plane joining two points A and B not on the same vertical level, such that the time taken by a particle sliding under the influence of gravity takes the least time. That year 1696, famous mathematicians like Newton Leibniz, Johnn Bernoulli and his brother Jacob Bernoulli and many other like (( )) – all these people solve the problem in that very year. So, the interest got created on the study of calculus of variations and many more problems were posed and solved using the techniques of the calculus of variations.

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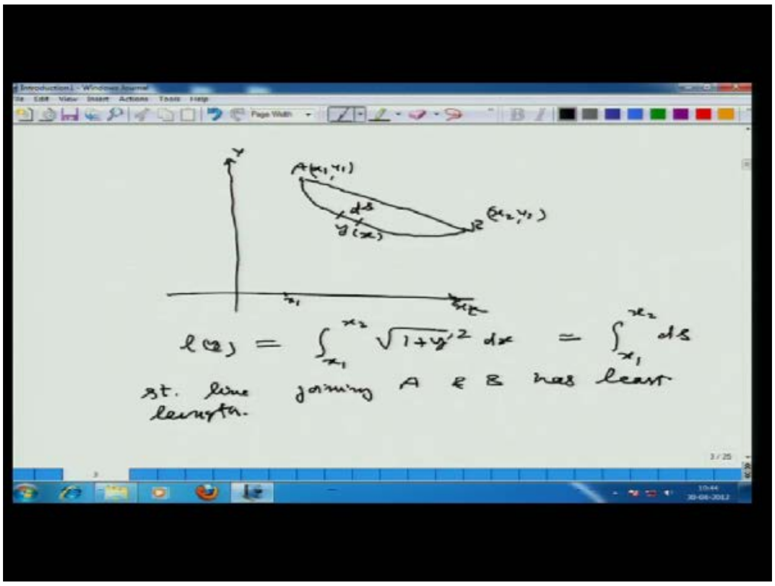

We saw the second problem, where in a plane, there are two points: A and B. Then, we asked to find a curve such that the length of the curve joining these 2 points is the minimum. And, we know the answer is the state line joining these two points: A and B in a given plane. And, we can see that it can be posed as a problem of calculus of variations like the minimization of the functional l, which is a function of y, given by integral x 1 to x 2 square root of 1 plus y prime square dx, which is nothing but the integral of the element length d s integrated over the interval x 1 to x 2. We will see that when these sufficient tools developed for this calculus of variations, we will see that the answer will be actually be the straight line joining these two points.

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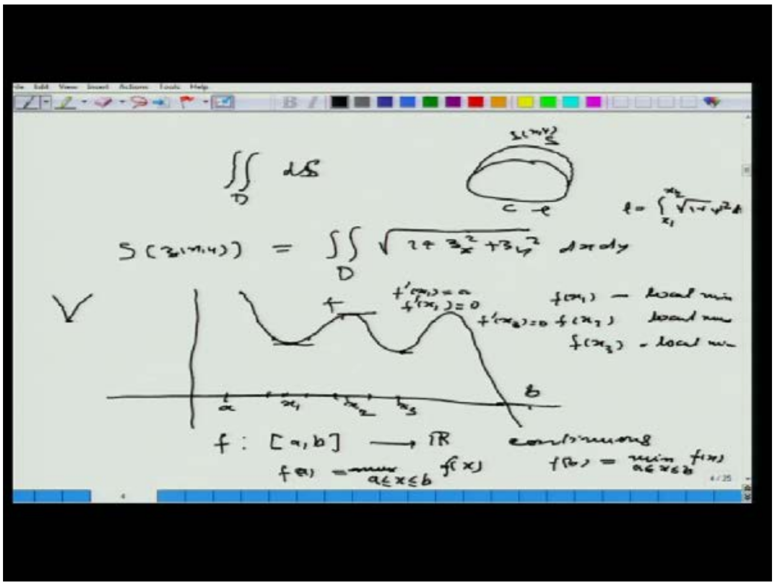

Then, other problems like isoperimetric problems, where curve is given in a plane, where its length is fixed, given by l and it is ask to find a surface, which is enclosed by this curve. So, this problem is also can be posed as a problem of calculus of variations. We will see that. Here in that case, the answer will come out to be the flat surface enclosed by the curve, whose length is fixed, given by the length l.

Now, this is the concept of the calculus of variations is similar. It is a generalization of the big concept of finding the another problem of calculus of variations, can be seen that given a curve… See here whose length is l, which is a fixed number and we are asked to find a surface enclosed by this curve c. Here this an isoperimetric problem, where the perimeter is fixed and you are asked to find a surface, which is enclosed by this curve c, whose length is fixed. So, in that case, here it can also be seen that s surface area given by z of x, y, where this surface will be given by z of x, y, is given by the double integral square root of 1 plus z x square plus z y square dx dy. And, here length of this curve will be fixed; that is given by the earlier integral, which is parametrically, we can see that the length of this will be given by x 1 to x 2 square root of 1 plus y prime squared dx. So, such a thing can be posed as the problem of calculus of variations, where you have one quantity, which is to be optimized under the given condition, which is known as a side condition like the length of the curve is fixed and we are asked to find the surface such that the area of the surface is the minimum or maximum.

Here, let us recall that certain concepts related to finding the points, where a given function takes the minimum or maximum value. Here there are certain points, where function takes the maximum value over the whole interval; such a point is called global maximum; or, at certain point, the function takes the minimum value over the whole interval; such a point is called the global minimum. And, there are certain other points in the interior of the interval or it may be at the end points also, where the function can have local minimum or local maximum. For example, at this point x 1, in the neighborhood of this, there are points, where the function is having larger value than the value at the point x 1.

Similarly, at the point x 2, in the neighborhood of this, there are points such that function f takes a lower value than the value at the point x 2. So, these are the points, where f takes a local minimum and local maximum respectively. So, here when function has smooth properties like its derivative is continuous, and you can see that at interior points, if the function takes local minimum or local maximum, then the derivative becomes 0; the tangent become horizontal at those points; whereas, at the endpoints, we may have to check the values of the function separately and compare it with other values, so that we can check whether at the endpoints, global minimum or global maximum is occurring or not. And, at other interior points, if function is smooth, then we just check for the equation a prime x equals to 0 and solve for those x such that the f takes a prime takes value 0. So, at those points, we will check then whether these are the points candidates for minimal or local minimum or local maximum. And then, we go for higher order derivatives, other tests to check whether we have minimum or maximum at those points.

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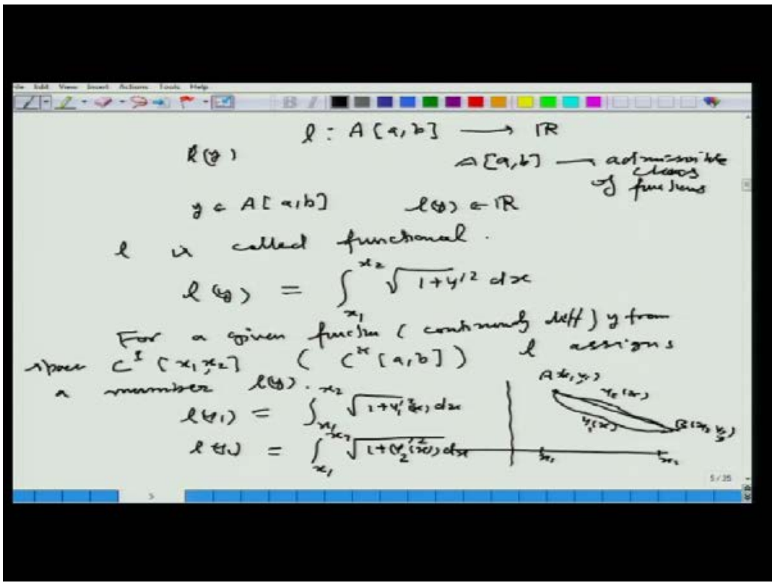

Then, here we defined the concept of functional. So, you know that in the case of (Refer Slide Time: 08:49) a function, a function assigns a number in this interval a, b to a number in the real line, R; whereas, in the (Refer Slide Time: 09:00) functional case, this functional l assigns a number to a given function in the class of admissible functions like y belonging to a admissible class over the interval a comma b; it assigns a number l y, which is a real number and it is given like in this example given by the integral x 1 to x 2 square root of 1 plus y prime squared dx. So, given y, we have the length l as a real number – positive real number, non-negative real number here such that it assigns to each curve smooth curve, so that y prime is piecewise continuous, so that this integral is well-defined. So, this l will assign a number to such admissible curve. Here admissibility is that y prime should be piecewise continuous, so that this integral is defined in the sense of remark. So, this functional l assigns a number l y to this function. Such a thing is called functional.

And, here (Refer Slide Time: 10:18) we will have when this integral or we have more general forms of integrals, where y prime y double prime appearing in the integrand, then we will take the higher order continuous spaces, which are spaces defined as c k, where k is non-negative integer over the interval a b and doubt with the supreme norm over the interval a b. So, for example, in this case, you have two curves: y 1 and y 2; and, l assigns a number l y 1 and l y 2 to each of these curves joining these two points: a and b. And, the these are… y 1 and y 2 are from the class of admissible functions. They are such that y 1 prime and y 2 prime are piecewise continuous, so that the integral defining their lines is well-defined.

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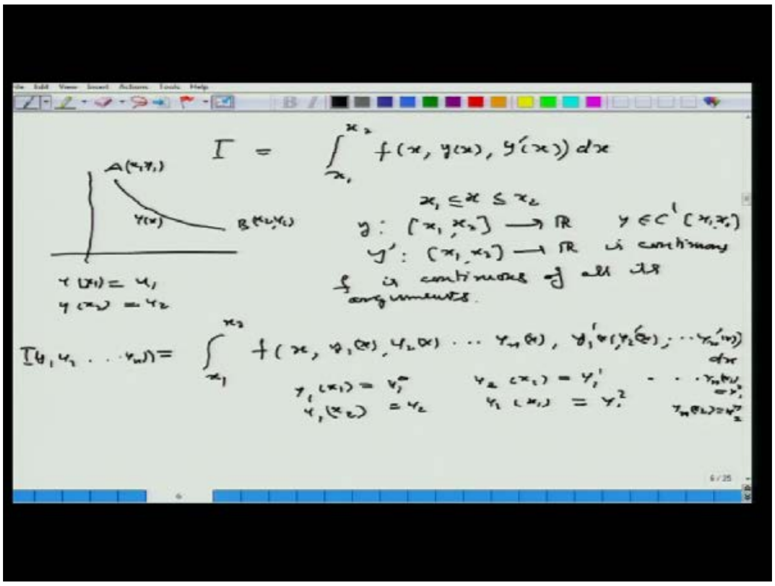

Then, next we consider the general form of integral I, which may appear in our analysis and the other integrals, which will involve higher order derivatives, will also appear in our analysis. For example, this I here is an integral defined over the interval x 1 to x 2 of a certain given function f, which has continuity property of that if f is continuous of its arguments; and, this y and y prime – y is continuous; whereas, y prime is piecewise continuous. So, then this integral will be well-defined in the sense of (( )) And, here the boundaries – these points A and B are fixed here. So, each of the admissible function belonging to the admissible class must satisfy these boundary that they are passing through these two points. So, each of the admissible function should actually in this class in such a way that these boundary conditions are satisfied.

Then, we are supposed to find a y, which will actually optimize the value of this integral whether it will minimize or maximize, we will be actually saying that I gets optimized by the function y. And then, other kinds of integrals, more general forms like where you have several dependent variables y 1 and y 2, y n appearing like this (Refer Slide Time: 13:03) integral over x 1 to x 2 f of x is the independent variable; whereas, y 1 and y 2 are dependent variables; and, their derivatives y 1 prime, y 2 prime and y n prime are appearing. So, here you would have these additional conditions like these are to be satisfied, so that each of these y i's are passing through the points A and B like you have y 1 at x 1 is y 1 0, y 2 at x 2 – y 2 0. Similarly, this also be satisfied. And, y n 1 and y n 2 at the point x 1 and x 2. So, these all the conditions to be satisfied by each of these y i's. And then, we are supposed to find the (( )) y 1, y 2, y n such that this integral is optimized.

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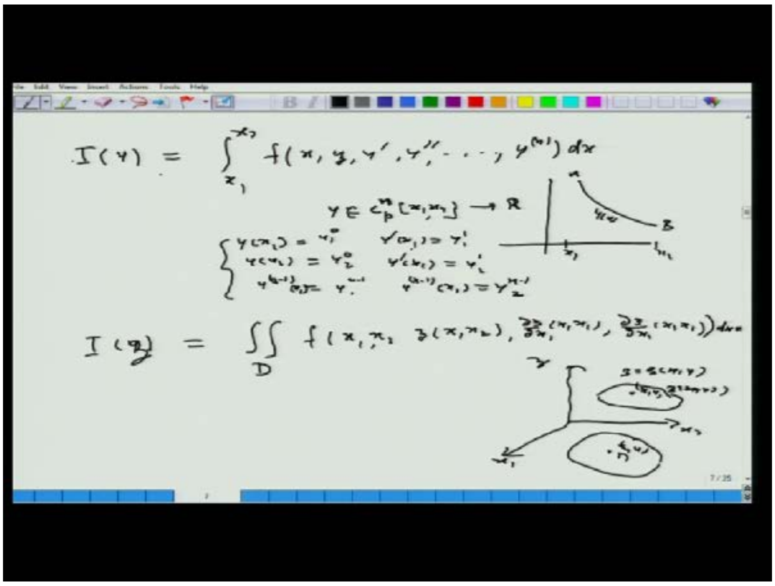

Then, there are other integrals in the form, where although the dependent variable is only 1 here, but then, there are higher order derivatives appearing. So, integral of x 1 to x 2 f of x, y, y prime, y double prime and so on up to y nth derivative; and, y is form c n x; or, piecewise, we can say that the highest order derivative can also be allowed to have a discontinuities of first kind owned on certain finitely many points inside the interval x 1 to x 2. And then, these are the conditions on y up to n minus 1 derivatives and… So, at each point, x 1 and x 2, these are the conditions to be satisfied by y, y prime and y up to y n minus 1.

Then, a more general problem would be where you have more independent variables appearing like x 1, x 2; and then, z is a function of x 1, x 2 surface such that here this is the derivatives with respect to x 1 and x 2 are piecewise continuous. Then, you can consider this in a integral of this f, which is a continuous function of all of these arguments; then, the integral is again defined in the sense of remand here over the domain d, which is an open connected subset of R 2. Here at each point x y, there is a point x y, z x y on the surface; and, this integral I is a function of z for… When we change the surface z, the value of I changes. And so, we are asked to find a surface z such that this I is optimized. Such problems appear in connection with the certain partial differential equations. And, those partial differential equations can be posed as optimization of problems as problems of calculus of variations equivalently like this. That is what we will see subsequently in our discussion.

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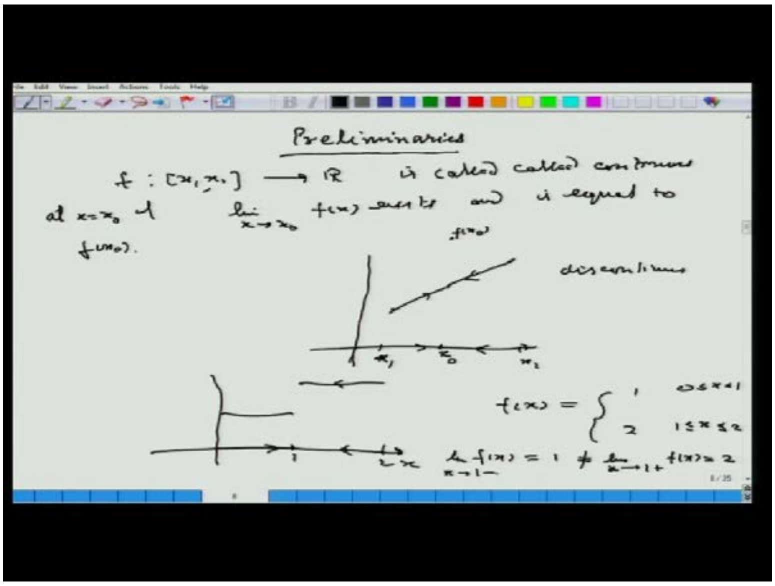

Then, certain preliminaries were discussed in the last lecture. We had certain function f from x 1 to x 2; and, it is called a continuous if the left and right limits exist and various examples were discussed in connection to that.

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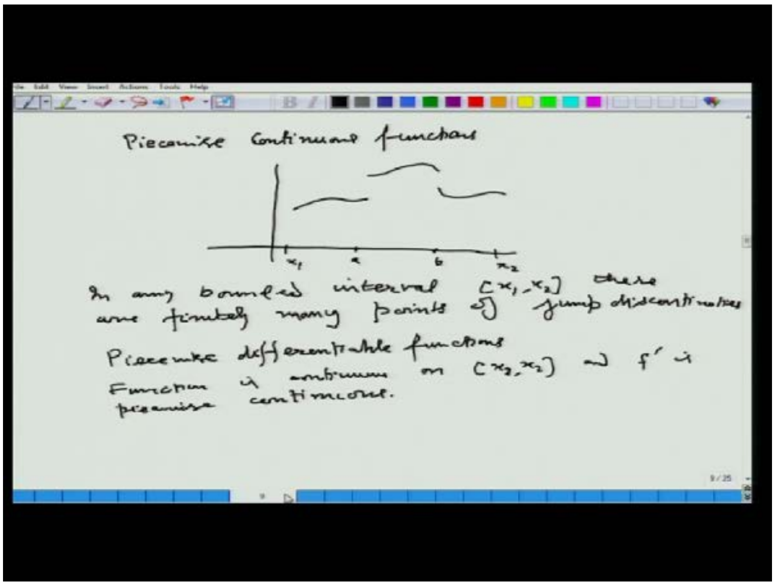

And then, piecewise continuous functions, where discontinuities of first kind are allowed; and, only at finitely many points inside the bounded interval x 1 to x 2; then, piecewise differentiable functions, where function is continuous and this f prime can have the discontinuities of first kind. Here prime is assumed not to change sign at infinitely mean in points inside the interval x 1, x 2, so as to avoid the case, where the derivative f prime; left derivative and right derivatives are equal. But, they are not equal to the value of the derivative at that point. Such situations will be avoided if we assume that f prime does not change sign at infinitely many points inside the interval.

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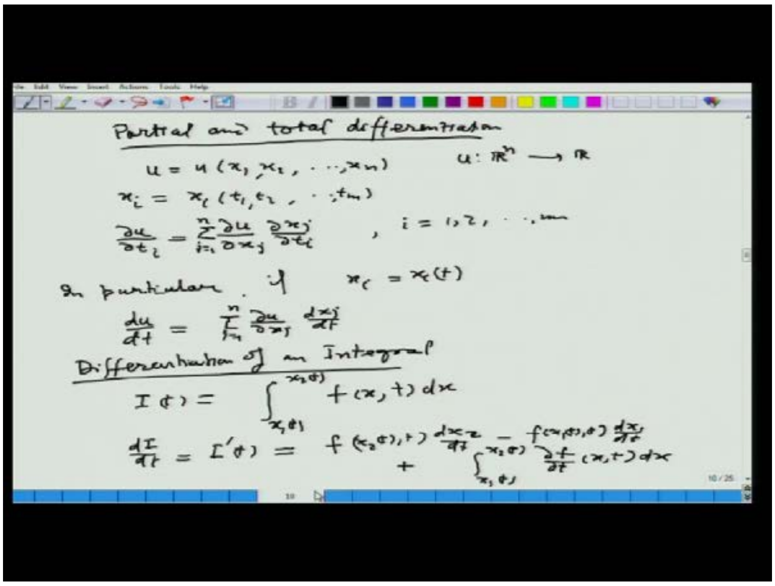

Then, we considered partial and total differentiation of a function of several variables; where, in the first case, when function u is a function of n variables x 1, x 2, x n. So, it is a function from R n to R such that the first order partial derivatives exists with respect to its variables. And then, x i's are in turn functions of several variables t 1, t 2, t m. Then, the partial derivative del u over del t i can be defined like this. And, in particular, if x i's or functions of only single variable, then this partial derivative with respect to t reduces to the ordinary derivative d u by d t in the following manner.

And then, we considered the Leibniz rule, which states how one actually differentiates an integral, where the limits are variable limits and this integrand is a function of several variables; like in this case, f is function of two variables: x and t; limits are functions of t; and, the variables of integration is x. So, here (Refer Slide Time: 18:56) I is treated as a function of t. And therefore, we can consider its differentiation with respect to t provided f has certain differentiability properties with respect to t. So, this can be defined like this I prime t is f of (( )) This t will be replaced by… Actually, it should have been x and x 2 t like that – and d x 2 over d t minus f of x 1 t d x 1 over d t and then x 1 to x 2 del f over del t x t d x. That is what we will have here. So, first term is f of x 2 t comma t d x 2 over d t minus f of x 1 t comma t d x 1 over d t. So, these differentiations of the limits are assumed here that x 1 and x 2 are differentiable functions of function t. And here inside this integral, the third term integral of x 1 to x 2 del f over del t of x t – here it is assumed that this f is the partial derivative of this f with respect to t exist; and, it is piecewise continuous function, so that this integral is well-defined.

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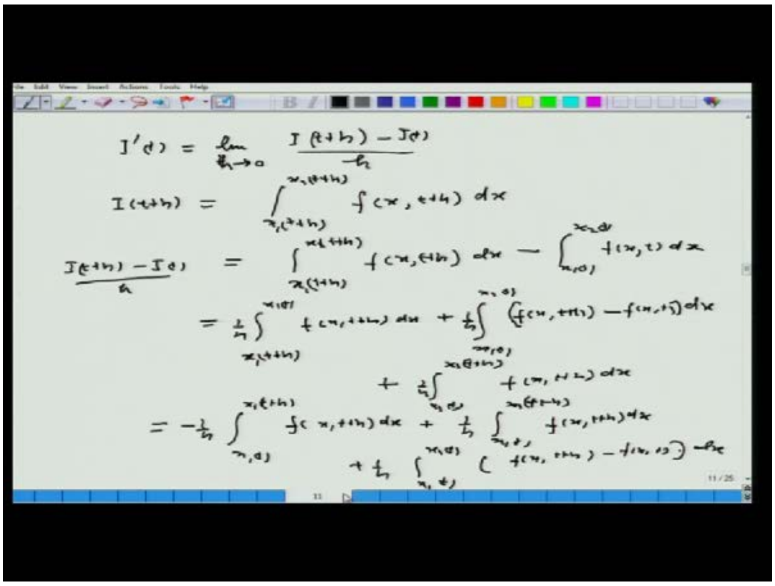

Now, this was the proof of that. Now, we start in this lecture on these remaining concepts, which will be required subsequently in our analysis.

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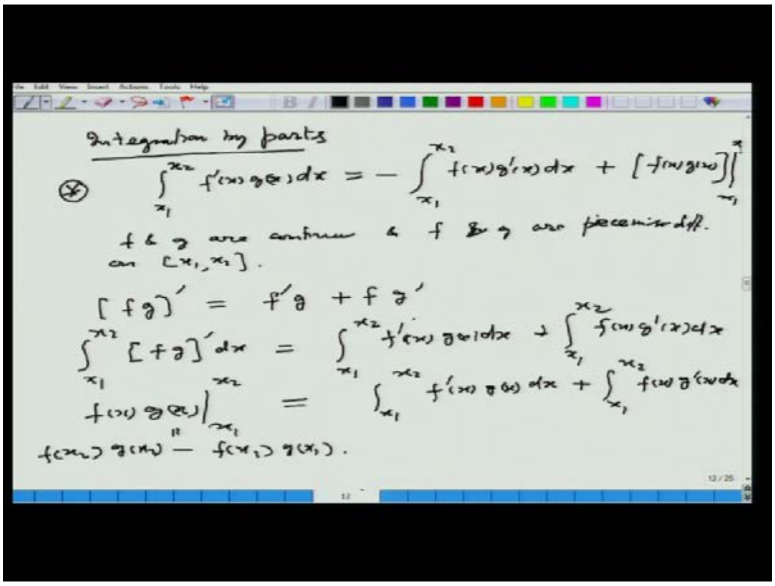

Now, the next one is integration by parts. It is a very useful formula, which gives us integral of one thing, where we have two functions appearing in terms of integrals of the same functions in a different form like you have x 1 to x 2 f prime x g x d x. This can be seen that this is equal to minus integral x 1 to x 2 f x g prime x d x plus f x g x evaluated at the bounded points x 1 and x 2. So, what it states that here you have two functions: f and g; where, f and g are assumed to be continuous; and, f and g are piecewise differentiable on x 1 to x 2. So, then, these integrals will be well-defined. Here this can be seen by the result that when you differentiate f into g prime, this is equal to f prime g plus f g prime. So, integrating this on the interval x 1 to x 2, is equal to x 1 to x 2 f prime x g x d x plus integral x 1 to x 2 f x g prime x d x.

Now, this is integration of differentiation of this term (Refer Slide Time: 23:23). So, this will cancel and it will give you the values at the end points. So, this side will be f x g x evaluated at x 1 to x 2. So, that is equal to x 1 to x 2 f prime x g x d x plus x 1 to x 2 f x g prime x dx. So, you can see that on the left side of this star, here we have x 1 to x 2 f prime x g x. So, this is equal to the values of f x g x at end points x 1 and x 2. So, that means, this is actually f of x 2 g of x 2 minus f of x 1 g of x 1; that is what actually this side is equal to. So, f x, g x evaluated at x 1 x 2; and then, minus evaluated at x 1. So, any of these terms can be seen that this term is this side minus the other term, which is appearing here – the first term on the right hand side of star. So, this is established using this fundamental concept that f into g prime is actually f prime g plus f g prime. And then, integrate this identity over the interval x 1 to x 2.

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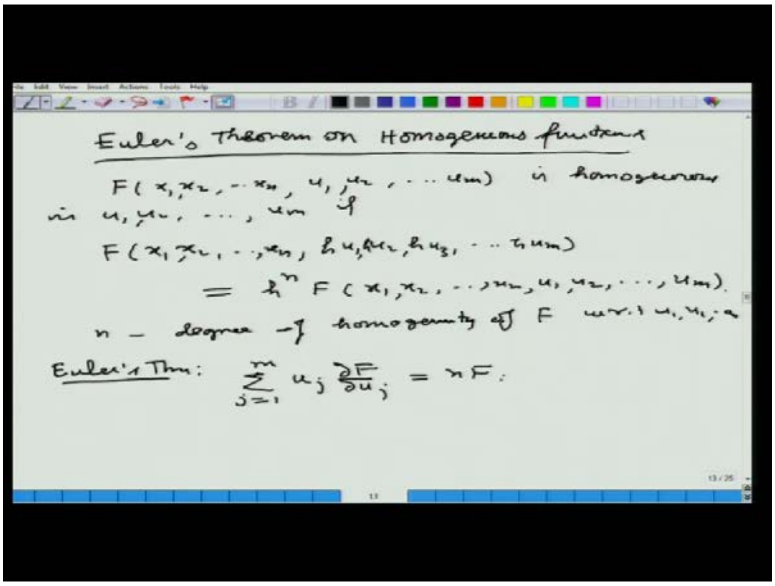

Next, we have the Euler's theorem on homogenous functions. So, what it says that first, the concept of homogeneous functions – a function F of several variables like you have x 1, x 2 and x n; and then, you have u 1, u 2 and u m. So, this F is a homogeneous function of these variables u 1, u 2, u m, if it satisfies… Is homogeneous in u 1, u 2, u m if F of x 1, x 2, x n, h of u 1, h of u 2, h of u 3 and so on h of u m, is actually equal to h to the power n – this n is called the degree of homogeneity of the function F x 1, x 2, x n; and, u 1, u 2, um. So, this function F is homogeneous in these variables u 1, u 2, u m if we have F of x 1, x 2, x n; and, h of u 1 comma h of u 2 comma h of u 3 and so on up to h of u m, is actually equal to… This h comes out with power n here. So, n is called the degree of homogeneity of F with respect to u 1, u 2, u m. So, for such a function, Euler's theorem states that del F is summation u j del F over del u j; j going from 1 to m. This is actually equal to n times F. This F is of course evaluated at x 1, x 2, x n and u 1, u 2, u m. So, this is what is the Euler's theorem for homogeneous functions, which will be very useful in our discussion.

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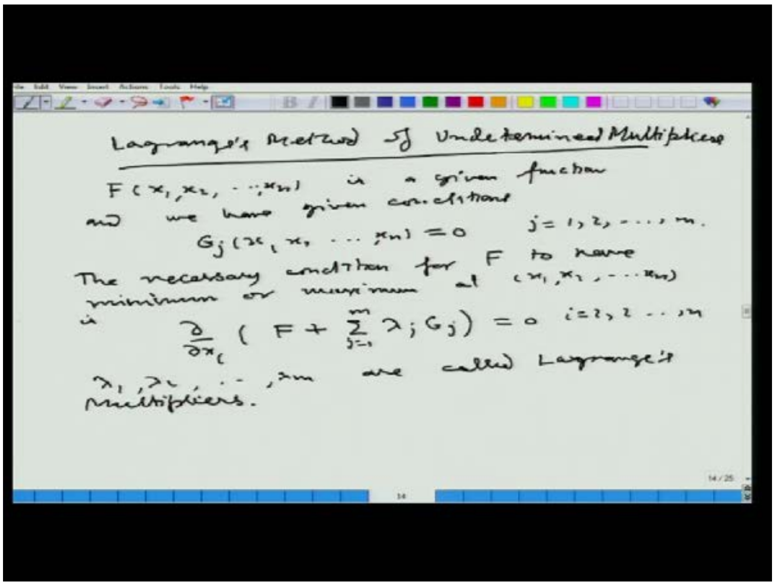

Next concept is the concept of method of undetermined Lagrange's multiplier – Lagrange's method of undetermined multipliers. Here in this case, we have this function F of several variables x 1, x 2 and x n here – and, we have certain conditions – is given function and we have conditions G j of x 1, x 2 to x n equal to 0; j is from 1, 2 to m. So, there are m conditions. So, these variables x 1, x 2, x n are supposed to satisfy these m conditions. Then, the necessary condition for F to have minimum or maximum at x 1, x 2, x n is that del over del x i of F plus summation j equal to 1 to m lambda j G j. This is equal to 0 at x 1, x 2, x n.

Here these lambda 1 are… So, the points… These (Refer Slide Time: 31:33) lambda 1, lambda 2, lambda m are called Lagrange's multipliers. These are unknown and they are to be found in addition with the points of minima or maxima. So, when we want to find the points of minima and maxima of this function F here, where these m conditions given, these are to be satisfied by those points, where F attains minima or maxima, then the necessary condition is that we should have these i equal to 1, 2 to n. These n conditions are to be satisfied at the point x 1, x 2, x n; where, this F attains minimum, maximum. Here the points x 1, x 2, x n are to be found; and also, these lambda 1, lambda 2, lambda m are to be found in this process.

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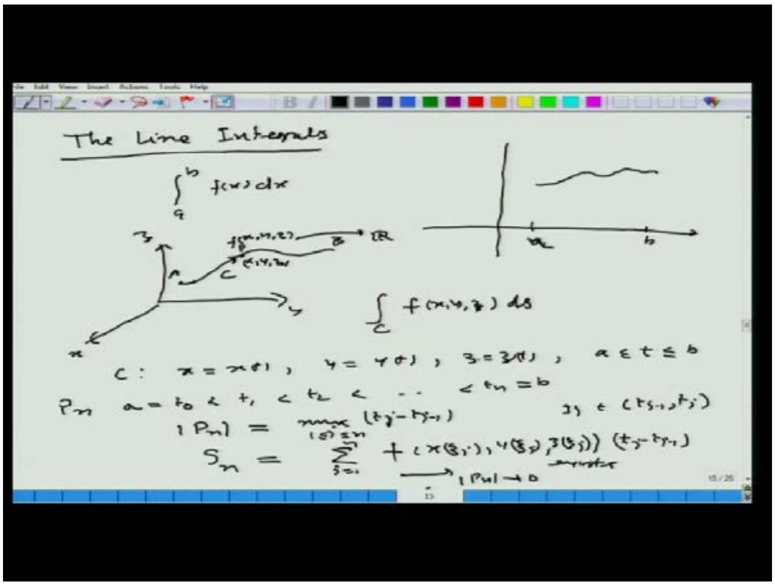

The next concept is the line integral. Here we have seen the integrals of this kind – a to b f x d x. So, here is this simple case, where we have certain interval here x 1; b here on the x axis and this function f is given over this interval. And, we know in the sense of remand, we can define this function a to b provided f is piecewise smooth function. So, the generalization of this is that you have in 3-dimensional let us say x, y and z; and, here instead of this interval a, b here, we have certain curve given here like this c; and, this is point A, point B. Like we move from a to b, we move from this capital A to capital B here. So, this defines the direction of the movement on the curve c. If we move from B to A, then that is a negative minus of the movement of what we do in the forward direction A to B. So, this clearly fixes the direction on the curve c here.

And, there is a function (Refer Slide Time: 34:32) f x, y, z defined from any point x, y, z here into this gives a value in R. So, we are supposed to find… Here we are supposed to define the notion of integral of this f over this curve c in certain sense here. And, we will have certain let us say over the arc length. So, s denotes the arc length here. And, this is the distance from any fixed point to a movable point here. So, let us say point A fixed. So, the arc length s is the length of this curve along the… See here; if p is here, now, this s will denote the length of this arc up to A to p. So, as p moves along this, the s increases here. So, that is the arc length parameter; or, in general, we may parameterize this curve like this. So, the curve c is parameterized as x t; y as y of t; and, z as z of t. Here t ranges between a to b.

Then, here this curve (Refer Slide Time: 36:10) c is assumed to be piecewise smooth in the sense that these x, y, z are piecewise continuously differentiable; their derivatives x y t are continuous; and, their derivatives are piecewise continuous functions. So, x prime t, y prime t, z prime t are defined at all points except at finitely many points, where they have jumped discontinuities. So, this curve is called the directed curve provided we have this kind of parametric representation here. Then, if we partition this curve like this, that is, like you partition the interval a equal to t 0 less than t 1 less than t 2 and like this; and, you have t n, which is b such that… So, this is the partition P n. And, the magnitude of the partition is defined like that – maximum of t j minus t j minus 1; where, j is from 1 to n. Then, we consider this kind of sum S n – summation j equal to 1 to n f of… Here on the interval, we have let us say xi j belonging to t j minus 1 to t j; then, you define f at x at xi j, y at xi j, z at xi j into t j minus t j minus 1.

Now, if this limit exists, if this limit (Refer Slide Time: 38:21) as P n tends to 0 as the size of this partition, which is the maximum of the largest length of some interval. So, if this tends to 0 and if this limit exists as this mod P n tends to 0; provided this limit exists, we define that to be…

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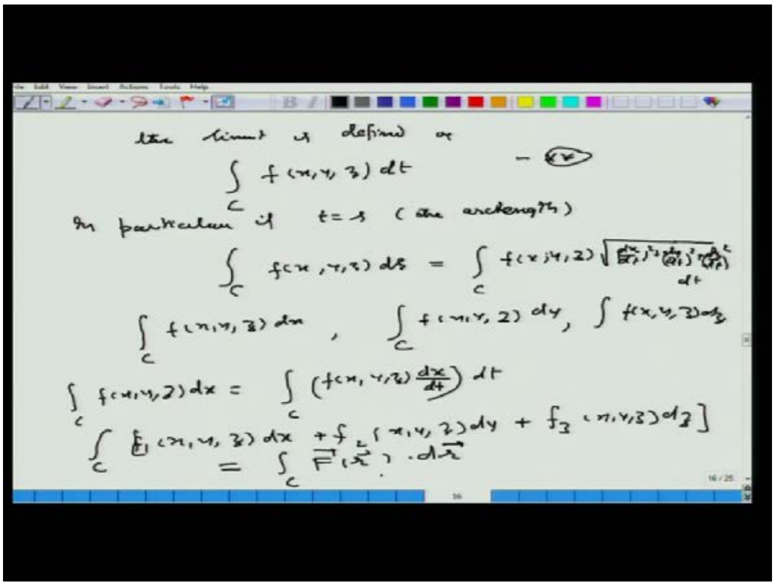

So, the limit is defined as integral over c f of x, y, z dt. So, x, y, z are functions of t. And so, this is defined in the sense of remand as in this case here (Refer Slide Time: 39:18). So, if this limit exists, then we say that this line integral exists (Refer Slide Time: 39:23). In particular, if t equal to s – the arc length, then we have integral over c f x, y, z. Then, x, y, z will be functions of s and we will have d s. And, we can see that this is actually equal to integral over this f x… So, if we change variable here, we will have x t then; and, some other x, y, z.; and, ds can be seen dx by dt square plus dy by dt square plus dz by dt square dt. So, that is the… From one variable to another, from arc length to any general variable, we have this relation here. So, the line integral here (Refer Slide Time: 40:41) for any given curve piecewise smooth curve here such that the function f is having the property that it is piecewise continuous in the sense that on this interval, the function – this f of x t, y t, z t – this is defined as g t. So, this will be a function of t only if this is a piecewise continuous on the interval ab. And, if the curve is piecewise smooth, we can see that this integral is well defined in the sense of remand. So, here any general parameter t is there, if the parametric t is replaced by the arc length as we have this form of the integral.

There are other forms of line integral like (Refer Slide Time: 41:39) f x, y, z dx over c; or, f x, y, z dy over c; f x, y, z dz. So, these are actually particular cases of this, because this can be written like this – f x, y, z dx can be written as f x, y, z dx by dt dt; or, in particular, we can take as here. So, if we treat the parameter as the arc length, then we have everything as a function of s; if we take the general parametric representation, where parameter is t, then we can have this. So, these are nothing but they are particular cases here. So, this f is actually replaced by f x, y t dx by dt here, which is of the form – we have this kind of general form of line integral. So, these all can be treated as particular case of the w star given here. In general, we have this form of line integral; where, we have f 1 x, y, z dx plus f 2 x, y, z dy plus f 3 x, y, z dz. So, this can also be… because each one can be defined like this. And so, each of these integrals are nothing but the sum of their respective… Here f is replaced by those f i's here. And so, this general form can be treated in the same manner. Actually the vector form of this is the following.

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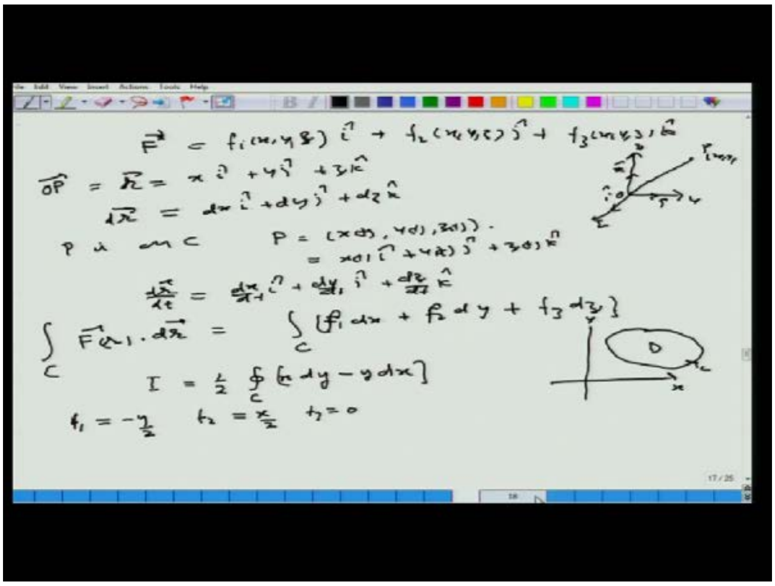

You have F r like this dot dr; where, vector F is f 1 x, y, z I plus f 2 x, y, z j plus f 3 x, y, z k; i j k are the unit vectors along the coordinate axis x i, j, k. So, this F is this vector function of x, y, z given like this. And, position vector r for any point P here is OP. So, this is x, y, z. So, position vector r is given by, which is nothing but vector like this OP – x i plus y j plus z k. So, d r element will be d x i plus d y j plus d z k.

Now, if x, y, z are parameterized by a parameter t along… So, the point x, y, z is lying on a curve, then this r will be a function of t. And so, we can have dr over dt, will be dx over dt i, dy over dt. If P is on c, that means, P is x t, y t, z t – this point; or, this is… Another notation can be x t i plus y t j plus z t k. Then, we can see that this dr by dt – the tangent vector at the point P, the curve will be given by dx by dt i plus dy by dt j plus dz by dt k. And so, this F r dot dr is integral over c will be nothing but f 1 dx plus f 2 dy plus f 3 dz. So, that is what is the line integral, which we considered here (Refer Slide Time: 46:51). So, it is actually the vector form of the line integral written in this manner; this is same as our line integral – the general form of the line integral.

Now, here one particular case of this, which we will be using the following, that is, 2-dimensional case, where I is taken as half of integral over c. So, here this is a plain curve in x y like this. And then, this domain d (Refer Slide Time: 47:35) is enclosed by the c. The curve c in the positive direction is taken counter clockwise. And so, when it is closed, we write the circle here and half of this x dy minus y dx. So, this is a line integral, which is of the form (Refer Slide Time: 48:00) this one, where f 1 is… So, in this (Refer Slide Time: 48:07) case, what f 1 is minus y by 2 and f 2 is x by 2 and f 3 is 0. So, we can see that this I is a particular case of this here (Refer Slide Time: 48:28). Now, this actually gives the area of the domain d enclosed by this curve c, which can be seen using the Green's theorem.

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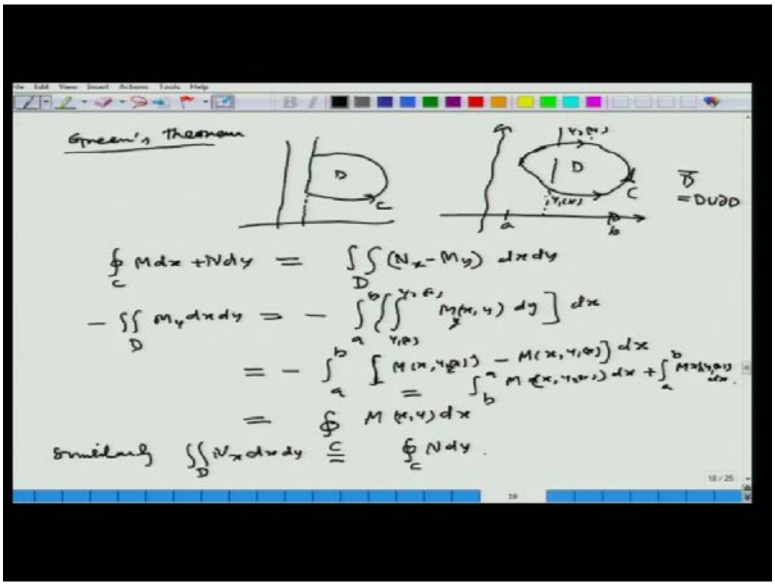

Green's theorem says that if you have any plain curve like this and this is the reason enclosed here, see here we will take a very special case, where any vertical line like this – it cuts at most two points. Like here it cuts only at one point; here it cuts at one point; and, at other points, this vertical line. And similarly, the horizontal line. The lines parallel to axis cut the area d at most at two points; or, situation may be like this also – the whole segment can also come like this vertical line here.

If this D is there and C is the curve and closing this D, this vertical line here should take them the whole segment as common points with the domain. So, we can allow this also. So, in particular, first, we will take… This is the simplest case, where we have this case – that the vertical line takes only at most two points common with the domain D. And, D – a closure. So, D is domain. So, the D closure is actually D union the boundary of D. So, only two points: this and this will be having intersection with this vertical line with this D closure here. So, for such simple areas, we can state the Green's theorem and then extend it to this and more general areas like this. So, Green's theorem says that the integral over this C M dx plus N dy is actually equal to this double integral N x minus M y dx dy.

Here (Refer Slide Time: 51:16) it is assume that the partial derivatives of N and M are continuous in this region, so that this integral will be well-defined here. And, on the boundary, M and N are continuous. So, on D, the partial derivatives exist and they are continuous in D and M and N are on the boundary – they are continuous, so that this line integral is defined; or, you can allow them to have piecewise continuity also, so that the integral as to be well-defined. So, this is what is this statement of Green's theorem. Under these restrictions on N and M, that M and N are here. Inside D, the partial derivative exists and they are continuous. And, on the boundary that means, on C, M and N are piecewise continuous. So, that is what is required. And, under these conditions, we have line integral M dx plus N dy equal to double integral over this domain D of N x minus M y dx dy.

This can be seen easily. Like in this simple case, first we prove it for this. So, let us say this is a here (Refer Slide Time: 52:35) and this b here is the x range like this – maximum range here; and, this a is minimum of this. So, x range. Then, we can define let us say this curve as y 1 x from here to here going like this and y 2 x like this. Then, the integral – this minus M y dx dy can be written as the iterated integral a to b and y 1 x to y 2 x of M y x, y – partial derivative of m – first, dy and then dx. So, here this differentiation integration will cancel each other and will give the values at the end points – M x, y 2 x minus M x y 1 x dx.

Now, here (Refer Slide Time: 54:12) we can see that minus sign is there. And, along this M, is integrated; along this curve y 2, it is integrated. And, because of this minus sign, we will have this direction gets reversed. And similarly, in the second term, here along this y 1, we will have – this minus, minus, will make it plus. And therefore, this will be the direction taken from a to b. So, this is actually nothing but the integral of this M x, y dx, because in the first term, we have minus here. So, direction gets reversed. So, it goes from b to a. So, this is actually equal to – from b to a, if have just this minus sign, it is like this b to a M x, y 2 x dx and then plus a to b M x, y 1 x dx. And so, the first term is going from b to a for the upper one. And so, it gives you this part minus of y 2 and… So, overall, this is the direction we have taken. So, this is the line integral M x, y dx on the close curve C.

Similarly, for this N x, we can do the calculation here, which will give us similarly, double integral N x dx dy over D will give you N dy. Now, this we did for the very simple case of this region D. Now, if we have this kind of situation, we will see that here x is not changing. So, dx will be 0. So, it does not contribute anything to this. And therefore, there will not be any contribution on this curve here. So, one extend this case to the more general situation.

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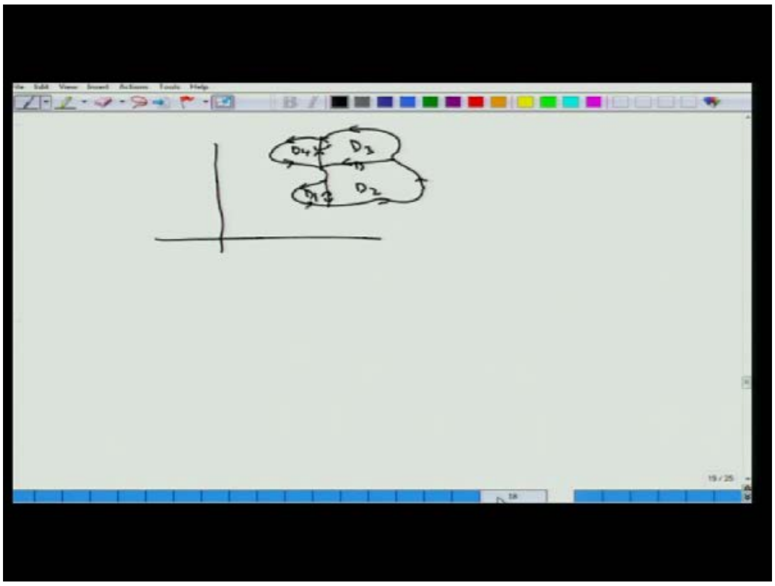

And, supposing you have this kind of region here D. Then, what we will do, we will partition this into this kind of region here and do integration over each part. So, D is partitioned as D 1, D 2, D 3 and D 4 like this. So, here each one we will do integration. And, these artificial boundaries introduced inside, where the integration over them will cancel. Like, when we move like this on this D 1, then on D 2, we move this way and we will go in the negative direction of this. And so, integration over this will cancel. Similarly, we go like this here and, when we integrate in this region, then we go in the negative direction of this. And therefore, the integration over these interior curves will get cancelled. So, here Green's theorem can be extended for more general situation like this. So, using this, we go back to the point here (Refer Slide Time: 58:12). We want to see that this integral I is actually giving you the area dx dy over the region D and closed by this curve C here. So, that is what we can take the Green's theorem here. And so, looking at this I, due to lack of time, we are not able to complete this concept here, which will be done in the next lecture.

Thank you very much for viewing this lecture.