Calculus of Variations and Integral Equation
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Lecture No. # 19

Welcome viewers to the NPTEL lecture series on the calculus of variations. This is the 19th lecture of the series.

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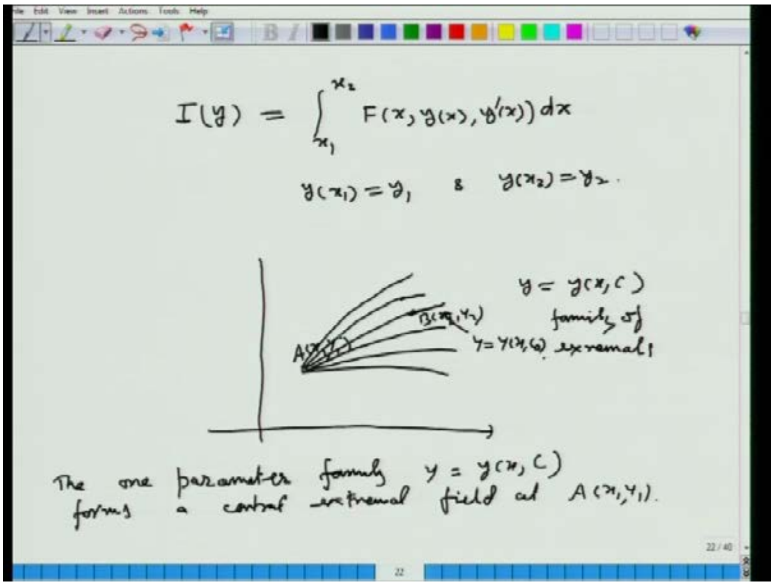

Recall that in the last lecture, we considered a functional of the type I(y) equal to integral x 1 to x 2 F of x y(x) y prime (x) dx, subject to the conditions that the point, the functional to y 2. So, we have the situation here that the point A, which is having coordinates (x 1, y 1) is having a central field here at this point. So, we have the extremals going like this and we assume that the extremals are given as y equal to y of x c. So, this is the family of extremals (no audio from 01:25 to 01:33) forming. So, the one parameter family (no audio from 01:38 to 01:46) y equal to y of x, c, where c is the parameter forms a central extremal field at the point A (x 1, y 1).

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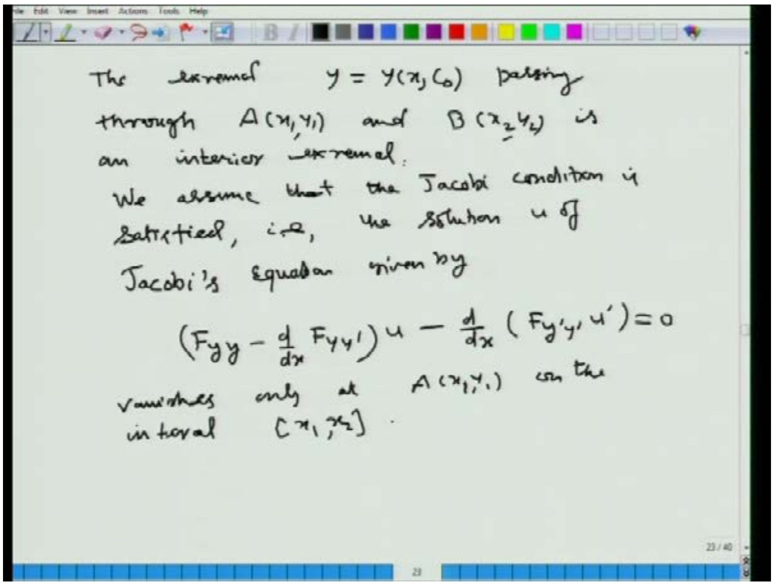

And the extremal y equal to y (x, c 0) passing through A (x 1, y 1) and B (x 2, y 2) is an interior extremal (no audio from 02:47 to 02:55) So that means, here this extremal, which is A, B (x 2, y 2). So, this is the extremal here, which is having the equation y equal to y (x, c 0). So, for the fixed value of c equal to c 0, we have this extremal passing through these two points A and B, and this exremal is not on the boundary. So, it is an interior extremal.

And so, here this family of extremals like this forms a central field at the point A. So, we assume that that the Jacobi condition is satisfied, that is the solution u of the Jacobi equation, Jacobi's equation, given by F y y minus d by dx of F y y prime times u minus d by dx of F y prime y prime u y prime equal to 0, vanishes only at the point A (x 1, y 1) and only at (x 1, y 1) on the interval x 1 to x 2 close. So, that means, at that point B does not vanish.

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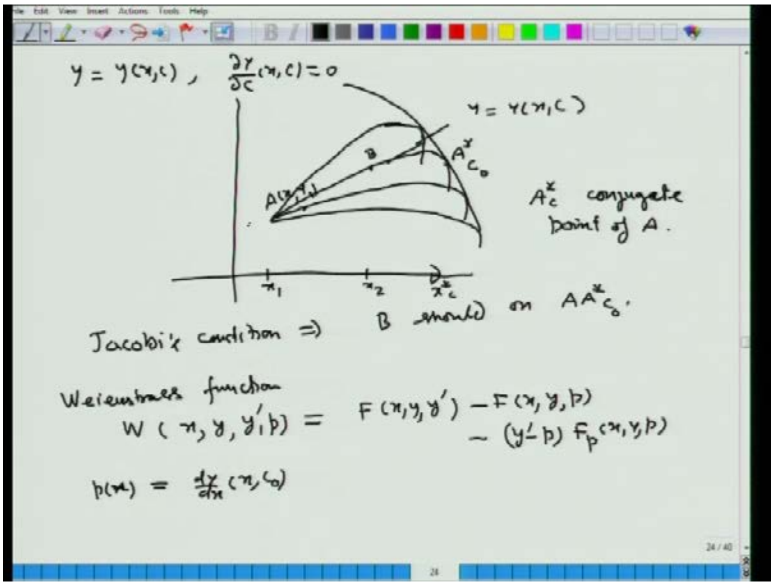

And so, we have like the situation here, that if we consider c discriminant curve, and so we have in addition to this point, there is an envelope like this, so that the extremals are going like this, and touching it tangentially likes this (no audio from 05:47 to 05:54) and so, this point here, so this point here, supposing that this is the extremal A star c. So, for given this this is the extremal y equal to y (x, c). And so, this A star c, which is on on the extremal passing through the point A, and having a intersection having tangential point here on the envelop of this family. The envelop of this family is given as y equal to y (x, c), and del y over del c (x, c) equal to 0.

So, here it is called c discriminant curve and the point certainly is on the c discriminant curve, other than this, there is the envelop of this family, if it exist, then we have the situation that the extremals from this center point of the central field will be having a points (( )) points here on the extremal.

So, the point on any extremal given by y (x, c) is called conjugate point, which is there on the extremal as well as on the envelope. So, that is called conjugate point of A. So, the Jacobi condition is Jacobi's condition implies that B, here this B point should be here. So, B should be on A A star c c 0 here. So, that is what we have for c equal to c 0, we are there on this, so, A star c 0 like this.

So, that is the Jacobi condition means that the Jacobi equation has the solution u, which vanishes only at this point, and then this point A star is c 0. And therefore, it does not vanish before, so, x 2 is here, so this this here on x, so that is let say, x star c here, which is the access of this point A star c 0.

So, x 2 should be such that; it should be prior to this x star c. So, that is the Jacobi condition, which we had considered earlier then we considered the Weierstrass Function, this Weierstrass

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Function W x, y, y prime, p, which we defined as F x, y, y dash minus F x, y, p minus y prime minus p times F p x, y, p, here p is actually p of x is dy by dx (x, c 0).

So, that means, on the extremal, we denote this y prime equal to p, which is the extremal y (x, c 0). So, we want to test whether this extremal satisfy certain conditions, which are called sufficient condition, which will ensure the functional I to have either minimum or maximum. So, we will see that under what conditions, we have the, at maximum value of the functional I, and under what other conditions on the minimum value of the function.

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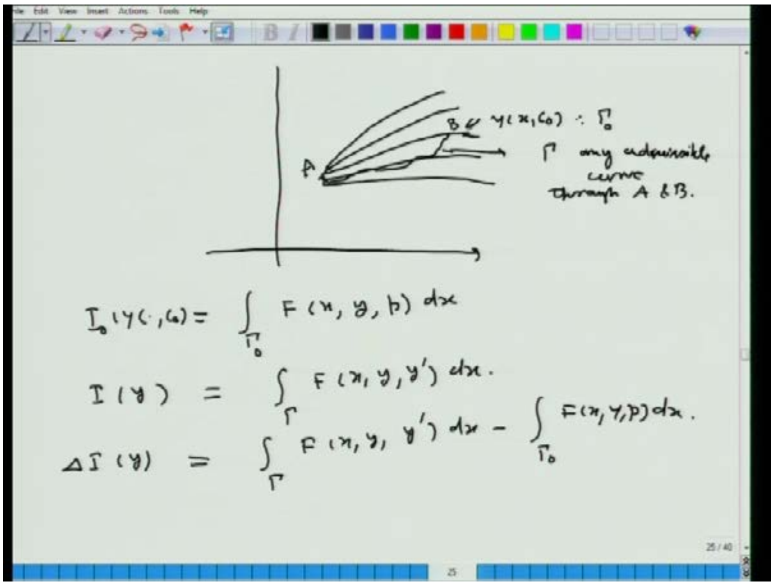

So, we have seen that here, we considered the following situation here, we have this extremals going like this from A, and here we have this B here. And so, this is the extremal y (x, c 0) and we consider any other curve, admissible curve. So, we will call this y (x, c 0) as gamma. So, this we denote as gamma 0 and this curve gamma any admissible curve through A and B.

And then, we consider this I 0 of that is y (c, 0) which is x 1 to x 2. So, we will write that as over gamma 0 F of x, y and here y prime is p. And I (y) any other curve y, admissible curve like this F x, y, and y prime dx, then we consider this delta I (y), which is the difference of these two gamma F of x, y, y dash dx minus gamma 0 F of x, y, p d x.

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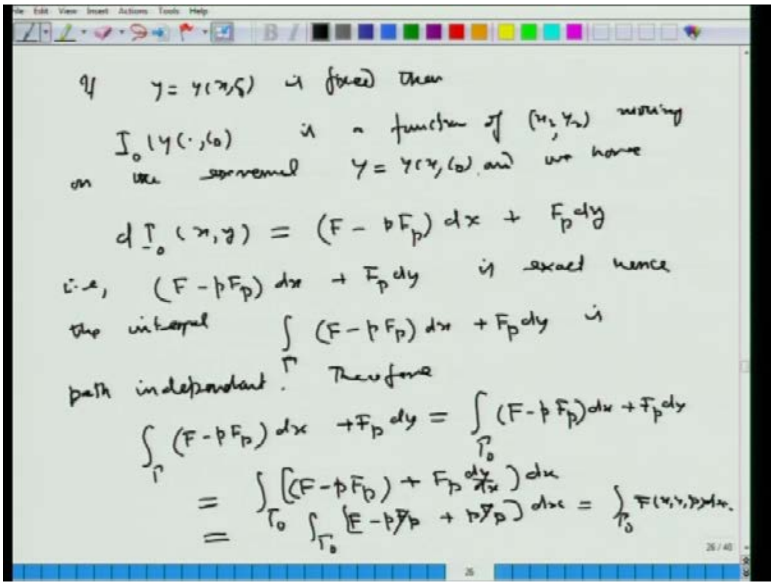

Observing that we have on, if that y equal to y (x, c) is fixed, then this I or other we take one this I 0 is fix then this I 0 (y), which is is c 0 is actually a function of of this (x 2, y 2) moving on the extremal y equal to y (x, c 0). And we have seen and we have this d of I 0 at let say (x 2, y 2), we guide as general (x, y) moving on the extremal then you have seen that this is actually F minus p F p evaluated at general point (x, y) dx plus at p dy.

And so, this is therefore, this is exact function here, that is F minus p F p dx plus F p dy is exact, hence the integral this over x 1 to x 2 F or n on any curve gamma F minus p F p dx plus F p dy is path independent. Therefore, we have seen that this gamma F minus p F p dx plus F p dy is same thing as gamma 0 F minus p F p dx plus F p dy , and then on this, we write this as gamma 0 F minus p F p, and taken this dx is taken out. So, F p then dy by dx, which is p here, which is over gamma 0 F minus p F p plus p F p dx. So, this cancels. So, you get for gamma 0 F which is x, y, p d x.

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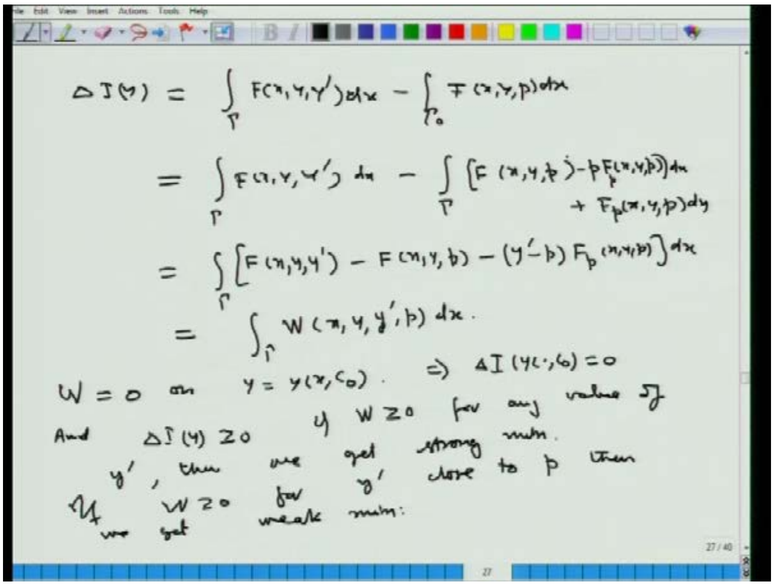

And so, we can therefore, we have seen that this delta I (y) which is the difference in gamma F x, y, p sorry y prime dx minus gamma 0 F x, y, p dx and that using this previous result here, we get this as gamma F x, y, y dash dx minus. Now, we can write from here this is same thing as on gamma. So, on gamma 0 F x, y, p is same thing as this. So, let us call this as 19.1. And so, using that 19.1 this integral, second integral can be written as gamma, integration over gamma F x, y, p minus p F x, y, p dx, here F minus p F p dx F minus p F p dx plus F p x, y, p dy . And so, this we would write as F x, y, y dash taking dx out, and on gamma here dy by dx is y prime. So, we get this F x, y, p minus this y prime minus, minus will make it plus, so minus of this F p x, y, p dx. So, this is what is our, Weierstrass Function x, y, y prime, p d x.

So, we see that this delta I (y), we know that W here, this W is 0 on y (x, c 0), because on this y prime is p. So, you get y prime equal to p and here also y prime is p. So, both the terms will be 0. And delta I (y) will be greater than equal to 0, if W is greater than equal to 0, and so, we will have I I (y). So, that is implies… So, it means that I is delta I (y, c 0) is 0. And in the neighborhood of this there are other extremals or any curve y, admissible curve y, which joins the points A and B, we see that delta I (y) is greater than 0, if W is greater than equal to 0.

And so, we have minimum or let us write it, if W is greater than 0 for any value of y prime, then we get strong minimum, because here, since y this is does not depend on y prime. So, we have zero order proximity, and if if this delta, if delta I (y) will… If W is greater than 0 for y prime close to p then we get weak minimum.

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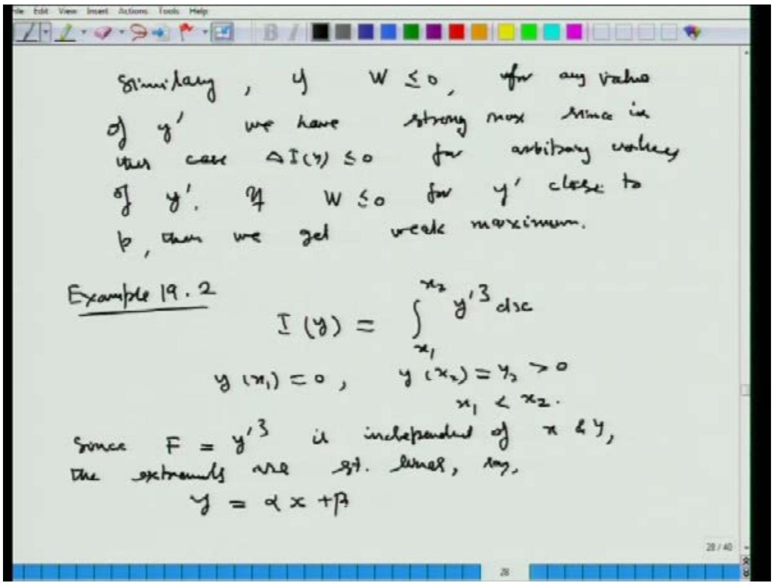

Similarly, (no audio from 21:10 to 21:17) If W is less than equal to 0 for any value of y prime, we have strong maximum, since in this case delta I (y) will be negative for arbitrary values of y prime. If W is less than equal to 0 for y prime close to p, then

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Then we get weak maximum. So, that is what we had considered last time through various examples. And so here, we have to just check, whether this W is positive or negative in the neighborhood of our extremal, which we want to test. So, will consider some examples here.

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So, let us call this 19.2, I (y) equal to x 1 to x 2 y prime cube dx here, y of x 1 equal to 0, y (x 2) equal to y 2. We assume that this y 2 is greater than 0, and this x 1 is less than x 2. So, here since this F, which is y prime cube is independent of x and y, the extremals are straight lines, say, y equal to alpha x plus beta.

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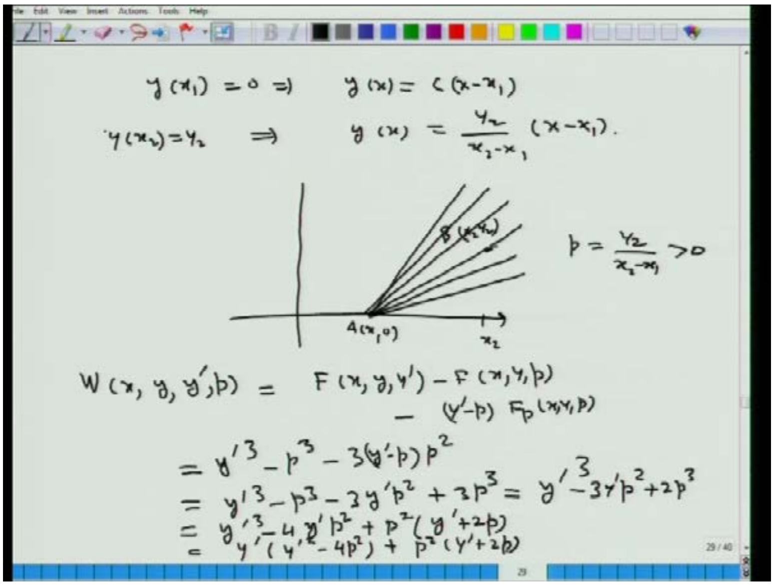

Now, y at x 1 equal 0 implies that y (x) equal to some constant times x minus x 1, and y at x 2 equal to y 2, which is positive. So, this is this will implies that c must be equal to y 2 over x 2 minus x 1; and so, we get these exremals like this. So, these are depicted here. So, this is a point A, which is (x 1, 0) and these are the straight lines. So, this extremal here this is x 2 here, and this is x 2 to y 2, and the slope of this p here, which is y prime on this. So, this is B here, this is B like this.

So, p is y 2 over x 2 minus x 1, which is positive by our assumption that y 2 is positive and x 2 is greater than x 1. So, there are other extremals going like this in the neighborhood of this.

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And so, we check that W x, y, y prime, p which is F of x, y, y prime minus F of x, y, p minus y prime minus p F p (x, y, p). So, this in this case will be y prime cube minus, now y prime we have to substitute p here on the extremal.

So, p cube minus this y prime minus p into F p means 3 p square. So, this is equal to y prime cube minus p cube minus 3 y prime p square plus 3 p cube. So, that is equal y prime cube minus 3 y prime p square plus 2 p cube here, we can see that, we can write it as y prime cube minus 4 y prime p square plus p square taking common in this. So, you get y prime plus 2 p.

And then, here we take, this as y prime, here we taking y prime common. So, you get y prime equal to y prime minus y prime square minus 4 p square plus p square into y prime plus 2 p.

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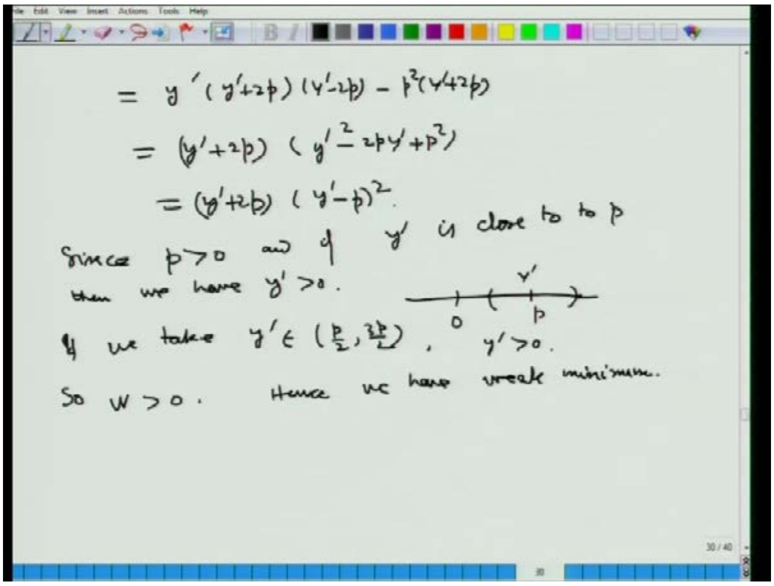

And so, factorizing the second factor y prime plus y prime plus 2 p into y prime minus 2 p minus p square y prime plus 2 p. So, this finally, gives us y prime plus 2 p into y prime minus y prime square minus 2 p y prime y prime square minus 2 p y prime plus p square, that is y prime plus 2 p into y prime minus p square. So, we can see that since this is nonnegative.

And so, since p is positive here, and if y prime is close to p p, then we have y prime positive, such like this you have 0 here, and p is here and supposing we take this small level around this. So, that y prime lies in this. So, than y prime will be positive here.

So, we can see that, that is, if we take, if we take y prime lying in the interval p by 2 to 3 p by 2. So, then we see that than y prime will be positive. So, if y prime is close to p like this. So, that it does not this neighborhood does not take this 0 inside, we see that y prime will be positive. And so, this W will be positive here, hence we have weak minimum.

Since, here the sign of W depends on y prime. So, it is actually in the first order proximity, because y prime will have to be close to p that means, derivatives will have to be… Derivatives of these nearby functions should be close enough. So, that is the first order proximity, which was defined earlier.

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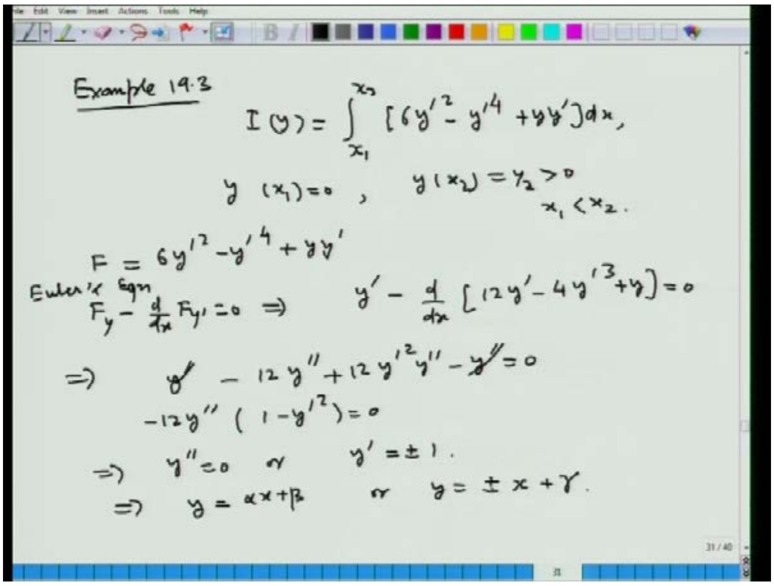

Now, the next example.

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This is 19.3 will write. So, here I (y) is integral x 1 to x 2 6 y prime square minus y prime to the power 4 plus y y prime dx. And the conditions are y of x 1 equal to 0, and y of x 2 to equal to y 2 again we assumed that this is positive and this x 2 is greater than x 1, so, x 1 less than x 2. And so here F is 6 y prime square minus y prime power 4 plus y y prime.

And the extremals are solutions of this F y minus d by dx of F y prime equal to 0 implies that is Euler's equation.

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So, it gives us, F y gives us y prime here and minus d by dx of F y prime means 12 y prime minus 4 y prime cube plus y equal to 0.

So, we get y prime minus 12 y double prime plus 12 y prime square y double prime minus y prime equal to 0. So, we get this cancels here, and so, minus. So, it is should have been F y minus 6 y prime and this… So, we take minus 12 y out, y double prime out. So, we get this minus 12 y double prime into 1 minus y prime square equal to 0. So, this implies that either y double prime equal 0 or y prime equal to plus minus 1, and integrating it again, we get y equal to alpha x plus beta or y equal to plus minus x plus gamma. So, we get straight lines as extremals.

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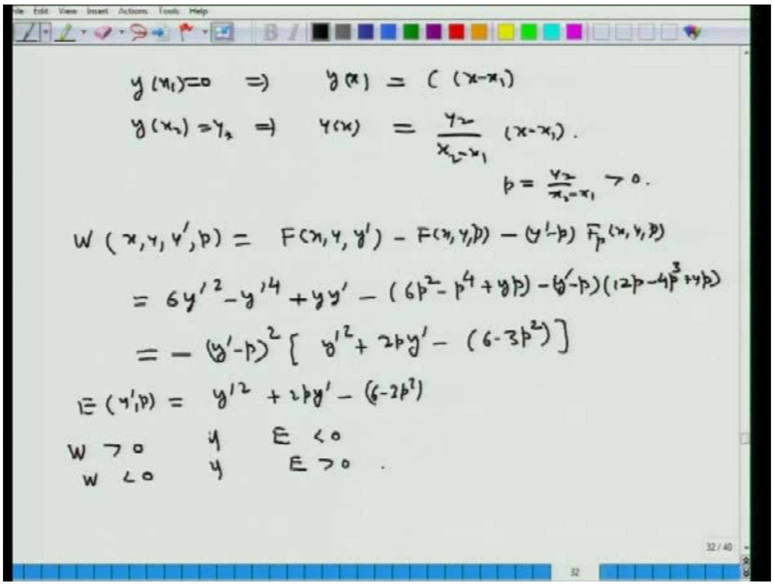

And y at x 1 equal to 0 would imply that y of x must be some C constant times x minus x 1, and y at x 2 equal to y 2 this implies that y (x) equal to y 2 over x 2 minus x 1 to x minus x 1 as we have got earlier. And so here so on this extremal p is y 2 over x 2 minus x 1, which is positive. Now, we look at this sign of this x, y, y prime, p that is the Weierstrass Function, which is defined as F of x, y, y prime minus F x, y, p minus y prime minus p times F p of x, y, p.

And so, it in this case, we get equal to 6 y prime square minus y prime to the power 4 plus y y prime, that is F here F x, y, y prime, and then same thing at y prime replaced by p. So, we get minus 6 p square minus p to the power 4 plus y p, and then minus y prime minus p and F differentiated partially with respect to p, here when y prime is replaced by p. So, we get 12 p minus 4 p cube plus y p.

So, this can be simplified finally, as minus y prime minus p whole square times y prime square plus 2 p y prime minus 6 minus 3 p square. So, if we denote this expression in the bracket as E, which is function of y prime p only, and so that is y prime square plus 2 p y prime minus 6 minus 3 p square.

And so, we have to check this, so, if W W will be positive, if E is negative and W will be negative, if E is positive. So, we have to see that if E is negative, then W is positive in this case, we we will have either strong minimum or weak minimum in the first case. And in the second case, we will have strong maximum or weak maximum depending upon, whether it is dependent on y prime or not.

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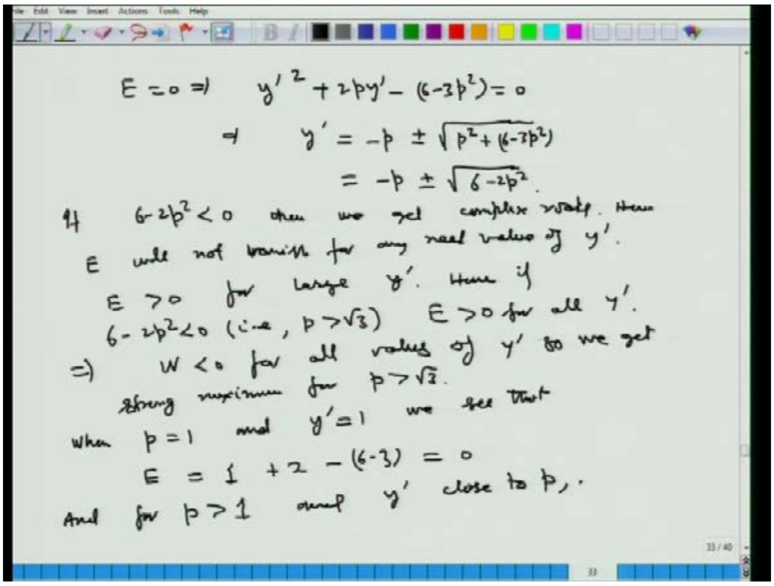

So, we just check the sign of E. So, E equal to 0 implies that this y prime square plus 2 p y prime minus 6 minus 3 p square equal to 0. So, the roots of this equation are y prime equal to minus p plus minus square root p square plus 6 minus 3 p square, that is minus p plus minus root 6 minus 2 p square.

So, we check the various cases, when 6 minus 2 p square is positive, when it is 0 and when it is negative. So, first case when if 6 minus 2 p square is less than 0, then we get complex roots.

Hence, E will not vanish on this line. So, here we have like this p and this is E here. So, on the p axis, here we E will not vanish on the... So, there is no point here on p axis, where E will be 0, will not vanish for any real p. So, we can remove this thing. So, here and this E is positive for large y prime.

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Any real value of any for real value of y prime, for large y prime is is positive, because you can see that y prime is the dominating term here. And so, if y y prime is… y prime square is the dominating term here, and if y prime is large, y prime square will decide the sign of E.

And so, for large values of y prime E will be positive. Hence, if p is that means, 6 minus 2 p square negative that is p greater than root 3 here, p will be positive, for all y prime. And so, this will imply that W is negative for all y prime values of y prime. So, we get strong maximum for p greater than root 3. So, that is the first case, and when…

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So, here we see that, if p is and p is 1 and y prime is also 1, we see that E equal to 1 plus 2. So, we substitute p and y prime equal to 1 minus 6 minus 3. So that is 0, so and for p greater than 1 and y prime close to p.

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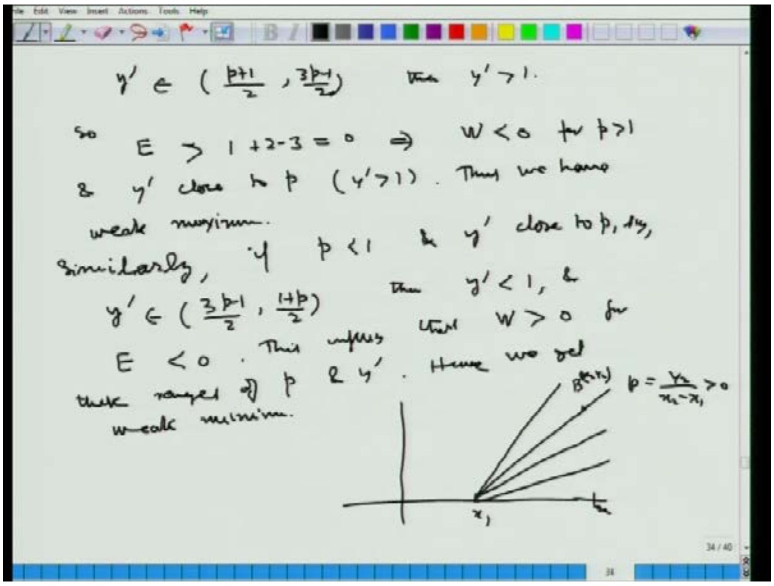

We take this intervals that y prime is in p plus 1 by 2 to 3 p minus 1 by 2, then y prime will be greater than 1. And so, E will be greater than E will be greater than 1 plus 2 minus 3 and that is equal to 0. And this will imply that W is less than 0 for p greater than 1, and y prime close to p that means, y prime y prime also greater than 1.Thus, we have weak maximum here.

Similarly, similarly, if p is less than 1 and y prime close to p, say, y prime lying in the interval 3 p minus 1 by 2 to 1 plus p by 2, then y prime will be less than 1, and then E will be negative. And this implies that W is positive for these ranges of p and y prime (( )). Hence, we get weak minimum.

So, here we have the following picture that we have here x 1, and then here we have x 2, and these are the extremals going here. And let say, this is the point here lying on this extremal p. which is (x 2, y 2) having p equal to y 2 over x 2 minus x 1 positive, and here we see that if in this range, if p is greater than 1.

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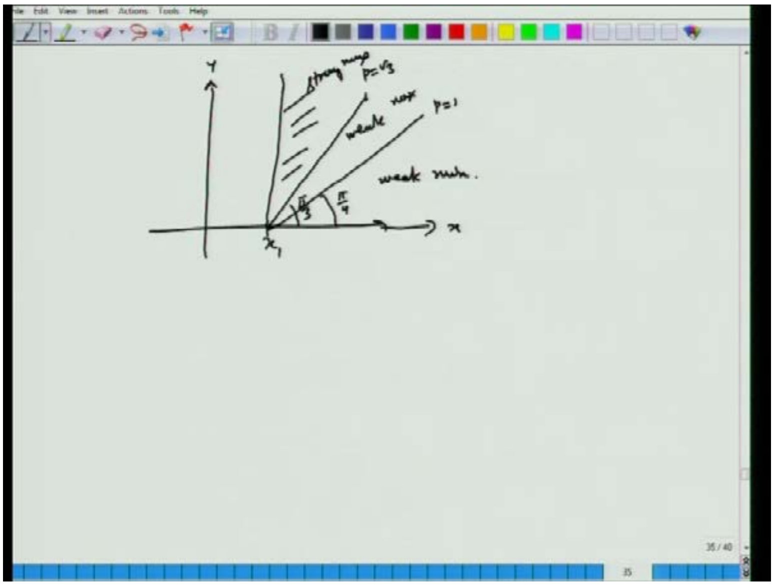

So, that is we have this situation that pi by 3, and then we have pi by 4. So, this corresponds to p equal to root 3, this corresponds p equal to 1, and so here and p is greater than, the slope is greater than root 3. So, here here we get strong maximum, here we get weak maximum, and here we get weak minimum in this range. So, that is x and y axis here and this is the point x 1. So, in this case, we have seen that the depending upon, what are the slope p, and in the neighborhood of this if y prime has close value close to p then we can decide that the functional, whether it will have strong maximum or weak maximum or weak minimum, within those ranges of p and y prime.

Next, in the next lecture, we will be considering various cases of these functional and we will see, we will get other condition, which is known as Lysander condition, which is similar to check. Here, finding that W is positive or negative is a difficult, if we straight away apply the definition of W, you see that we will (( )) use the Taylor series expansion of W. And we will be able to get a nice condition, easy to verify, which is known as Lysander condition that we will consider in next lecture. Thank you very much.