Calculus of Variations and Integral Equations

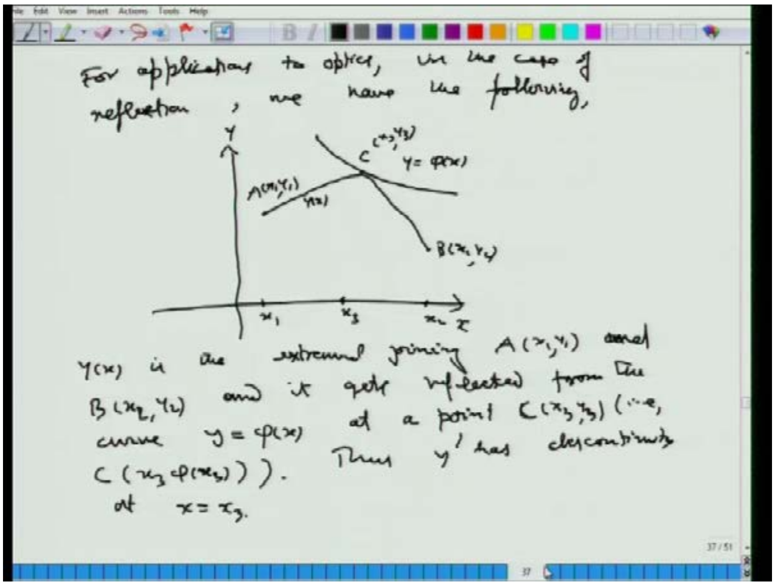
Prof. Dhirendra Bahuguna
Department of Mathematics and Statics
Indian Institute of Technology, Kanpur

Module No. # 01

Lecture No. # 17

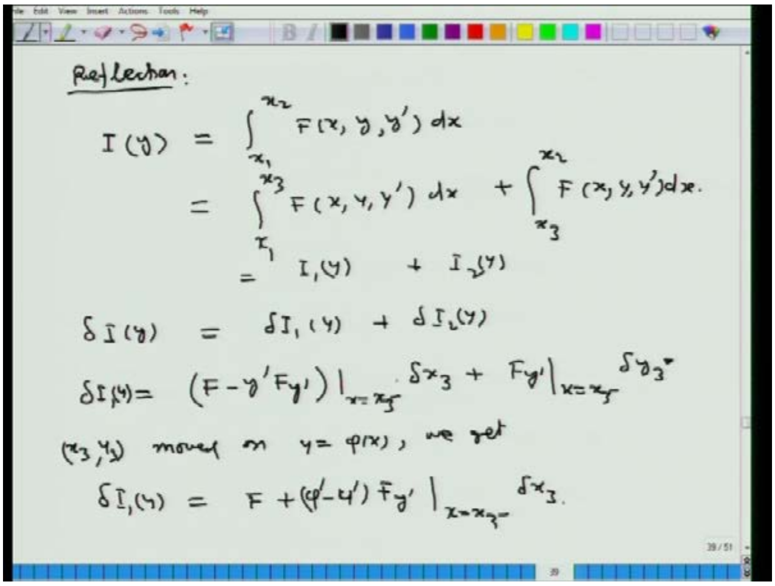
Welcome viewers to the NPTEL lecture series on the Calculus of Variations. This is the 17th lecture of the series; in the last lecture that is in 16 lectures, we allowed discontinuities to occur in the derivative of extremals. So, that we can apply these results to problems appearing in a optics, where we considered reflection from a given function y equal to phi x. The functional, the extremal got reflected at a point C, this extremal joins two points A and B, which are given and C is a point moving on the function y equal to phi x and the extremal is getting reflected at C. So, it is starts from A and it gets reflected from C and joins the point boundary point B. So here, we are supposed to get that point C, so that the functional I gets optimized. So here, also we considered the problem of refraction.

(Refer Slide Time: 01:32)



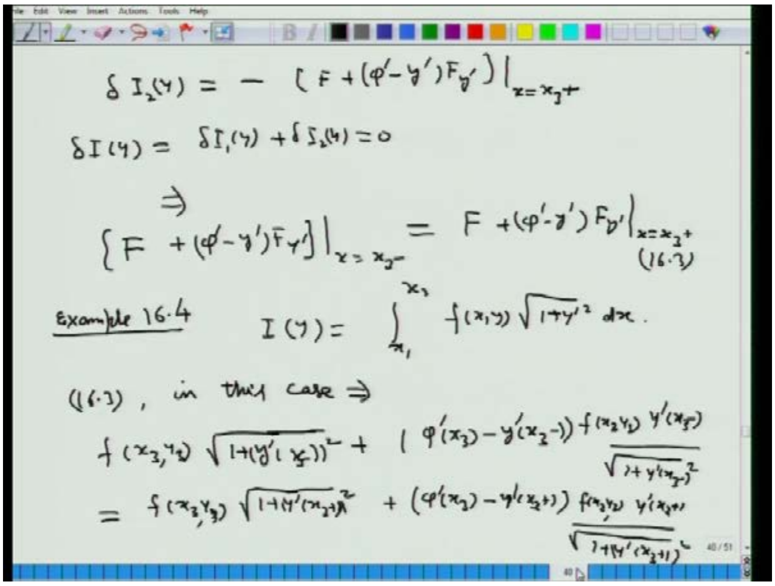
So, here in the first case, we considered this picture here. So, this is a point A and the point B, these are given points here and C is a point, which is moving on the curve y equal to phi x. And so, this extremal which joins these two given points A and B gets reflected at C. And so, here although y in the extremal y is continuous but it is derivative is discontinuous at C, and so here in the first case reflection case both A and B are on the same side of the curve y equal to phi x.

(Refer Slide Time: 02:29)



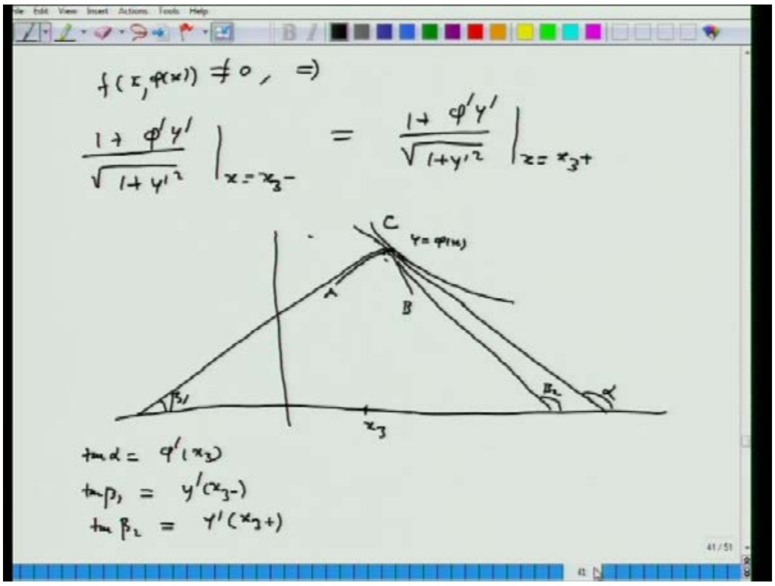
And so, we use the conditions here, this delta I which is some of these variations delta I 1 plus delta I 2, we break the integral x 1 to x 2 at x 3 which is the point x 3 y 3. And so, we get this delta I, evaluated at point x 3 minus that is a left limit and x 3 plus those in the 2 values are then equated.

(Refer Slide Time: 02:45)



And that is the condition 16.3, which we have which we had got in the last lecture. And then we considered this example 16.4, where this f (x,y) is non 0 on the given function of y phi y equal to phi x.

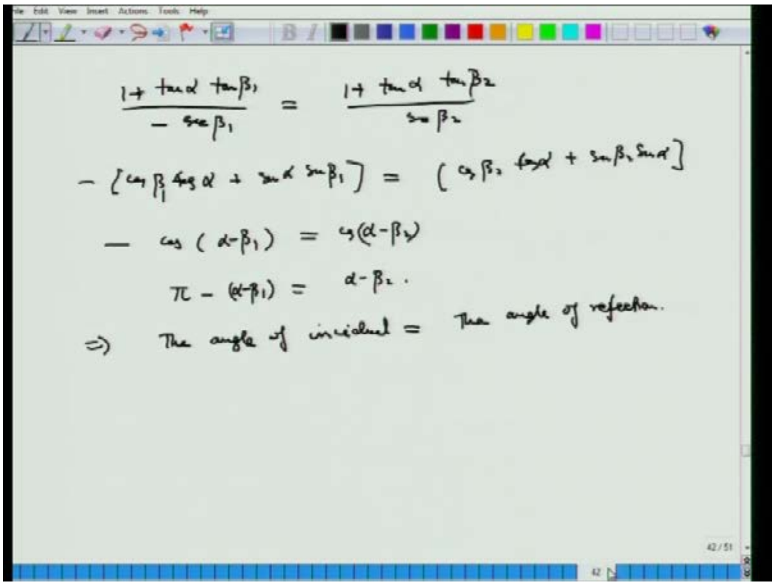
(Refer Slide Time: 03:06)



And So, we got the condition here that this 1 plus phi prime y prime over a square root 1 plus y prime square at x equal to x 3 minus; that means, the left limit and same thing at the point x 3 in the right limit.

And so, here we denote these tangent at tangent to the curve y equal to phi x at C makes the angle alpha and tangents at the extremals at the point C, because those at point at the point C, we will have two tangents that is why the derivative is discontinuous at c. So, denoting those left tangent and right tangent as tan beta 1 and tan beta 2, so that the angles made by those tangents at the on the x axis as beta 1 and beta 2. So, we denote them like this tan alpha is phi prime at x 3, phi is assumed to be smooth, so that derivative at x 3 exists and a tan beta 1 is y prime at x 3 minus and tan beta 2 is y prime at x 3 plus.

(Refer Slide Time: 04:25)



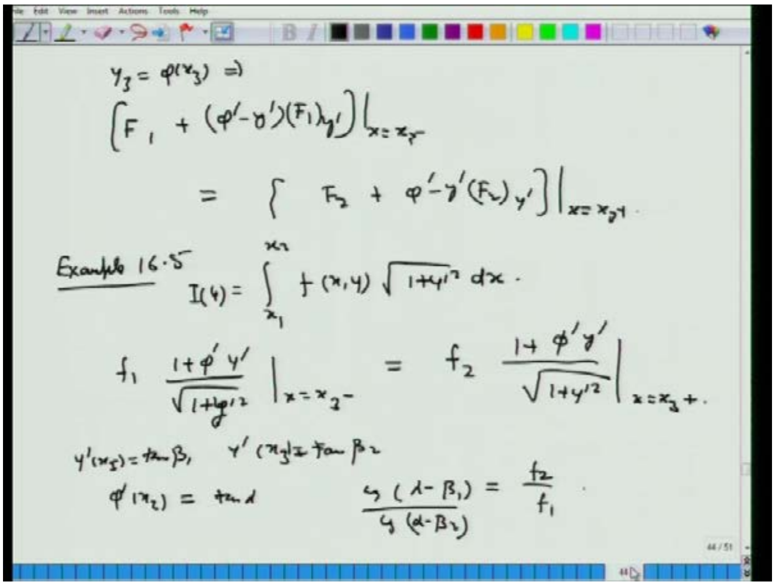
So, we got this condition here since these two angles are here beta 1 is like this and beta 2 is like this (Refer Slide Time: 04:24). So, we get here minus sign in this sec beta 1 here and then multiplying by cos beta 1 and the left side and cos beta 2 the right hand side, we get here, this minus cos of alpha minus beta 1 equal to cos of alpha minus beta 2. So, this is phi minus alpha minus beta 1 equal to alpha minus beta 2.

So, these are the angles at the point here and so, the incident angle, because alpha minus beta 2 is this angles, which in made to the tangent by the tangent on the curve by the tangent on the extremal. Similarly, you have alpha minus beta 2 give this angles, so phi minus alpha minus beta 1 will be the other angle, which is the incident angle. So, we get here the famous law of reflection that is the angle at the incident is same thing as the angle of the angle of the incident is equal to angle of the reflection.

In the case we have refraction, so then the figure is explained here, in the case of refraction the point A is on one side and the point B is on the other side of the function y equal to phi x. And so, here we have a different mediums the light rays travelling from one medium to another and obviously, then the function the integrant will be discontinuous at C.

So, we take two different functions f 1 and f 2 there, so is integrant and although y may also have discontinuities of it is derivative at c and f also has discontinuities at the point C. So, we break the this integral again in 2 parts x 1 to x 3 and x 3 to x 2 here, first we have f 1 and the second we have f 2 and then proceeding the same manner we get here delta I(y)(Refer Slide Time: 06:44) Finally, we get that delta I(y)that should be equated to 0 the first variation must be 0 and therefore, it leads to the condition that f plus phi prime minus y prime f y prime evaluated at x 3 minus that is the left limit is equal to the same value of the same thing at x 3 plus, so that is what was obtained in the last lecture as 16.3.

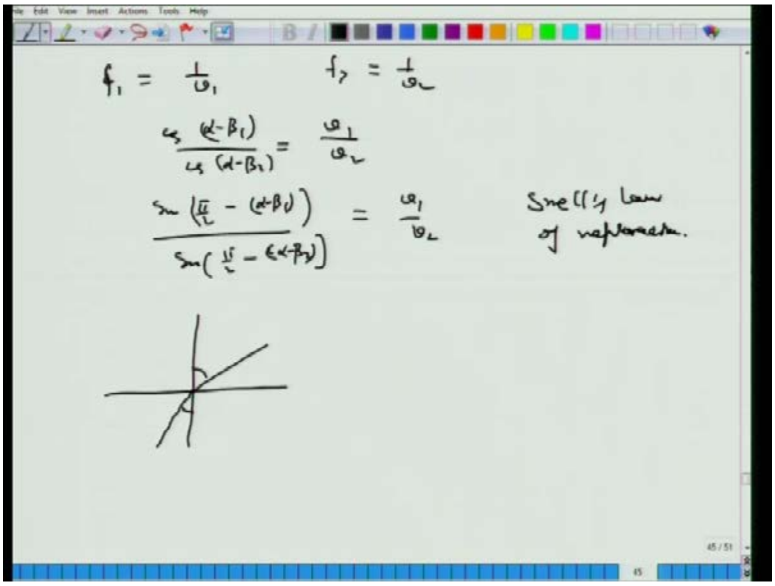
(Refer Slide Time: 07:20)



And. So, here if I(y)is taken as x 1 to x 2 to f x y square root of 1 plus y prime square d x then, we will have two different functions f 1 and f 2 there. And so, we get here in the case refraction, we get two different functions f 1 and f 2 there and so, we get this condition here. And so, in this example we have f has f 1 on the left side of the curve and f 2 on the right side of the curve y equal to phi x.

So, we get this condition here and denoting those angles as we had noted earlier. So, we get here that cosine of, now actually beta 1 and beta 2 both are on the same side in the refraction case (Refer Slide Time: 07:59). Here this tangent to this will come here like this that is beta 1 and like this it will come somewhere here beta 2, so they are on the same side here and alpha is like this and so, we get both the signs will be same here.

(Refer Slide Time: 08:45)

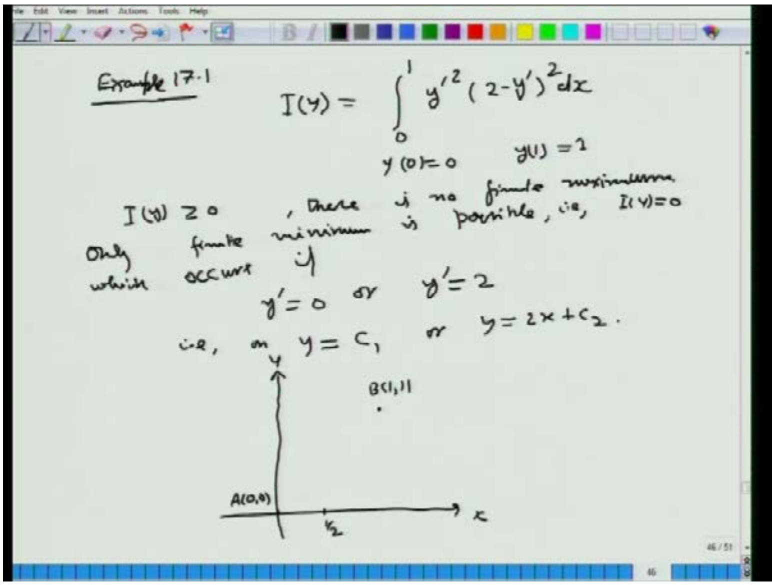


And we get in this case cos of alpha minus beta 1 over cos of alpha minus beta 2 and if these functions f 1, f 2 are 1 over v 1and 1 over v 2 respectively, where these v 1and v 2 are the velocities of the light ray in those two different mediums. Then we get this thing and since for in the law of refraction the angles are measured from the vertical line like this. So, if angles are measured, we are measuring angle at the tangent whereas, it we need to measure those angles from the vertical 1 and therefore, we take phi by 2 minus of these these angles.

So, we get sin of phi by 2 minus alpha minus beta 1 upon sign of phi by 2 minus alpha minus beta 2 equal to v 1upon v 2. And so, what it says that the signs of the angles the ratio of the sins of the angles of at the point of refraction, we get the ratios of those this ratios of the sins of those angles must be equal to the ratio of the velocities of the particle in the different mediums.

And so, this is what is known as famous law of Snell's law of refraction, that it what is obtained in this analysis. Now here not only the discontinuities in the derivative of an extremely can occur only in reflection or refraction that can instances, where these may occur in other instances also like if you consider this example.

(Refer Slide Time: 10:07)



So, let us call it 17.1 now, so this is I (y) equal to integral 0 to 1 y prime square 2 minus y prime whole square d x and here this y 0 equal to 0 and y 1 equal to 1. So, here situation is; obviously, I(y)is greater than equal to 0 and there is no there is no finite maximum maximum is no finite maximum because y prime can take as large value as possible. And so, only finite minimum is possible, so here extremal value is actually minimal value and that is I(y)equal to 0 that is I(y)equal to 0 and which occurs if y prime equal to 0 or y prime equal to 2.

So, that is, so that is on y equal to C 1 or on y equal to 2 x plus C 2 and so here, what we have is the following, so x y axis and here this point A is 0,0 and b is 1 1 and so, here and so, this is half. So, here of course, y equal to x is a line we joining here which is the straight line here, because this functional the functional here the extremes are straight lines. So, we have a line joining these two points that is y equal to x, but on that y prime is not 0 nor y prime equal to 2, so this I(y)will not be 0 for that.

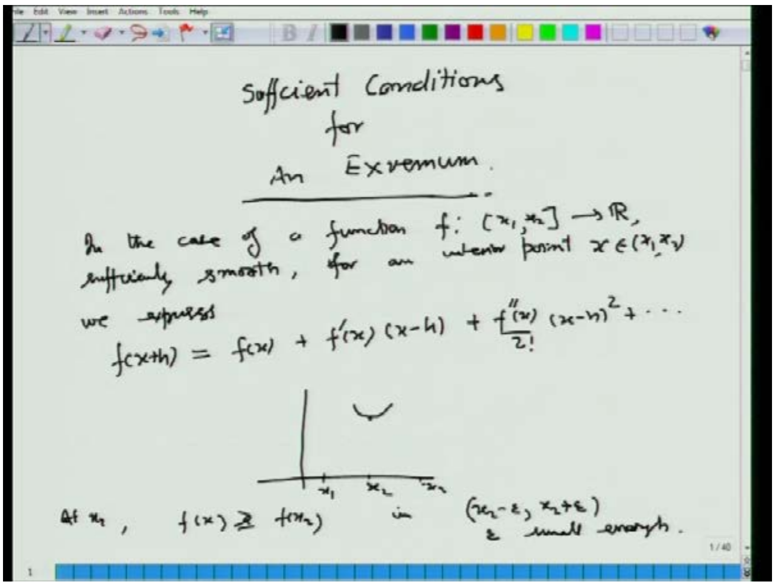
So, here what should happen that we should allow only y equal to some constant that is horizontal line and then y prime equal to 2, so that we have this line like this and so, we can go like this or we can move like this and then go up like this or we can go to any level and move like this (Refer Slide Time: 13:25). So, like this we can move or we can go like this or we can go like this, but these extremists will have then a corners appearing here, here and all these points corners will appear and so, y prime will be discontinuous.

So, these are instants this an example where this functional attains minimum value only on the extremals, which are having discontinuities in the derivatives, that is they have corners in the domain. So, I mean here these such functions can occur in practice that where the extremals will they will be continuous but, the derivatives will be discontinuous; so they will have corners at a certain number of points.

So, like this we have that naturally these these kind of functions occur in practical problems, now we will move onto the cases where we should have certain conditions derived, so that it is ensured that if those conditions are satisfied. Then the extremals where we have the value of first variation equal to 0, so on that extremals the it is a candidate for optimizing the functional but, whether it minimizes or maximizes that is to be checked like we check it in case of functions the derivatives, higher order derivatives.

And check whether the higher order double derivative is positive then we have minimum if double derivative is negative than second derivative is negative then we have maximum.

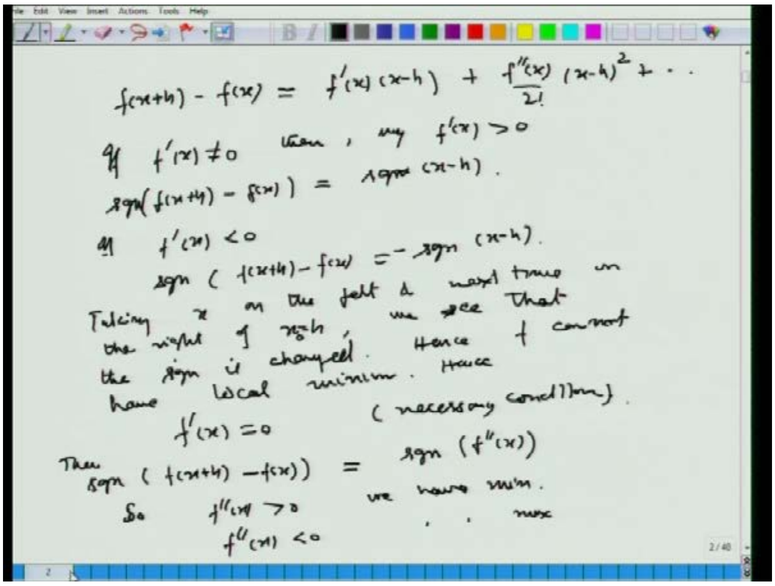
(Refer Slide Time: 15:20)



So, like you will recall that, so will actually now consider sufficient conditions for an extremum. So, here in the case of function the function f from x 1 to x 2 into R and sufficiently smooth then we see that at any point, interior point for an interior point x in x 1 to x 2, we express f of x plus h. But Taylor series is f of x plus f prime at x minus h plus at double prime at x on factor 2 x minus h square and so on.

So, here we see that if it it has maximum, let us say the case when this x 1 is here x 2 is here and x is here, so suppose that the function has minimum, so it should go like this and therefore, for minimum. So, a here f x can be expressed as the Taylor series if it is assumed to be sufficiently smooth and equal to f x plus a prime x minus h f double prime x upon factor 2 x minus h square and so on. And then we see that at this point x 2 at x 2 f of x must be less than must be greater than equal to f at x 2 in certain in x 2 minus epsilon to x 2 plus epsilon epslon is sufficiently small all enough.

(Refer Slide Time: 17:49)

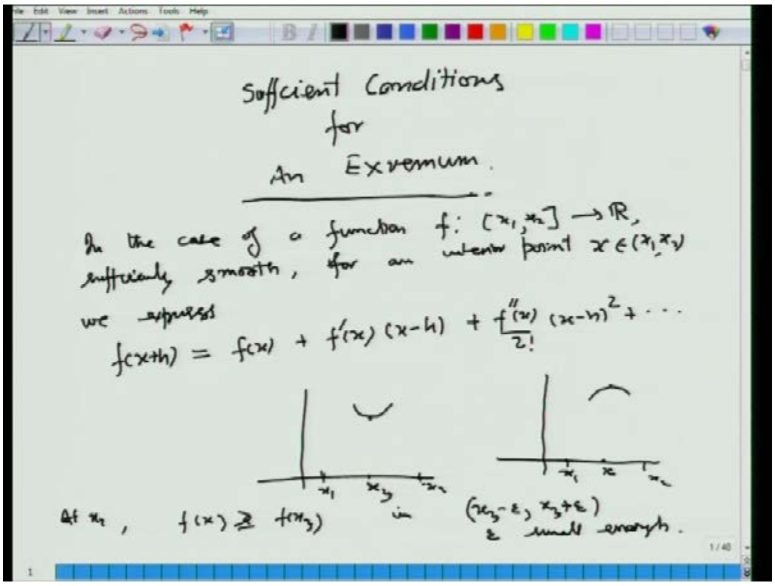


And so, and so, we see that the f of x plus h minus f of x equal to actually f prime at x to x minus h plus f double prime x upon factor 2 x minus h square and so on. Now, if if a prime x is not 0 then f of x plus h plus, then the sine of this is same thing as sine of this term. So, if this and say f prime at x is positive then the sign of this is same thing as sign of x minus h, if a prime at x is negative then the sign of f of x plus h minus f of x equal to sine of minus sign of x minus h.

Now if you take x to be in the left and the next time you take x in the right of h, then we see that the sign changes. And therefore, this in this case, so taking x on the left and next time on the right of x equal to h at right of x 0 equal to x h, we see that this sign is changed; hence f cannot have local minimum in this case.

So, therefore, hence a prime s must be 0 necessarily, so this is a necessary condition, so likewise in our case in the functional case, we saw that the first variation. So, here the derivatives like first derivative is gives like our first or first variation gives like first derivative in the case of function. And for the now here in this case, so necessary condition if this is satisfied then the sine of, then the sine of f of x plus h minus f of x is then equal to sign of f prime f double prime x.

(Refer Slide Time: 21:33)



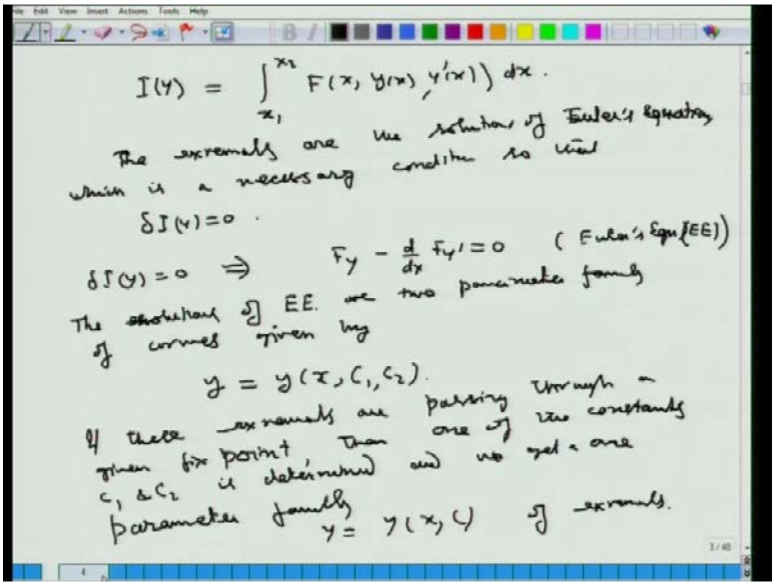
So, if f double prime is positive then we have minimum. So, if f double prime (x) is positive we have minimum, if f double prime (x) is negative then we have maximum, because in this case then you should have the situation like this. In the case of maximum the function should be like this at x 1, x that is why this is x 3 this is x 3. So, x 3 minus x 1 x 2 for the interval boundary points of the interval and x 3 is an interior point, so that is what should happen here.

And the so, in the case of a minimum you should have a double prime at x positive and f double prime at here we should actually have this, let us write it as that is x 3 here.

And so, in this case if f double prime is also 0, then we see that we have to go to higher derivatives to check the sign of f of x plus h minus f of x. So, in the same manner you for the functional will have to go to higher order variations in this case.

Now, also we have seen that (Refer Slide Time: 23:06) we first solve this f prime equal to 0 and the solution of this equation those roots of this equations are the points, which are called critical point. Because the at those interior points only minimum or maximum can occur. And therefore, of course, at the leaving the boundary points, because we are taking only the interior points here.

(Refer Slide Time: 23:31)

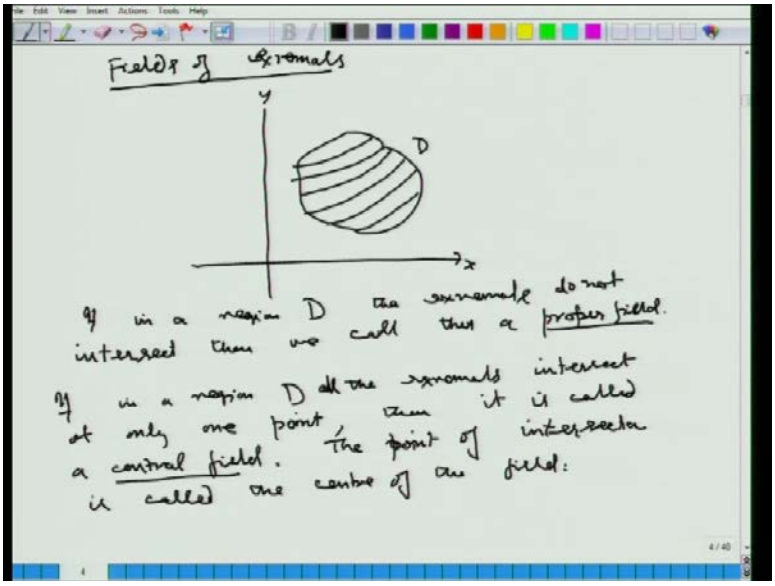


So, same manner we for the functional if we consider I (y) equal to x 1 to x 2 f of x, y (x), and y prime x. So, we see that like the points here of this equation, the extremals of the equation should be. So, the extremals are are the solutions of Euler's equation, which is obtained which is a necessary condition which is a necessary condition condition, so that this first variation delta I(y)equal to 0.

So, this so this delta I(y)equal to 0, we have got that this implies that f y minus d by d x of f y prime equal to 0 that is our Euler's question. And so, here the solutions the solution of these Euler's equation, let us call it E E in short we will called it E E or two parameter family of curves given by y equal to y (x,c 1,c 2).

Now, here what we do, we assume that this point one of the point let us say is fixed with these extremals are passing through a given point in fix point. Then one of the constants the this constant c 1 and c 2 is determine and we get one parameter. Finally y equal to y (x,c) of extremals. Now we consider various notions of fields of extremals.

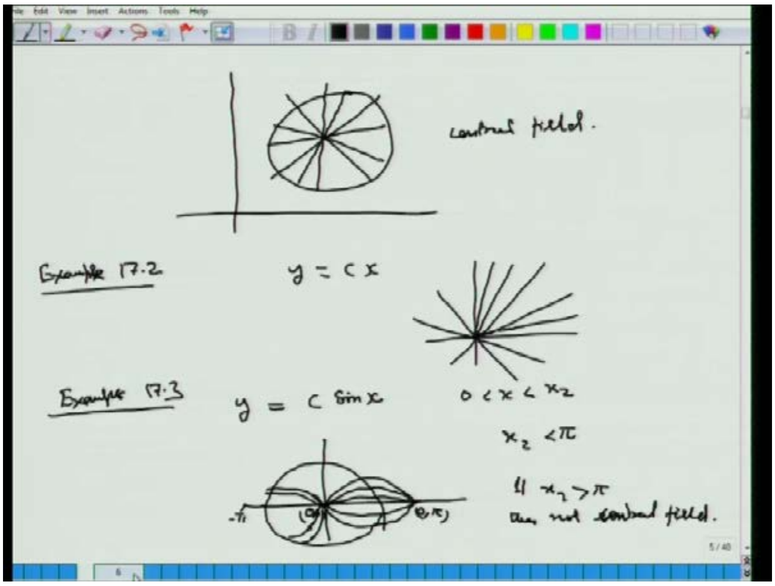
(Refer Slide Time: 27:21)



So, fields of extremals (No audio from 27:20 to 27:32), so we say that if certain domain D like these extremals are occurring like this and this is domain D in if in a region D the extremals do not intersect then, we call this an a proper field.

So, if there is a point two extremals cross then such a field will not be a proper field. So, this is the definition of proper field, if in a region D the extremals intersect all the extremals all the extremals intersect at only one point, then it is called then it is not a proper field but, it is called a central field. And the point of intersection is called the centre of the field like for example.

(Refer Slide Time: 29:54)



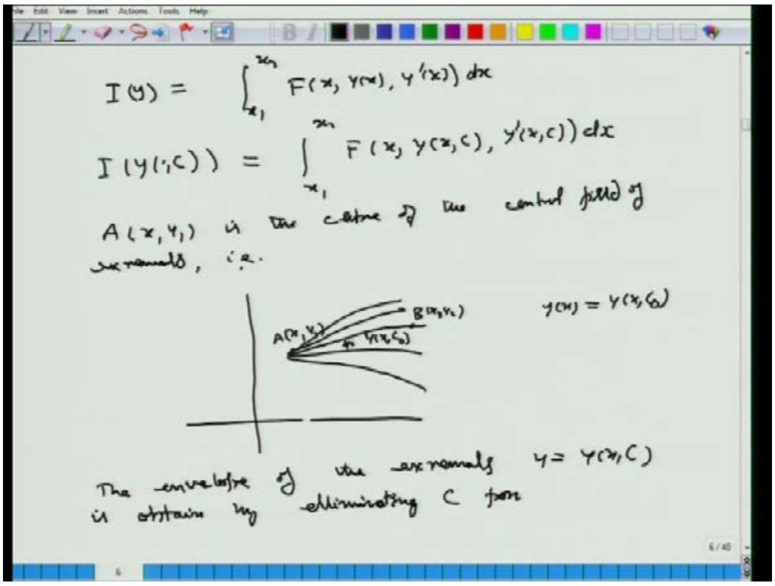
Here it could be like straight lines passing through this point. So, they form of course, this whole region D must be covered by the extremals and so this is a central field. So, for example, the practical examples if you take this, y equal to let say for example, y equal to c x, this 17.2.

So, y equal to c x, so you have this from this origin all these straight lines going to any region containing this so, any region containing the whole this is called a central field, so this is an example central field. And if you take y equal to c sin x and then we see that here, this also when c equal to 0, we get let say this is 0 and here it is 0 pi, so all this things will be there.

So, if we, so in that field we should avoid this pi 0 pi, so any region containing this one, so like that on the left hand side also you will have this going there. So, any region like this, where we avoid this pi and minus pi, we will have the central field and 0 is going to be its centre. But, if a pie is also included in that then it is neither a central field nor a proper field.

So, here we will have like this 0 less than x less than some number x 2, so this x 2 must be less than pi, if if x 2 is greater than pi, then not a central field. So, like this we need the notions of these proper fields and central fields. Now we consider the family of these extremals is like this.

(Refer Slide Time: 33:09)



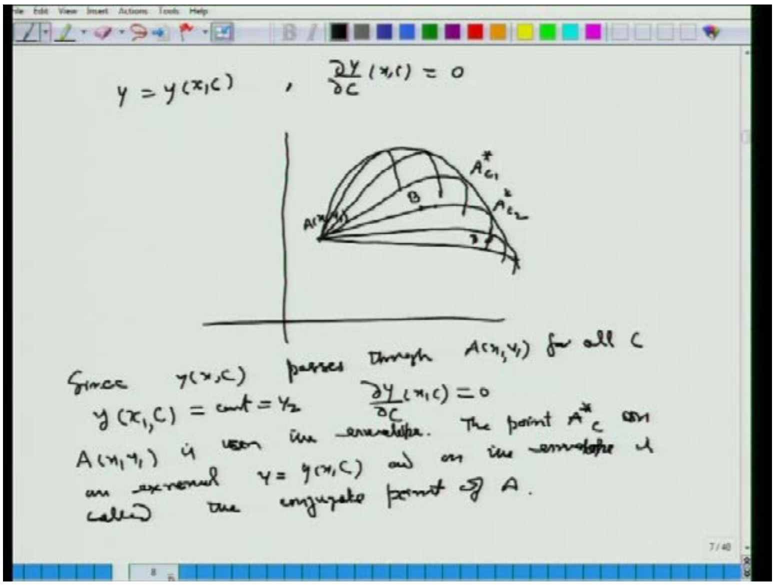
So we have this I(y)which is x 1 to x 2 f of x y x and y prime x d x. So, we consider here this I(y)on extremals only as before we considered only on the extremals, because; and this extremals should be an interior extremals like in this case this one it should not be boundary extremals. Like we have in the case of function the critical points are all those critical points are interior points not boundary points, boundary points minimum and maximum can occur in other ways also.

So, we consider that are functional on the extremals only and here the point A assume that this point A x 1 y 1 is the centre of the central field of extremals, that is we have this picture, that the point a this is x 1 y 1. And so, from here various extremals are passing. And so, this extremals which passes through this point in A and B that is x 2 y 2 this forms a central field and this extremals that is y x c we will call this given extremals as y x y x equal to so, this y x equal to y x c 0.

So, fixed value given to the constant c to obtain this extremals and when we change the c we get all the different extremals and we assume that this is a central field and the between a point a is the centre of this field. Now, we need to see that, the necessary sufficient conditions are to be satisfied we will obtain from this, so that the external is, so that the functional is extremized on this functional and you know, whether it is a minimum or maximum value if that sufficient condition is satisfied.

Now, we can consider here the envelope of this family of externals, so here if at all it has an envelope the envelop of the externals y equal to y x c is obtained by eliminating c.

(Refer Slide Time: 36:54)



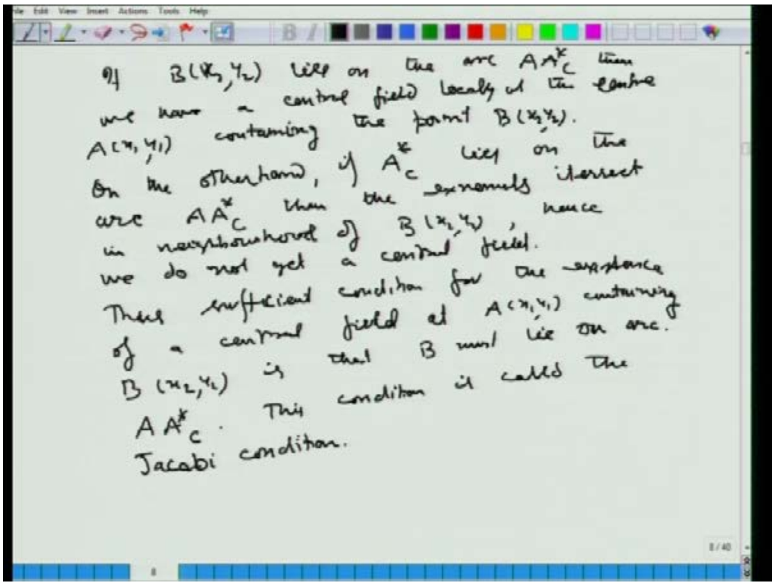
From these two questions that y x c is and equal to 0 and del y by, so eliminating by y sorry this y equal to y x c and this del y by del c at x c this should be 0. So, eliminating c from this the curve we get the envelope like this we have the following situation here this A, which is x 1 y 1. And here like this A will also be on because, we know that here at A see A is passing through all the extremals so; that means, a x 1 c 0 is the this point. And so, here we c that del y by del c at x 1 because at this, since y x c passes passes through A x 1 y 1 for all c; that means, this y x 1 c is constant, that is y 2 that is equal to y 2. And therefore, here you see that del y by del c at x 1 c will be then 0 and so, this A x 1 y 1 is the is on the envelop.

Now, other points will be obtain like this you have this extremal which is getting gets reflected here like because, this envelope should go tangentially to these members. So, this the envelopes obtain here in this manner. And it could be here or it can be on the other side here and the point here this point is called like is called conjugate point of A, so c 1 and this will be a star of c 2 and so, on.

So, the point the point A star c is on an extremal y equal to y x c and on the envelope is called the conjugate point conjugate point of A like this.

So, here we see that if we are considering central field. So, if the point B is before this arc A B is somewhere here then we can consider the central field in an domain like this. If B is on the other side, if B is somewhere here then; obviously, we c that the extremals intersect here and therefore, it will not form a central field.

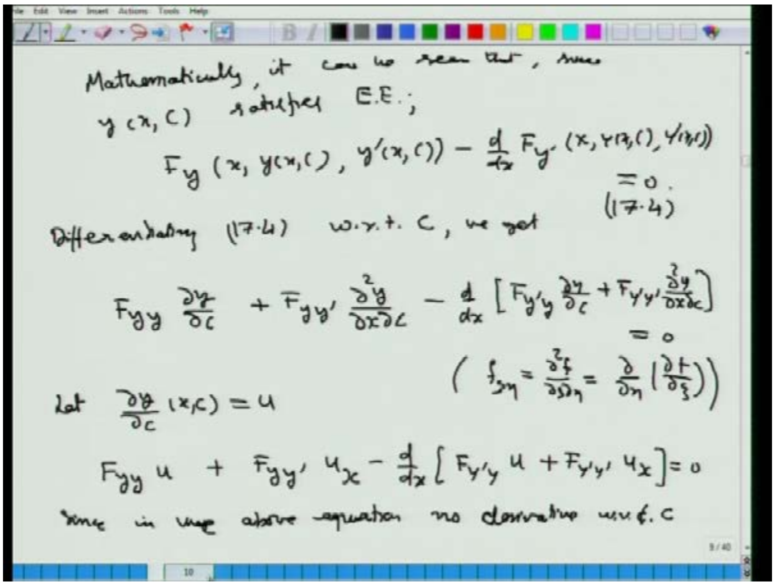
(Refer Slide Time: 41:40)



So, if B that is x 2 y 2 lies on the arc A star C, then we have central field locally at the centre A x 1 y 1 containing the point B that is next to y term. So, here you C that in this case if B here (Refer Slide Time: 42:40), you can consider this centre field like this and this B we will be included in that.

On the other hand if B x 2 y 2 lies on rather this if this conjugate point is A star c lies on the arc A star C then the extremals intersect in the neighborhood of B hence we do not have we do not get a central field. Thus the sufficient condition for the existence of a central field at A that is x 1 y 1 containing to be x 2 y 2 is that that B must lie on A star C on the arc this is called this condition is called the Jacobi condition.

(Refer Slide Time: 45:40)



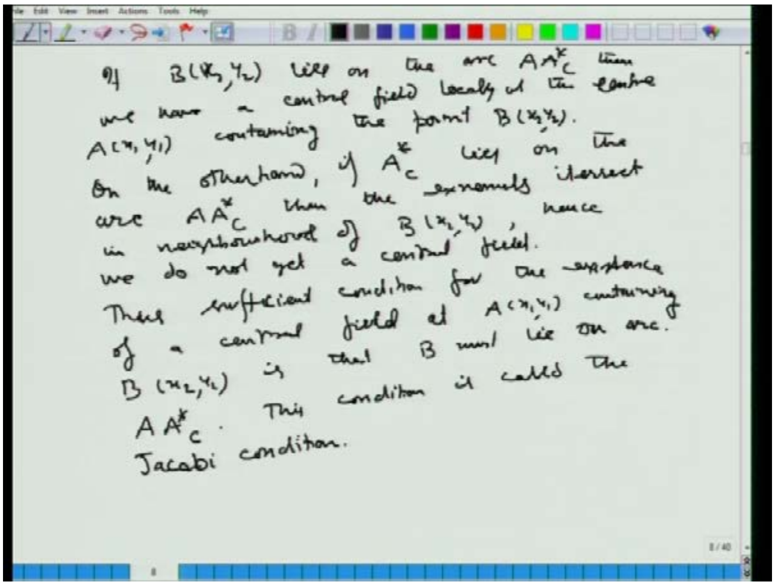
Now, mathematically this can be seen that it can be seen that since here we get that since y x c satisfies Euler that is F y which is x y x c y prime x c minus d by d x of F y prime at x y x c y prime x c equal to 0.

Differentiating with respect to, so let us say this is 17.4, differentiating 17.4 with respect to C, we get F y y we will not write these dependence here F y y and then y prime with respect to C, so that this del y by del c plus F y y prime del y prime del 2. So, this is with respect to x and so, we get another one with respect to C. So, del x del c minus d by d x of here F y prime y and del y by del c plus F y prime y prime del 2 y by del x del c equal to 0.

Here we have use the convention that is del 2 f xi eta is del 2 f by del xi del eta that is first we are differentiating with respect to xi and then eta; rather than other way around if it is eta xi. Usually books might following the other way around but. we will follow this convention that f xi eta which is actually equal to del 2 f del xi del eta, means we are first different with respect xi and then with eta.

So, that is del over del eta of del F over del xi, so that is the convention we are following here in the notation. And so, let that this del y over del c of x c denote denoted as u, then we get here in this equation F y y u plus F y y prime u x minus d by d x of F y prime y u plus F y prime y prime u x.

(Refer Slide Time: 50:21)



Now, here there is no C derivative appearing, since in this above question no derivative with respect to C is appearing explicitly although it is denoting u it is C derivative appearing that but not explicitly. So, we have thus we denote this u x as u prime, prime means with respect to x keeping other variables fixed.

We get we denote which gives finally, F y y minus d by d x of F y y prime u minus d by d x of F y prime y prime u prime. So, this is this is Jacobi equation, this is 17.5 17.5 is called Jacobi equation. And we solve we solve this we solve 17.5 is a second order equation in u, we solve the 17.5 for u clearly u at x 1 y 1 that is at a is 0. If u is not 0 in the interval x 1 less than x less than x 2, then the Jacobi condition is satisfied.

And so, we need to solve just this equation here. So, mathematically that whatever was explained here that they at extremal that point B should be before this conjugate point of a extremal that conjugate point of A. And so, we see that u is 0 at A, and then subsequently you should not be 0 anywhere in that interval, so that before thus the extremals will not touch the envelope. And so, that is what we will see in various examples in the next lecture, thank you very much for paying attention.