Calculus of Variations and Integral Equation

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Module #01

Lecture #16

Welcome viewers to the NPTEL lectures series on the calculus of variations. This is the 16th lecture in the series. Recall that in the last lecture, we had considered various cases of the functional where the points A and B can move. They can move freely in this space or they can move in a constraint way. If A and B are in a plane, then they can move along curves. Or if A and B are in three dimension space they can move either along curves or on surfaces.

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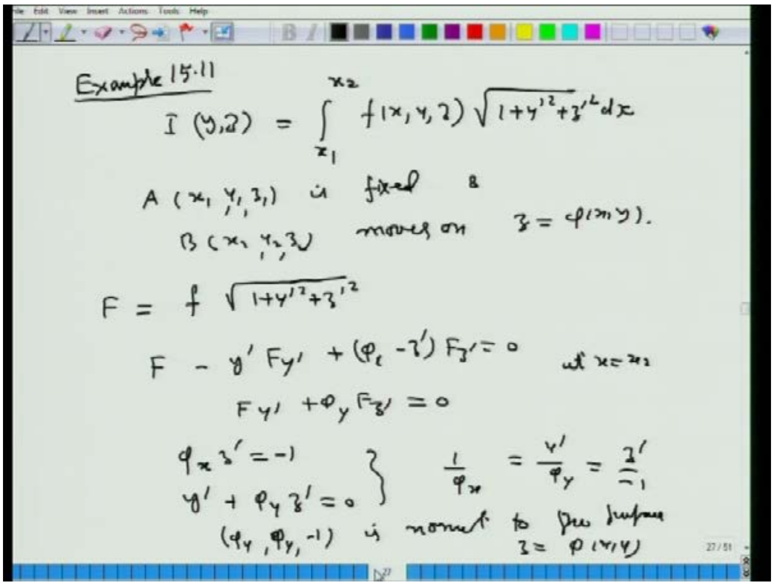

So, in the last part of the last lecture, we had stopped at these example 15.11, where we considered the functional I y z integral x 1 to x 2 F x y z square root of 1 plus y prime square plus z prime square d x.

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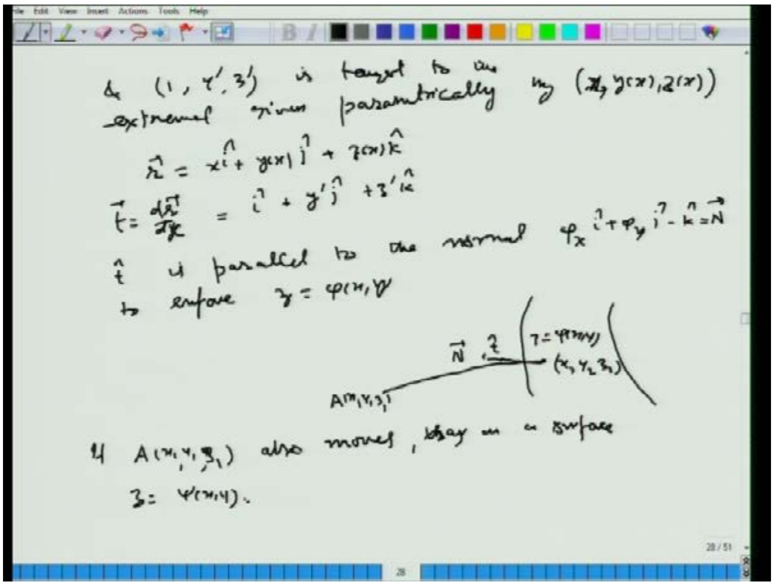

Here, this point A is taken to be fixed. That is in three dimensional space, A having coordinates x 1, y 1, z 1 and B having coordinates x 2, y 2 and z 2, where the point B is moving on the surface given by z equal to phi of x y. So, here the integrant is capital F, which is given by small f times square root of 1 plus y prime square plus z prime square. And the condition here is the following that f minus y prime F y prime plus phi x minus z prime F z prime equals to 0 and f of y prime plus phi y F z prime equals to 0 add the other end x equal to x 2. And so, these conditions imply that phi x z prime equal to minus 1 and y prime plus phi y z prime equal to 0.

Solving these for z prime, we get here z prime by minus 1 equal to 1 over phi x and solving it for z prime we get z prime over minus 1 equal to y prime over phi y. So, this is what we get that 1 over phi x equal to y prime over phi y equal to z prime over minus 1. Here, we see that this phi y phi x phi y minus 1 is the normal to is the normal to the surface z equals to phi x y and 1 y prime z prime is tangent to the extremul given by given parametrically by y x comma y of x z of z and therefore, the tangent will have. So, here the position factor on this is given by x plus y which is function of x and z of x like this. So, the tangent here will be d r by d x which will be I plus y prime j plus z prime k.

So, this is what is tangent to the extremul and here what it says at this tangent is parallel to so, this tangent t is parallel to the normal that is phi x I plus phi y j minus k to the surface z equal to phi x y. So, here this is the surface that is given by z equal to phi x y and at this point this x 2 y 2 and z 2 here, this extremul is such that, this tangent here at this point is actually parallel to the normal. So, they are in the same direction. Here, t as well this n, which is this one normal to the surface. That means, here the extremul, this is the point at which is x 1 y 1 and z 1 it is here at this point x 2 to surface orthogonal.

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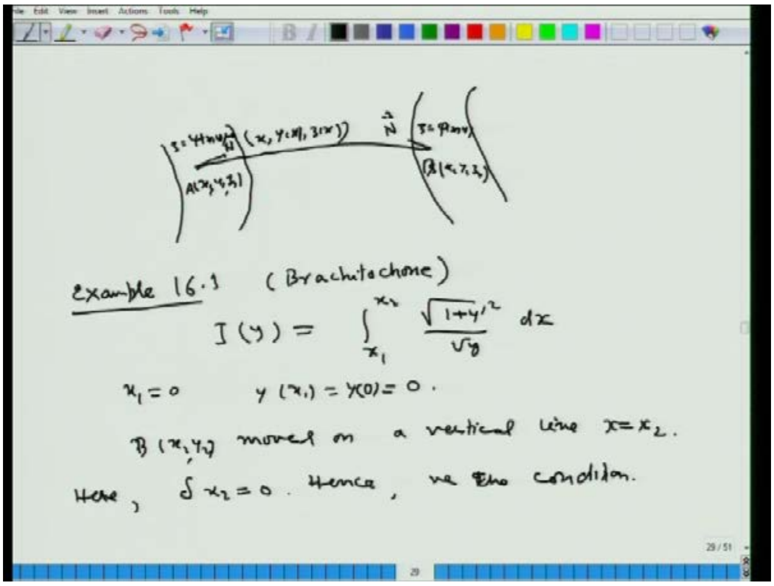

So, that is the result which we have got in the earlier case also and the in the case of the when x 2 y 2 z 2 moves along a curve there also we got the orthogonality condition and here also we get the similar condition here. So, if this A x 1 y 1 z 1 also moves, then say on a surface z equal to phi x y. So, we here we get this situation that this is the surface z equal to psi x y and A is here that is x 1 y 1 z 1 is moving on the surface and we have this another surface here that is z equal to phi x y and B is moving on this x 2 y 2 z 2.

So, this extremely joining this, that is (x, y, x, z x) will be such that here also we have the orthogonality condition and here also we have will have orthogonality condition. The tangent here will be parallel to the surface and here also we will get the similar condition. Now, let us take the case, where this example this we will call 16.1. Here, the brachitochone functional, that is I y equal to integral x 1 to x 2 square root 1 plus y prime square over root y d x. And here we take x 1 equal to 0 and y at x 1, which is y at 0 equal to 0 and this point x 2. Here, so, the point B which is x 2 y 2 moves on a vertical line, which is given by x equal to x 2 the constant x 2.

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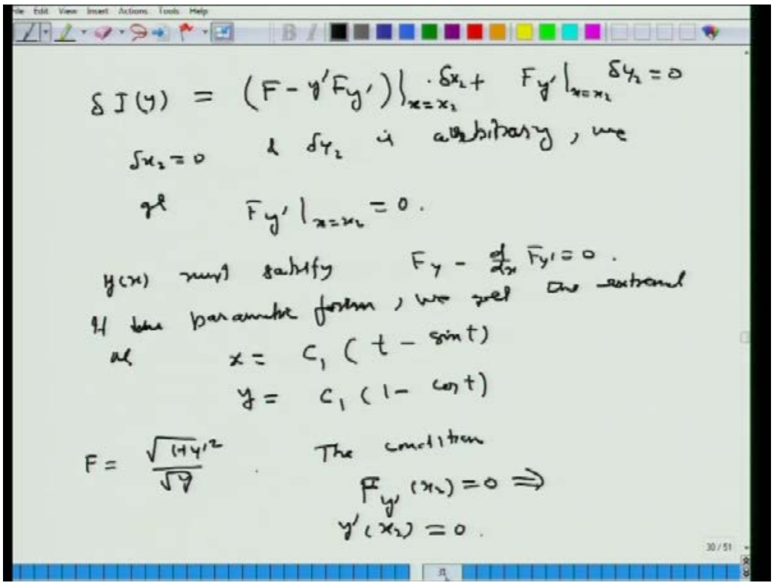

So, in this case here delta x 2 will be 0. Hence, we get hence, the conditions that the variation delta I here it is a function of y only, which is F minus y prime F y prime evaluated at x equal to x 2 because here the point is fixed. Only b is moving plus times delta x 2 here plus F y prime evaluated at x equal to x 2 delta y 2 equal to 0. And since delta x 2 is 0 and delta y 2 is arbitrary, we get F y prime evaluated at x equal to x 2 equal to 0. So, that is what we get here. In this case, we know that here the extremul are here, y of x must satisfy this F y minus d by d x of F y prime equal to 0, which gives us in the parametric form.

We get the extremul as x equal to this we had already solved, t minus sin t and y equal to c 1 minus cos t. And so, here F y prime here since f is here root 1 plus y prime square over root y. So, the condition F y prime at x 2 equal to 0 implies that y prime at x 2 equal to 0.

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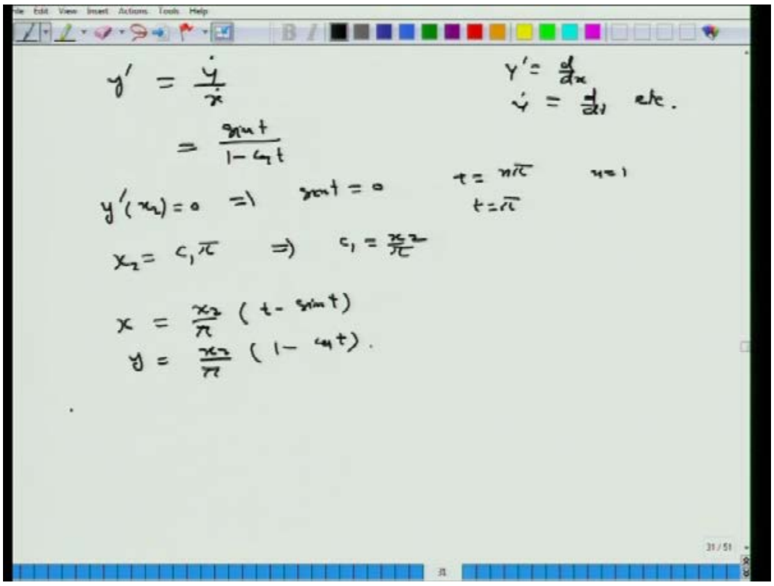

Now, here y prime which is actually y dot upon x dot. Dot means d by d t and prime means here y prime means d by d x and y dot equal to d by d t etcetera. Here, so, this is what will give us this will be sin t over 1 minus cos t. And so, y prime at x 2 equal to 0 implies that sin t must be equal to 0. And so, we get t equal to n pie and so, we get here x 2. So, x from here the first equation, we get t equal to n pie so, x equal to c 1. So, the first solution we will take t equal n equal to 1. So, if we take n equal to 1, we get t equal to pie and so, x equal to c 1 pie and this imply that c 1 equal to x over pie.

And so, we get x equal to sorry x 2 here x 2 equal to c 1 pie and so, c 1 equal to x 2 over pie. So, x equal to x 2 over pie t minus sin t and y equal to x 2 over pie 1 minus cos 2. So, that is what we will get the extremely in terms of x 2 here.

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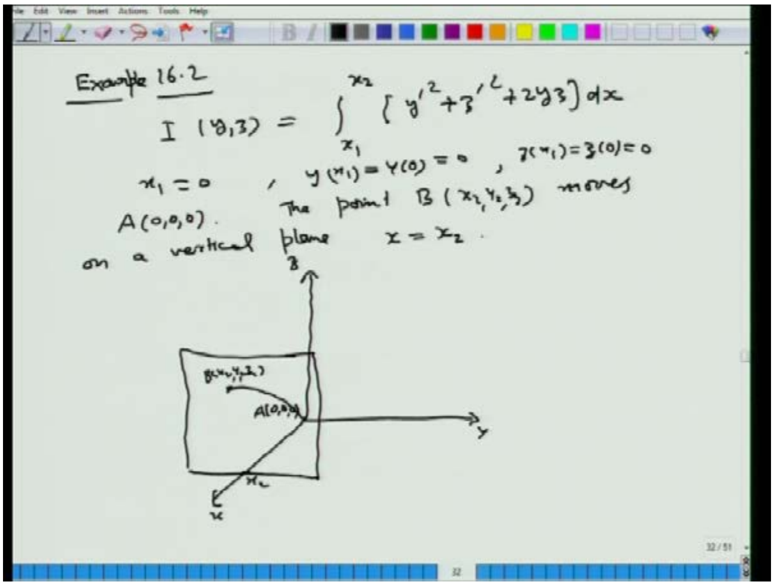

Now, the second case where F I is a function of y and z, we get the following. So, that is the next example 16.2. So, this we take as I of y z equal to x 1 to x 2 y prime square plus z prime square plus 2 y z d x and here again we take x 1 equal to 0 and y x 1 which is y at 0 equal to 0 and z at x 1 that is z at 0 equal to 0. So, a is actually (0, 0, 0) here and these point and point b which is x 2 y 2 z 2 moves on a vertical plane, that is x equal to constant. So, x equal to x 2. So, here this is the following picture we have x y and z. So, as is here that is (0, 0, 0) and we have x equal to x 2 here. x 1 is 0 so, plane like this. So, this point B is here that is x 2 y 2 and z 2 this point B is on this vertical plane given by this.

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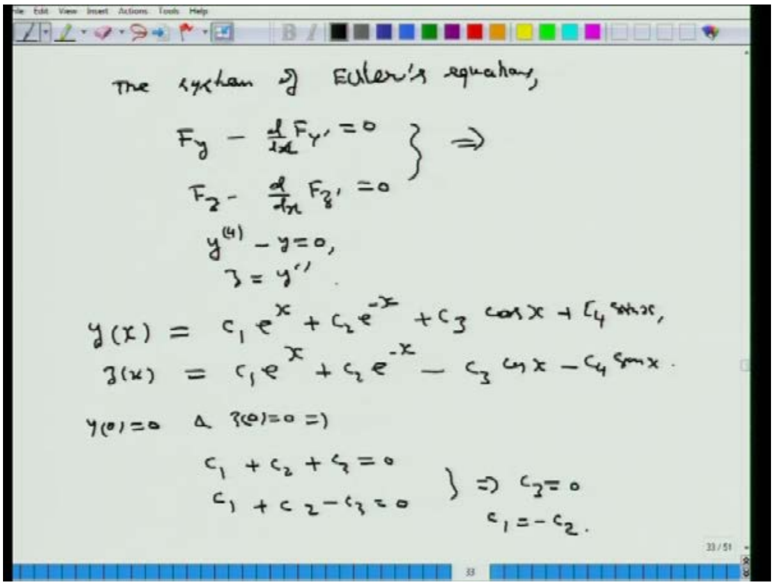

So, here we get first this system. So, this system of Euler's equation F y minus d by d x of F y prime equal to 0. F z minus d by d x of F z prime equal to 0 implies that, y fourth derivative minus y equal to 0 and z equal to y double prime. So, this solution of this system is the following that is c 1 e to the power x. So, y equal c 1 e to the power x plus c 2 e to the power minus x plus c 3 cos x plus c 4 sin x. And since y since z equal to y double prime, so, z x is c 1 e to the power x plus c 2 e to the power minus x minus c 3 cos x minus c 4 sin x.

Now, here this y 0 equal to 0 and z 0 equal to 0 imply that c 1 plus c 2 plus c 3 equal to 0 and c 1 plus c 2 minus c 3 equal to 0. So, adding these will imply c 3 is 0 and c 1 equal to minus c 2.

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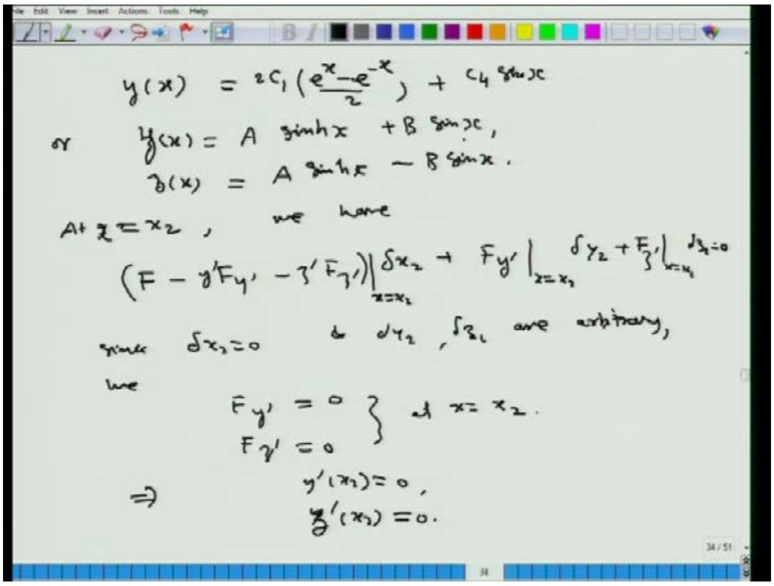

Write y x as, twice c 1 e to the power x minus e to the power minus x by 2 plus c 4 sin x. Similarly, z x equal to so, y x or y x can be written in the hyperbolic form. So, A will change this constants c 1 c 2 to A and B. So, A sin hyperbolic x plus B sin x and z x is then a sin hyperbolic x minus B sin x. Now, for at x equal to x 2, we have the condition that f minus y prime F y prime minus z prime F z prime delta x 2. This evaluated at x equal to x 2 plus F y prime x equal to x 2 delta y 2 plus F z prime x equal to x 2 delta z 2 equal to 0.

Now, since delta x 2 is 0 and delta y 2 delta z 2 are arbitrary variation, we get F y prime equal to 0 and F z prime equal to 0 at x equal to x 2. So, these 2 imply that y prime at x 2 equal to 0 and z prime at x 2 equal to 0. Now, here y prime at x 2 will be a cos hyperbolic x and plus B sin hyperbolic B sin x at x equal to x 2.

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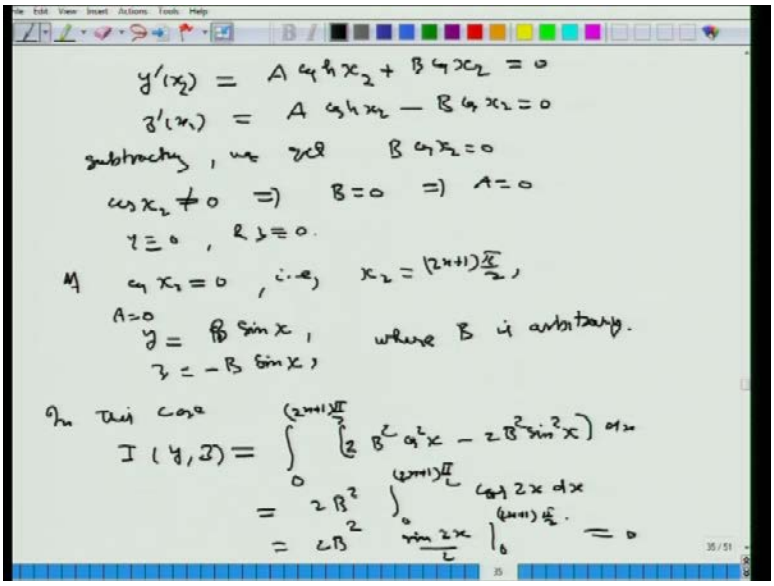

If that is so, y prime x is actually a cos hyperbolic x plus b cos x. So, at x 2 you get this if this is implies to be 0 and similarly, z prime at x 2 is a cos hyperbolic x 2 minus b cos x 2 equal to 0. So, subtracting this we will get cos x 2 0 and so, subtracting us get B cos x 2 equal to 0. So, if cos x 2 is not equal to 0, this would imply that B equal to 0 and then this would also imply a equal to 0 and so, y identically 0 and z identically 0 will be the solution. And if cos of x 2 equal to 0, that is x 2 equal to 2 n plus 1 pie by 2, then in this case, we get the externals y equal a equal to 0 and here y becomes B sin x and z equal to minus b sin x, where B is arbitrary.

But in this case this I y z turns out to be 0 since we get 0 to x 2. Here is 2 n plus 1 pie by 2 and we get y prime here and substituting y and z here, we get 2 b square cos square x minus 2 B square sin square x d x which gives us 2 b square integral 0 to 2 n plus 1 pie by 2 of. Here, this is cos 2 x cos square x minus sin square x gives us cos 2 x d x and integrating gives us 2 b square sin 2 x by 2 evaluated at 0 to 2 n plus 1 pie by 2 and so, this will be 0. Because here 2 x will make it 2 n plus 1 pie and sin will be at 0 there and at 0 also this sin is 0. So, we get this extremul the value of the extremul at any of those points x 2 given by 2 n plus 1 pie by 2 equal to 0.

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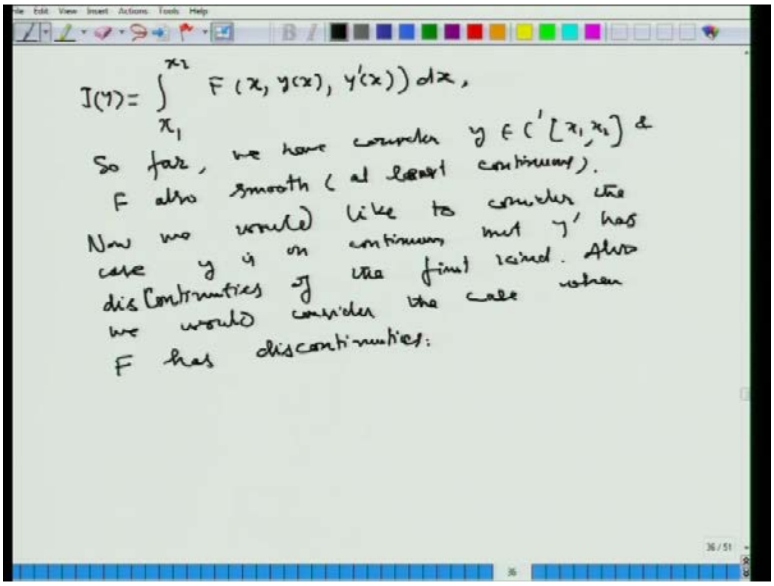

So, that was the case we had considered in the last lecture. Now, what we considered here were function we have considered this functional I y equal to x 1 to x 2 F(x, y (x), y prime x) d x. So, far we have considered y to be c 1 on this x 1 to x 2 and f also smooth at least continuous. Now, we would like to consider the cases where this y is only continuous and, but, y prime has discontinuities of the first kind. Also, we would consider the case when f has f has discontinuities.

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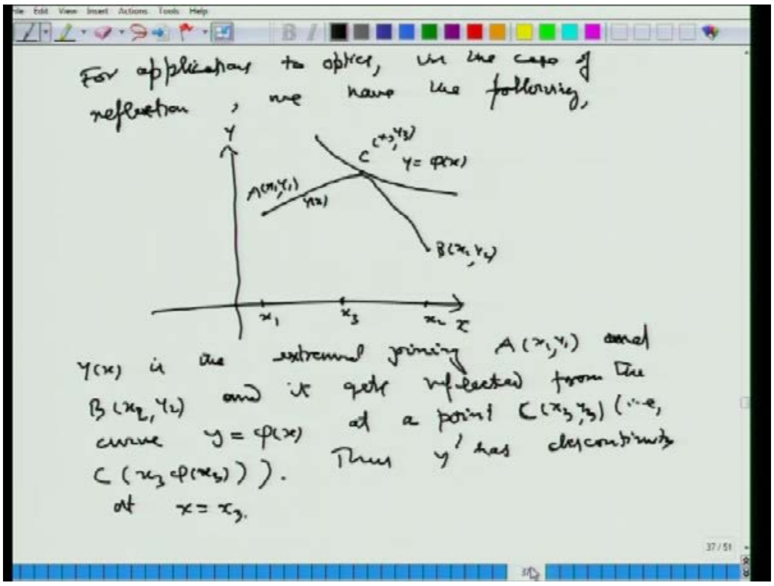

Here, for applications to optics, in the case of reflection, we have the following. That is here you have some curve here like which is given by y equal to phi x and we have these 2 points A which is x 1 y 1 and B x 2 y 2. These are fixed, but, then here like light ray goes here and get reflected, and then it goes to point B. So, there is a point c that is x 3 y 3 on this which will be moving on this curve y equal to phi x. So, here this extremely this is the extremely y x which joins these 2 points. So, y x is the extremely joining a x 1 y 1 to a and b x 2 y 2 and it gets reflected from the curve y equal to phi x at moving point c, at point c which is x 3 y 3. That is here c is x 3 and phi of x 3.

So, obviously, here thus y has y prime has discontinuity at x equal to x 3. So, here this is x 1, here this is x 2 and this point is x 3 here. So, which is the abscess or of the point c and y 3 is the ordinate of this point c. So, c is moving here. So, this ray goes here somewhere here and hit gets reflected and joins this. So, like that the c can move on this curve and we need to see that for this extremul y it here, what is that, it optimizes our functional given by this.

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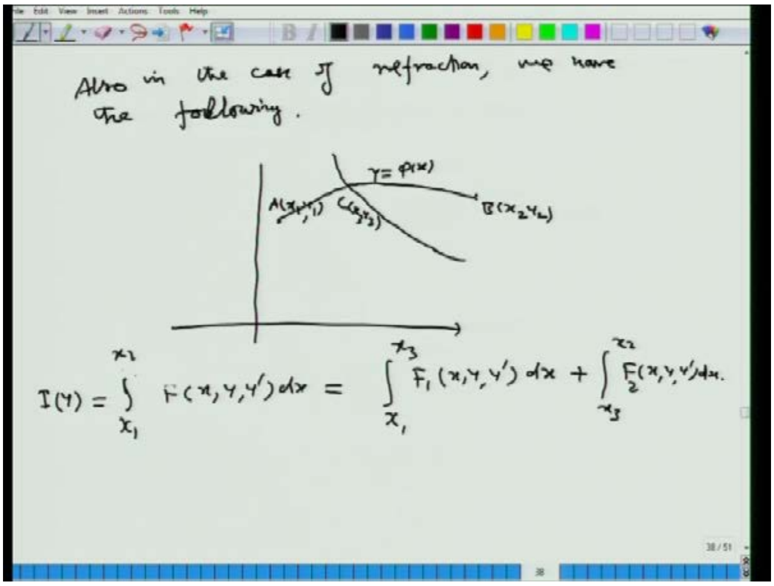

So, that is what is to be considered here and also in the case of refraction, we have the following. That is here again you have a curve like this which is y equal to phi x and A is here x 1 y 1 and B point is on the other side. So, this curve goes like this and then joins this B, which is x 2 y 2 and this point C is again on the curve y equal to phi x which is x 3 y 3. So, here it is goes to C and then refracted and goes to B. Here, there is a one medium on this side, which here A C is considered and C B is considered in other medium and so, this f will be f on here. So, we will have up to A to C. So, this I y here will be x 1 to x 3 x 1 to x 2 f of x y prime d x, will then have to be broken in this way. x 1 to x 3 F 1 of x y prime d x plus x 3 to x 2 f of F 2 of x y prime d x.

Because here this is a different medium and the on the other side of this curve, we have a different median. And then these integrants will be involving velocities of the light ray in one medium. And at the other time, on the other side, it will be involving velocity of the light ray in the other medium, which will be different and will apply this thing to those refraction cases. Also, there can be discontinuities in the derivative of the extremul in other ways, which we will explain through some examples. So, let us consider the first case the reflection.

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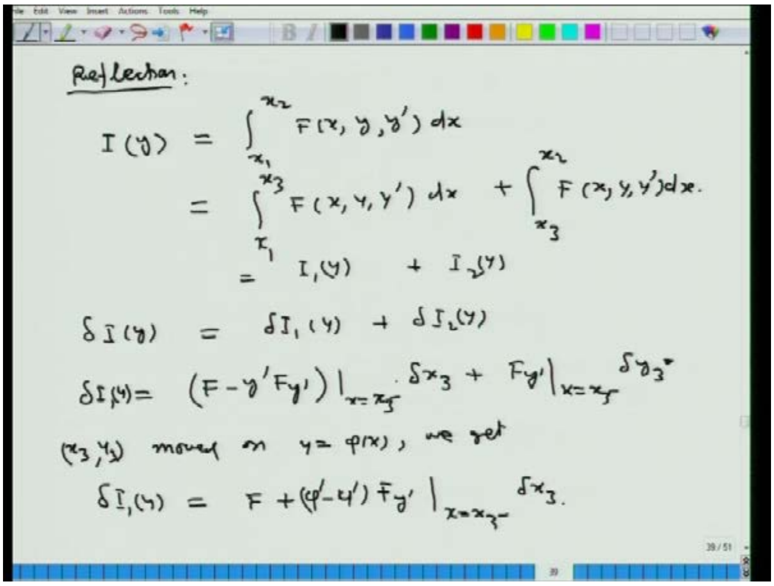

So, in the reflection case we have I y given by x 1 to x 2 f of x y prime d x and this is to be then since y prime has discontinuities. So, at x 3 so, we break it x 1 to x 3 f of x y prime d x plus x 3 to x 2 F x y prime d x. And so, here if we call this as I 1 y and plus I 2 y and so, the variation of this I y delta I y will be delta I 1 y plus delta I 2 y. That will be the newer part in the increments of this and it will have the linearity property. And so, delta I 1 here you see that it is the case like earlier this part, I 1 involves integral up to A x 1 to x 3.

So, here c is moving on this curve. So, it is like our earlier case. So, we can apply this delta I 1 will be then f minus y prime F y prime evaluated at x equal to here. So, here the points A and B are fixed, only the point C is moving. And so, that is what we have that x 3 times delta x 3 plus y F y prime. Here, we will have to take this x 3 this here y prime has discontinuities at C. We will have to take the left limit and here we will have to take the right limit. So, we will write x 3 minus denoting that we will take the left limit here. Similarly, x x 3 minus delta y 3 this thing that is delta I 1 and similarly, here delta I 2 this y will have since here the direction is reversed.

Here, we are going from A to C here, we will be moving from C to B and so, the direction is reversed. We get minus sign here minus f of n here. So, here since this y 3 since x 3 y 3 moves on y equal to phi x we get delta I 1 y is f plus phi prime minus y prime F y prime evaluated at x equal to x 3 minus into delta x 3 because delta y 3 then will be in terms of delta x 3 as before.

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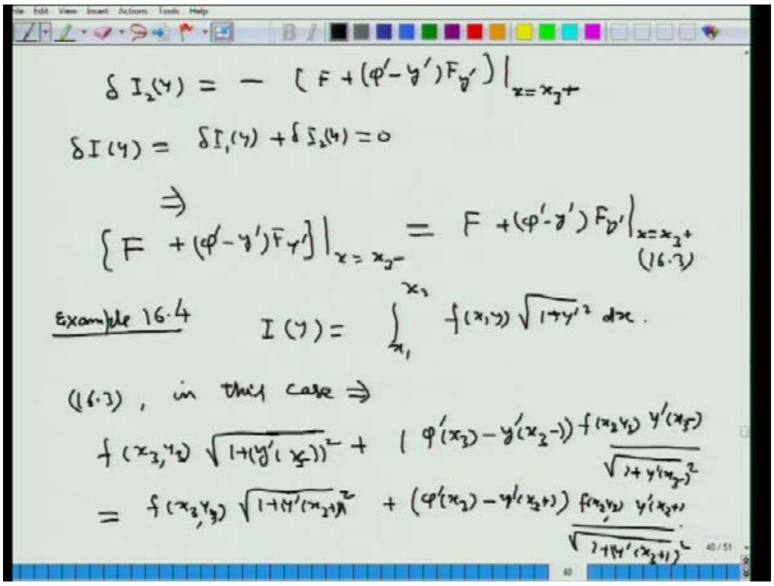

And similarly, we get delta I 2 at y as minus here because direction is reversed. f plus phi prime minus y prime f of y prime evaluated at x equal to x 3 plus and so, the condition is that delta I y equal to 0 means delta I 1 y plus delta I 2 y equal to 0, which gives us the condition that f plus phi prime minus y prime F y prime evaluated at x equal to x 3 minus must be equal to f whole thing f plus phi prime minus y prime F y prime evaluated at x equal to x 3 plus. So, this is the condition we get here. Now, so, let us see this example.

This is 6.3, we can use. So, I y is x 1 to x 2 f of x y root 1 plus y prime square d x and here the situation is that we have a curve like that, which is already explained here. That we have this A here and B here and there is a curve from which, this extremely is getting reflected. So, in this case we get using this let us call it 16.3 like this and this as 16.4. So, 16.3 in this case imply that, f at x 3 y 3 square root 1 plus y prime x 3 minus square plus; this phi and all they are continuous. So, x 3 phi prime is also continuous, only y is y prime is having discontinuous.

So, y prime at x 3 minus into F x 3 y 3 y prime x 3 minus divided by square root 1 plus y prime at x 3 minus Whole Square. This thing is equal to f at x 3 y 3 root 1 plus y prime at x 3 plus square putting this square here plus phi prime at x 3 minus y prime at x 3 plus f at x 3 y 3 y prime at x 3 plus divided by square root 1 plus y prime at x 3 plus whole square.

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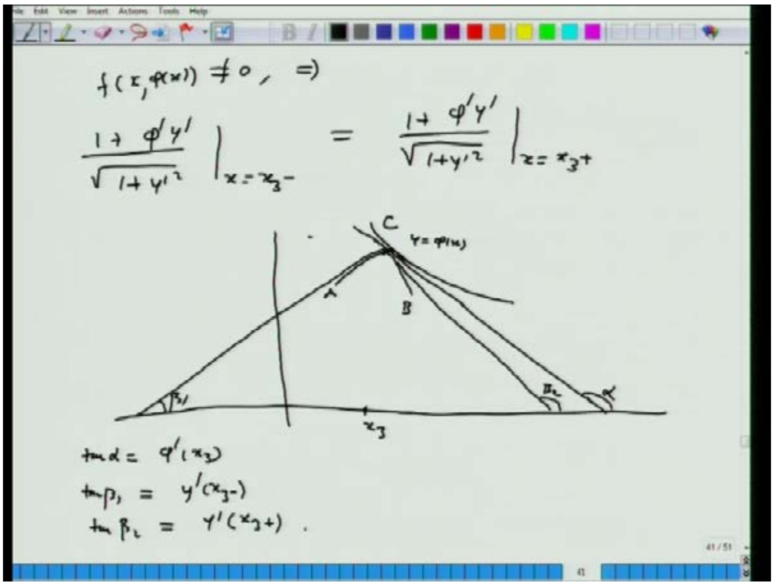

Now, to assume that, this f of x phi x as before, we assume that this is not equal to 0. So, we can divide by this and so, this implies that 1 plus phi prime y prime over square root 1 plus y prime square at x equal to x 3 minus equal to 1 plus phi y prime over root 1 plus y prime square evaluated at x equal to x 3 plus. So, that is the condition we get here. So, if we denote here so, from this we want to derive these standard laws of reflection and refractions. So, let us consider the case here. So, I have to draw the figure. So, that this is the curve y equal to phi x. So, here the tangent to this let us say this makes the angle alpha here.

So, this angle is alpha and this curve from going from A has tangent here. So, makes let us say the angle beta 1 and this curve going here, so, B here. So, let us say this tangent makes angle beta 2 here on the x axis. So, we see that the angle of incident will be so, let us say this alpha. So, tan alpha is phi prime at x 3 and tan beta 1 is y prime at x 3 minus and tan beta 2 equal to y prime at x 3 plus. So, this is the point C this is the point x 3. Here this is of course, the point x 1 y 1 this is the point x 2 y 2 like that.

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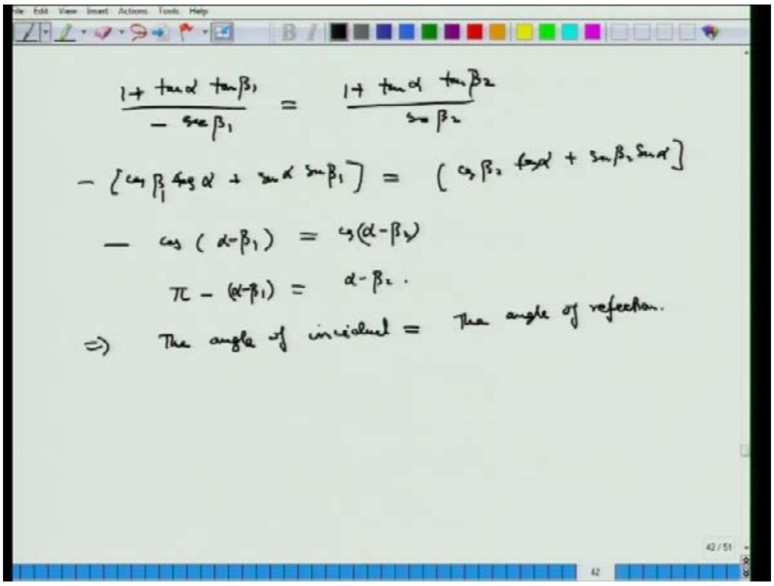

So, this is the point abscissa of c which is given by x 3. Now, using these notations, we see that putting it here 1 plus. So, we have 1 plus phi prime means then at x 3 is tan alpha. And you get tan beta 1 over you get minus sec beta 1 equal to 1 plus tan alpha tan beta 2 over sec beta 2. So, multiplying by cos beta 1 and here and cos beta 2 there and cross multiplying it gives us the following that is or taking it up there as cos beta 1. So, you get minus cos beta 1 plus tan alpha here, that we can write as sin alpha and cos alpha. And multiplying by cos alpha both the sides, you get cos alpha plus sin alpha sin beta 1 equal to cos beta 2 beta 1 here and plus cos beta 2 cos alpha plus sin beta 2 sin alpha like this.

So, this can be written as minus cos of alpha minus beta 1 equal to cos of alpha minus beta 2. So, this means that pie minus alpha minus beta 1 equal to alpha minus beta 2. So, here alpha minus, you look at the figure, which we had drawn. This alpha minus beta 2 gives us this angle here and similarly, alpha minus beta alpha minus beta 1 gives us this 1 and alpha minus beta 2 will give us this angle which will be then taken more than as phi minus that thing. So, we get this pie minus alpha minus beta 1 equal to pie alpha minus beta 2. So, here angles so, this actually implies that the angle of incident equal to the angle of reflection, which is the famous law of reflection here. Now, we go to the case, where we have refraction here.

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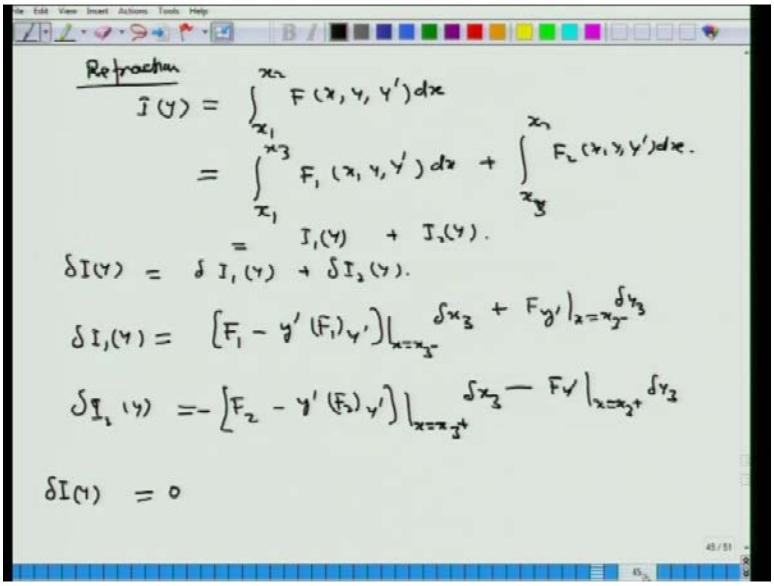

So, in the refraction case, we get I y as integral x 1 to x 2 F x y prime d x will then be having 2 parts x 1 to x 3 F 1 x y prime d x plus x 3 to x 2 F 2 x y prime d x. So, here in the previous case, this F is having a cert standard form which will consider now in the second case. So, then we considered this delta I y again this will be I 1 y in this way and I 2 y. And so, delta I y will be delta I 1 y plus delta I 2 y and here, so, delta I 1 y because now, x that point at which is a x 1 y 1 which is fixed, only this point x 3 y 3 is moving. Similarly, here x 2 is fixed and only this x 2 y 2 is fixed and x 3 y 3 is moving.

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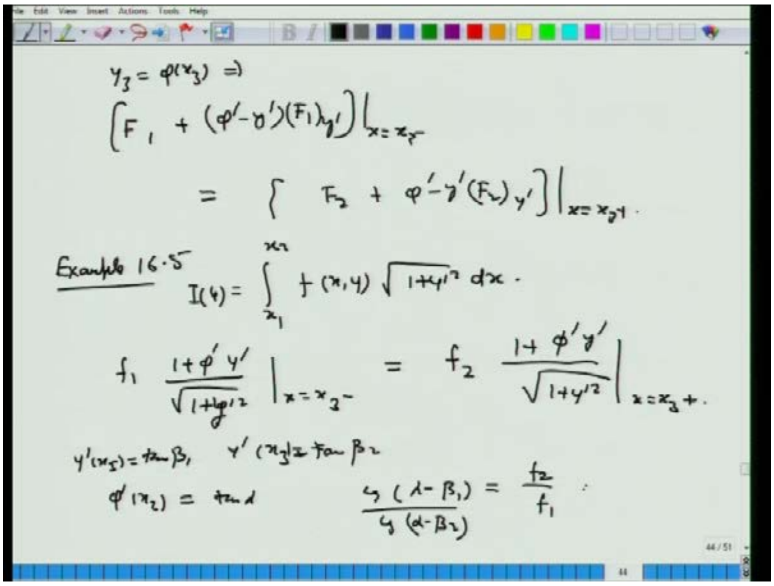

So, we get as before here that is F 1 minus y prime F 1 y prime this evaluated at x equal to x 3 into delta x 3 and plus the minus here, same way because direction is reverse and s 3 here. F 1 minus y prime at F 1 F 2 sorry no F 2 y prime evaluated to x equal to x 3 plus 2 delta x 3. We have x 3 plus 1 sorry here, we have one more term delta y 3 plus F y prime evaluated x equal to x 3 minus delta y 3 delta I 1. And similarly, delta I 2 y is F 2 minus y prime F 2 y prime. Here, will minus sign it will be plus delta x 3 delta y 3 with the whole of minus sign. So, this also we write minus here.

So, this substituting it here so, delta I y will be then equated to 0. So, we will have these things here and we get since now, delta y 3 will be written in terms of here, this delta y 3 will be since y 3 is x 3 y 3 is moving. So, delta I y equal to 0, gives us the condition that and since y 3 equal to phi of x 3.

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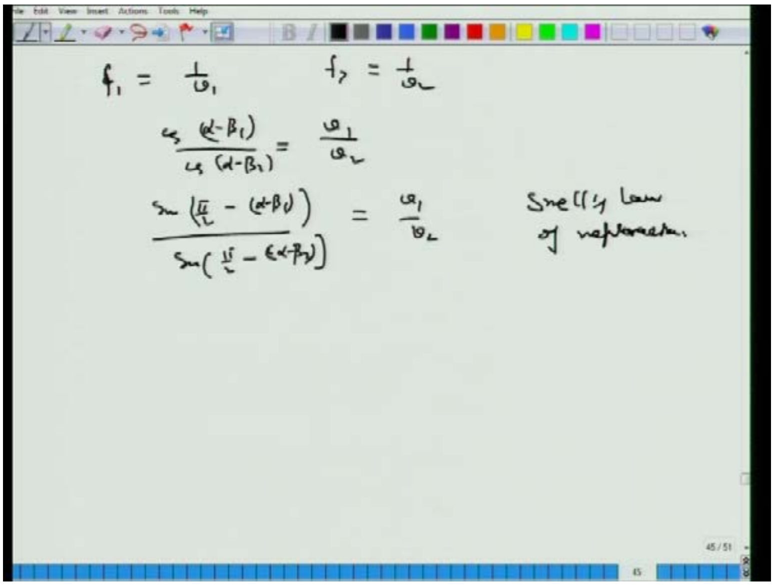

So, this implies that we get F 1 plus phi prime minus y prime of F 1 y prime. This thing evaluated at x equal to x 3 minus equal to F 2 plus phi prime minus y prime F 2 y prime x equal to x 3 plus. So, that is the required condition now. So, in this example 16.5, we have this integral I y equal to x 1 to x 2 f of x y square root 1 plus y prime square d x gives us. That same way, we get F 1 1 plus phi prime y prime over square root 1 plus phi prime square y prime square, evaluated at x equal to x 3 minus equal to F 2 1 plus phi prime y prime over square root 1 plus y prime square evaluated at x equal to x 3 plus.

And so, here using the same notation that y prime at x 3 is beta x 3 minus is beta 1 y prime at x 3 plus as beta 2 and phi prime at x 3 equal to tangent tan beta 1 and beta 2. So, this is tan beta 1 tan beta 2 and this is an alpha. We get same way that cos now, cos of beta cos of alpha minus beta 1 equal to over cos of alpha minus beta 2 comes out to be, here, this F 2 divided by F 1. And so, here in this case the this F 1 is 1 over velocity v 1 and F 2 is 1 over v 2 we get this cos alpha minus beta 1 over cos alpha minus beta 2 equal to v 1 over v 2, which is the Snell's law. Because this can write it as sin phi by 2 minus alpha minus beta 1 over sin phi by 2 minus alpha minus beta 2 equal to v 1 over v 2, which is the smell's law of refraction.

So, we stop here and in the next one, we will consider more cases of the discontinuities of the extremuls. Thank you very much for viewing this.