Calculus of Variations and Integral Equation
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Lecture No. # 15
Calculus of Variations and Integral Equation

Welcome viewers, to the NP-TEL lecture series on the calculus of variations. This is the fifteenth lecture of the series. Recall that in the last lecture, the fourteenth lecture we had considered the functionals with moving boundaries.

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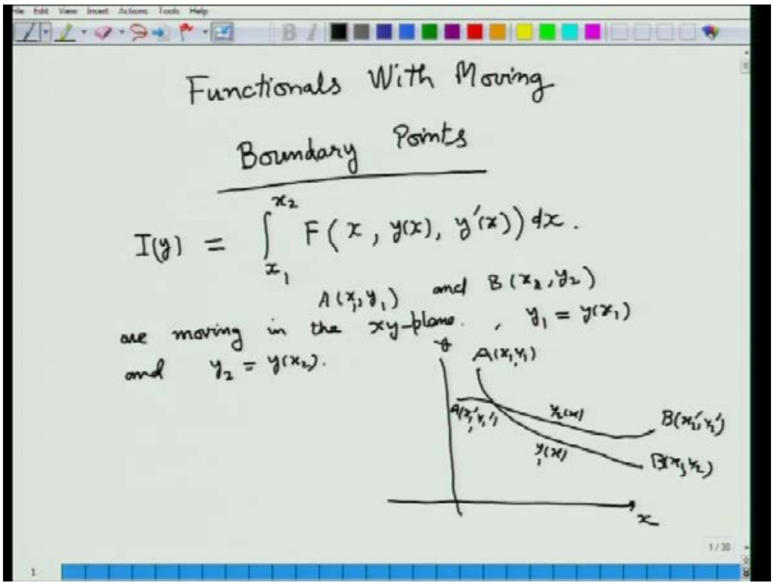

Here, the integral we had considered is of simplest type that, I y equal to integral x 1 to x 2 F of x y x y prime x dx, where these boundary points A x 1 y 1 and B x 2 y 2 can move freely in the xy-plane or they can move in a constrained way that they are moving along a curve.

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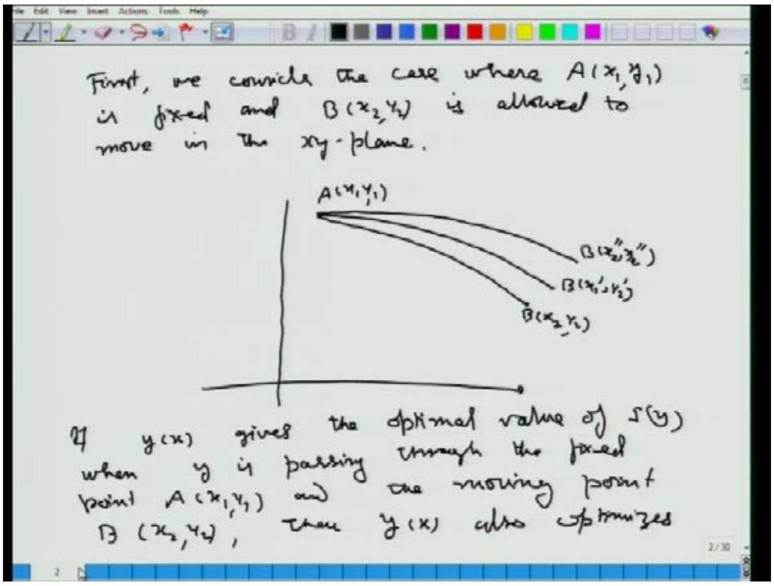

In the last lecture, we had started with the introduction of these type of functionals. Here, we had considered first, the case when one of the points let us say the point A(x 1, y 1) is fixed and the point B is moving freely in the xy-plane or moving along a given curve. Here, B(x 2, y 2) moves to neighboring point B x 1 dash y x 2 dash y 2 dash and then again it moves to B x 2 double dash y 2 double dash; like that it keeps on moving or it may move along a given curve. As we know, that if the functional is optimized for given points A and B, where A and B are moving then it also gets optimized when those two points were assumed to be fixed.

Therefore, any functional which is optimizing it whether the points are moving or they are fixed, that extremal should be solution of Euler's equation; that is the necessary condition which should always be satisfied. So, we can restrict our consideration of variation of the functional over the family of extremals.

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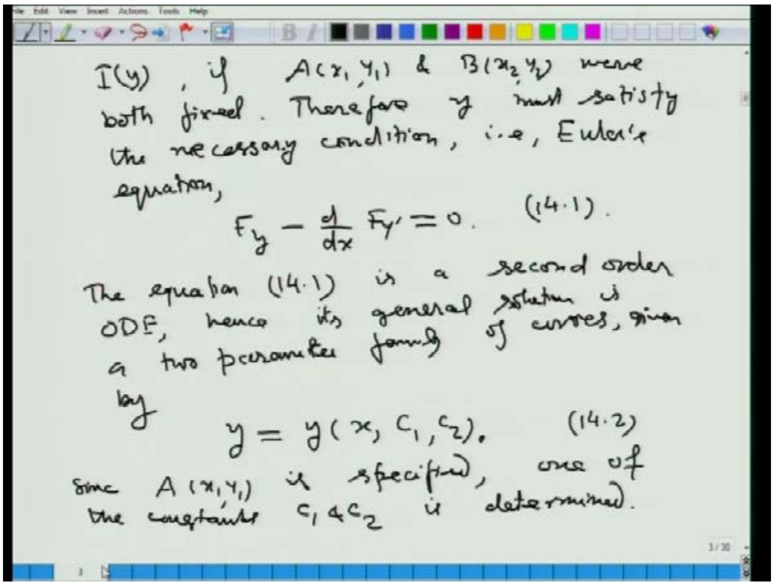

Here, you know that this Euler's equation F y minus d by dx of F y prime equal to 0 is a second order differential equation which gives us the family of extremals as two parameter family of functions, where y equal to y of x, c 1 and c 2 ; c 1and c 2 are parameters. And so, we assume that the point A(x 1, y 1) is fixed. Then, one of these two constants gets determined and then we get this sub family y equal to x of c s family of extremals taken as one parameter family; that means, c is given different arbitrary values and we get different extremals here.

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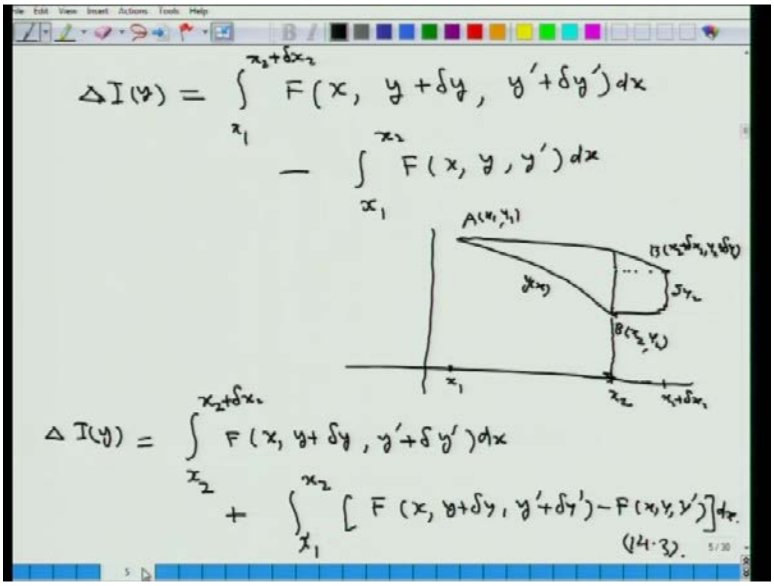

We can consider now I on this sub family. I y is a function of x and this c is the constant coming here in the equation 4.2 and then, we have the integral x 1 to x 2 F of x y x comma c y prime x comma c dx. So, we need to consider this functional only on the sub family of extremals; one parameter sub family of extremals. Then, we consider its increment , delta I y which is the difference of these two integrals at x 1 to x 2 plus delta x 2 F of x y plus delta y y prime delta y y prime plus delta y dx minus x 1 to x 2 F of x y y prime dx. Here, we break this integral into two parts: x 2 to x 2 plus delta 2 and x 1 to x 2. These are written here, x 2 to x 2 plus delta x 2 of F x y plus delta y y prime plus delta y prime dx plus x 1 to x 2 F of x y plus delta y y prime plus delta y prime minus F of x y y dash dx and these two integrals are then written like this:

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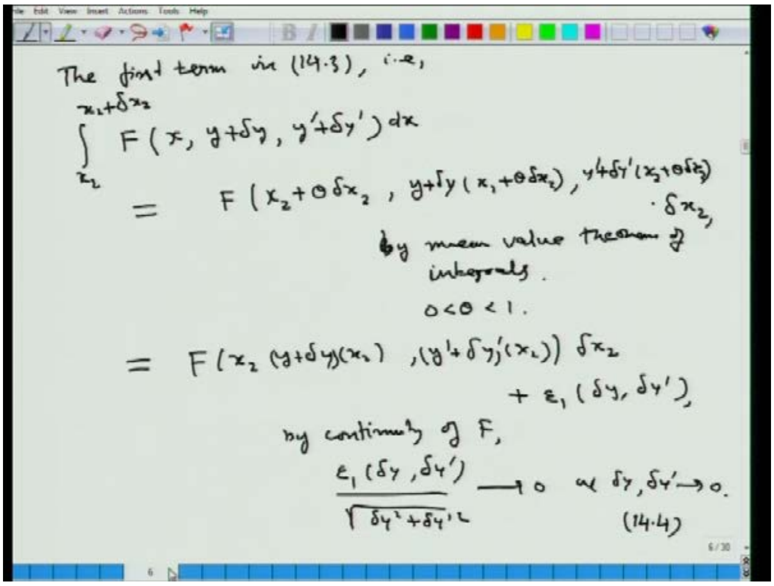

They are approximated by the mean value theorem and Taylor series expansion. So, first integral is approximated; F evaluated at some intermediate point where theta lies between 0 and 1. F of x 2 plus theta delta x 2 y plus delta y evaluated at x 2 plus theta delta x 2 y prime plus delta y prime evaluated at x 2 plus theta delta x 2 and at times delta x 2 here. And then, by continuity it is written like F of x 2 y plus delta y at x 2 y prime plus delta y prime at x 2 times delta x 2 plus some function epsilon 1, which is function of delta y and delta y prime of higher order in delta y and delta y prime.

This is the condition required here that epsilon delta y delta y prime divided by square root of delta y prime plus delta y prime square tends to 0 as delta y and delta y prime tend to 0.

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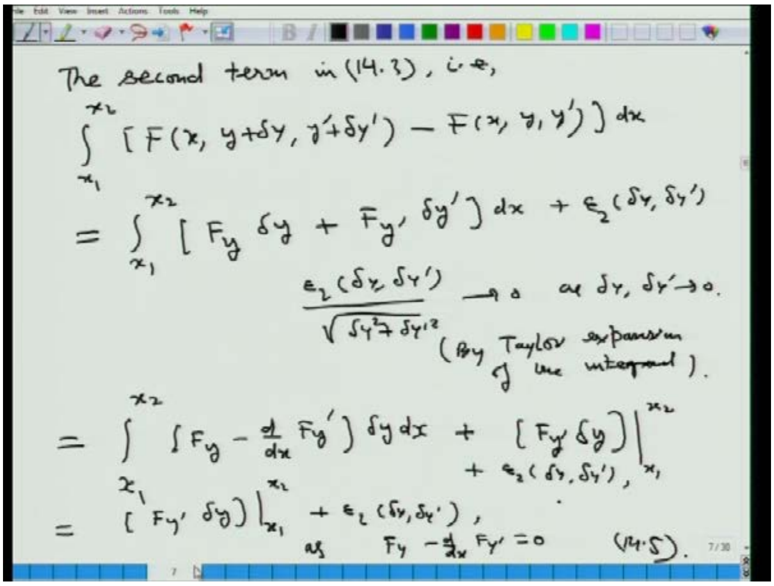

Here, the second integral using the Taylor series expansion can be written in this manner that, x 1 to x 2 plus F y delta y plus F y prime delta y prime dx plus epsilon 2 of the same time as epsilon 1, and then shifting this derivative on delta y here. This derivative here is shifted onto to y prime and so, we get this term x 1 to x 2 integral F y minus d by dx F y prime delta y dx plus the boundary term F y prime delta y evaluated at x 1 to x 2 plus here epsilon two which is coming from the top. Here, since y is an extremal , this integrand is 0.

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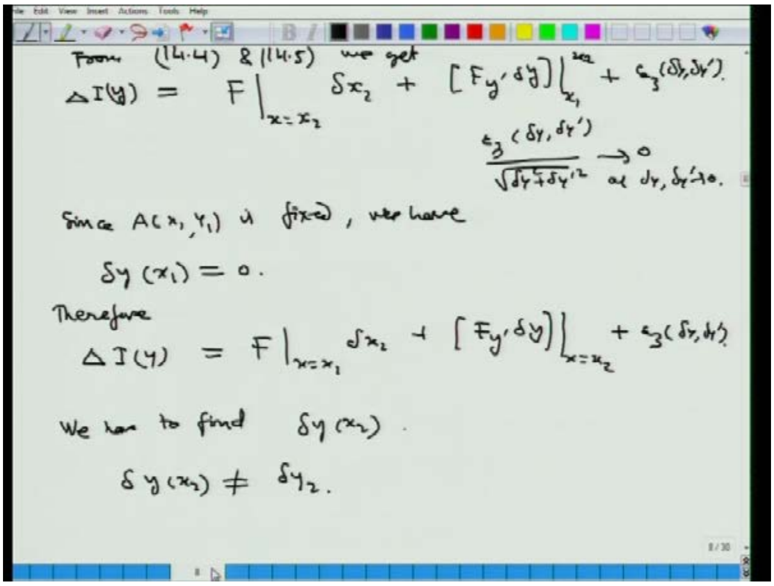

We are left with this 14.5 and which then gives us delta I the increment of the functional as F evaluated at x 2 times delta x 2 plus F y prime delta y evaluated at x 1 to x 2 plus some epsilon 3 which is sum of epsilon 1 and epsilon 2; since the point A is fixed. So, we get delta y at x 1 is 0. Finally, we get delta I y as F evaluated at x 2 times delta x 2 plus F y prime delta y evaluated at x equal to x 2 plus epsilon 3. Now, here delta y at x 2; we need to calculate that which is we know that delta y at x 2 is not equal to delta y 2.

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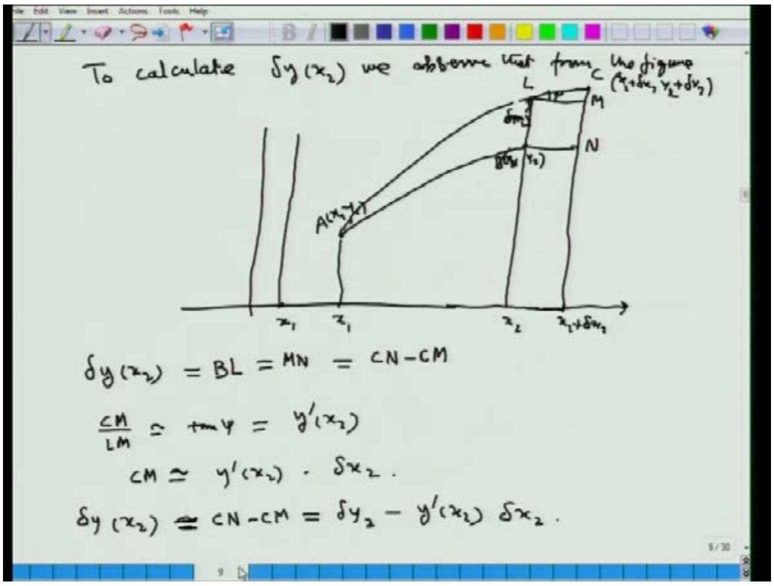

Here, in the given figure A is fixed; only B is moving. So, B has moved from this to let us say point C. Here, at x 2 plus delta x 2 and y comma y 2 plus delta y 2 and this is B x 2 y 2. Here, we can see that this BL is actually delta y at x 2 which is not as delta y 2. And so, we need to calculate this delta y at x 2; we use here, that BL is MN which is CN minus MN CN minus CM.

Here, we approximate this CM by assuming that this is a triangle, since these quantities are small. We assume that these are straight lines and so, CM over LM is approximated ten tangent of psi where psi is this angle. We get this as y prime evaluated at x 2 and so, we get finally delta y equal to delta y 2 delta y at x 2 equal to delta y 2 minus y prime at x 2 delta x 2.

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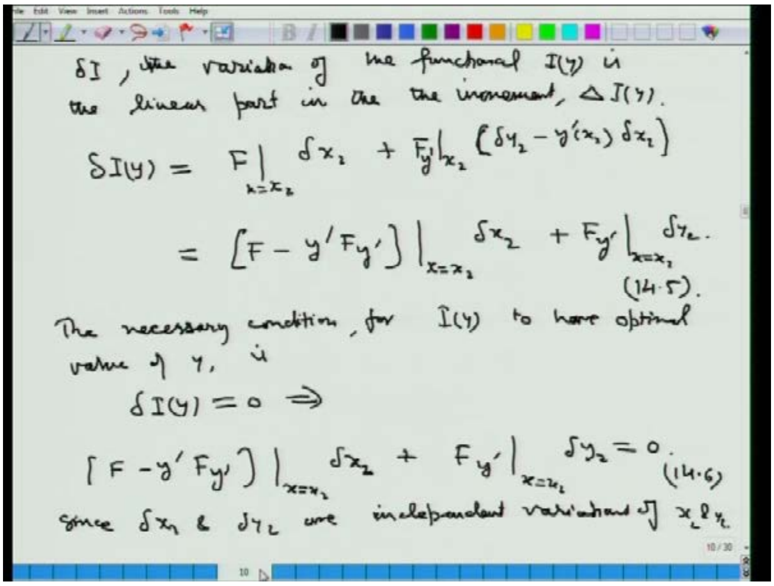

Using this finally, we get delta ; now since this small delta, see here capital delta is the increment and small delta I is the linear part in the variation linear part in the increment delta I and so, dropping those epsilon terms we get finally delta I, like this . Here, we collect this y F y prime y prime times delta x 2 in the first term and so, we get F minus y prime F y prime evaluated at x 2 times delta x 2 plus y prime delta x 2.

Finally, we get this variation delta I; small delta I which is the linear part in the increment capital delta I like this as 14.5 here. Now, the necessary condition is that this variation must be 0. We get this 14.6 here and now assuming that these delta x 2 and delta y 2, that is the increment at the movement in the point x 2 and y 2. If this movement is independent, then we see that delta x 2 and delta y 2 can take arbitrary values. If we take delta y 2 as 0 and delta x 2 as 1, we get this F minus y prime F y prime equal to 0. Similarly, if we take delta x 2 equal to 0 and delta y 2 1 in 14.6, we get F y prime evaluated at x equal to x 2 equal to 0.

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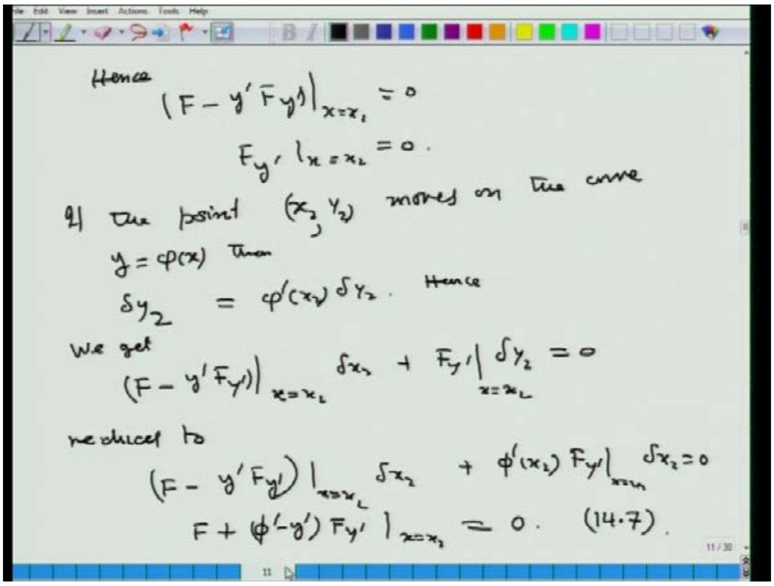

So, independent variations of delta x 2 and delta y 2 will lead to this. And, if we have the constant movement that, the point x 2 y 2 moves along a curve given by y equal to phi x, then we see that delta y 2 will be phi prime x 2 delta sorry this should be delta x 2. And hence, using this in delta y 2 here we get F minus y prime F y prime delta x 2 plus phi prime x 2 F y prime times delta x 2 equal to 0.

Now, again delta x 2 is independent is arbitrary variation of point x 2 and then the coefficient of that must be 0. So, we get this 14.7 which is F plus phi prime minus y prime times F y prime evaluated at x 2 equal to 0.

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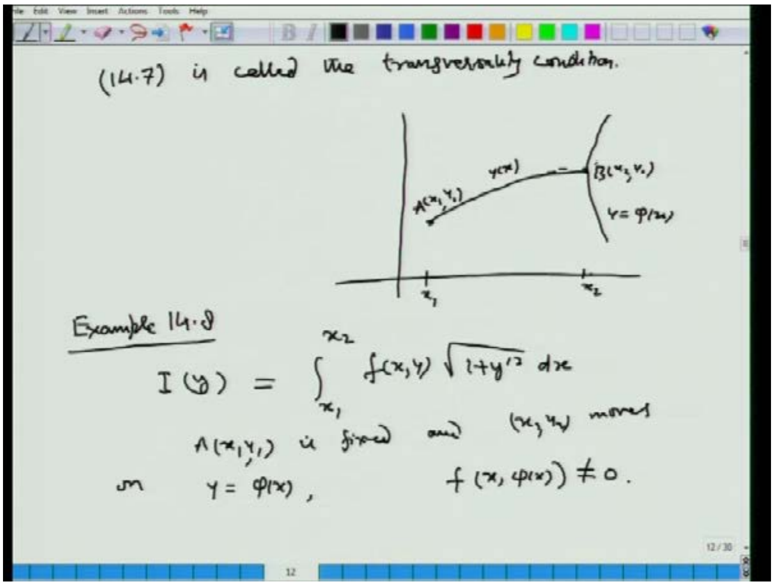

This is the condition known as transversality condition and that is what is used here in this example 14.8; and we take here the point A(x 1, y 1) is fixed and this point (x 2, y 2) is moving on this curve. Here, we need to find that point x 2 such that this extremal y x gives us the optimal value of the functional I(y) defined integral x 1 to x 2 F x y square root of one plus y prime square dx. We assume that the value of F on the points on this curve is not 0.

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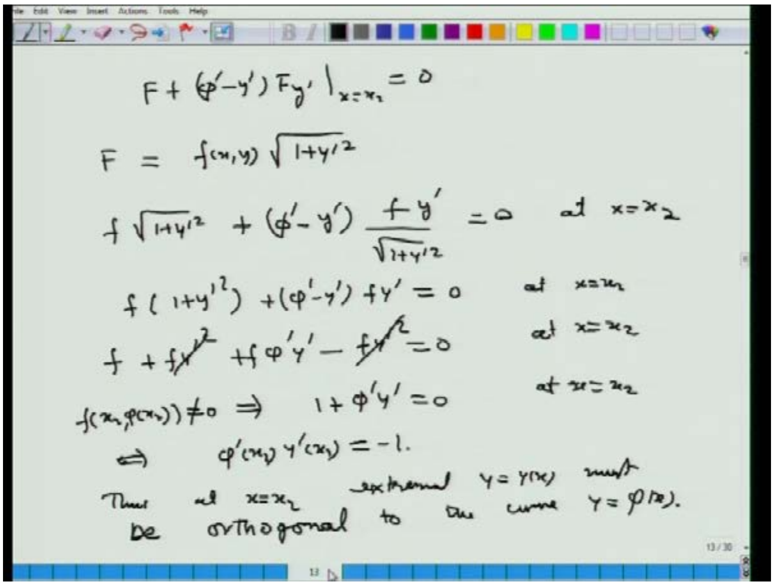

Using this transversality condition we get F plus phi prime minus y prime F y prime evaluated at x equal to x 2 equal to 0. Here, F is small f times square root one plus y prime whole squared y prime squared. And then finally, we get this after simplification that one plus phi prime y prime equal to 0 at x equal to x 2, which gives us that phi prime at x 2 y prime at x 2 equal to minus 1; this is the orthogonality condition. So, the transversality condition reduces to the orthogonality condition here; that is what it means that here this point.

Here. this y x tangent to this and tangent to this phi that is, phi prime and y prime; this should be orthogonal. Here it should have the 90 degree angle here; it should hit this. So, So, we should take that x 2 which this tangent to this extremal and tangent to this curve y equal to phi x must be orthogonal to each other.

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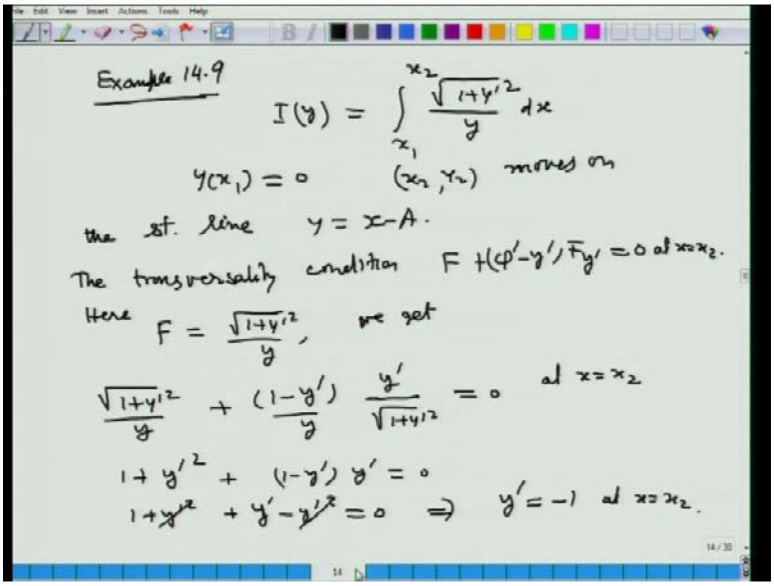

So, in the next example we have taken I(y) equal to x 1 to x 2 integral square root of 1 plus y prime square over y and here, y at x 1 equal to 0 and (x 2, y 2) moves on the straight line y equal to x minus A. So, here the transversality condition reduces to y prime equal to minus one at x equal to x 2

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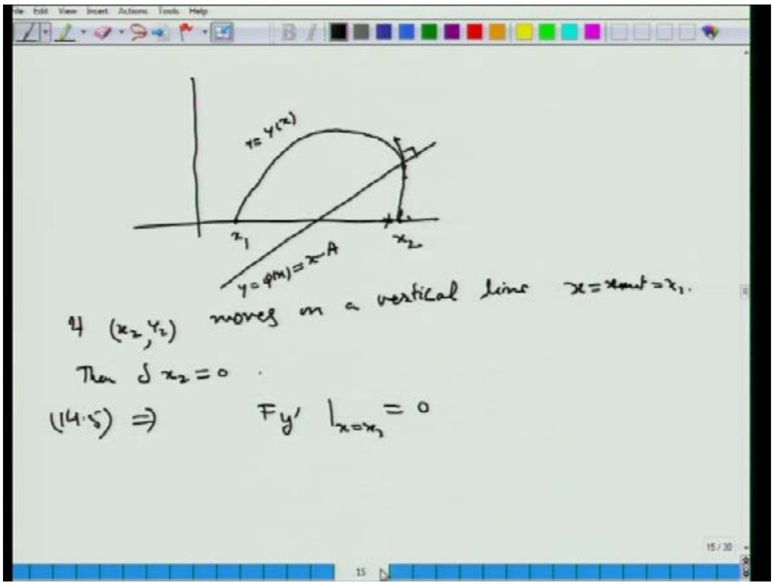

Which says that; that means, y equal to minus x. It should be that line and this is y equal to x minus a and therefore, this extremal should hit here on this line at this point x equal to x 2 y 2 orthogonally. So, that is what we get here. And if the movement is on vertical line, then we know that then delta x 2 must be 0. So, this condition here delta x 2 is 0. We are left with F y prime evaluated at x 2 delta y 2 0. Since delta y 2 is arbitrary, we get F y prime at x 2 equal to 0. So, that is what is obtained.

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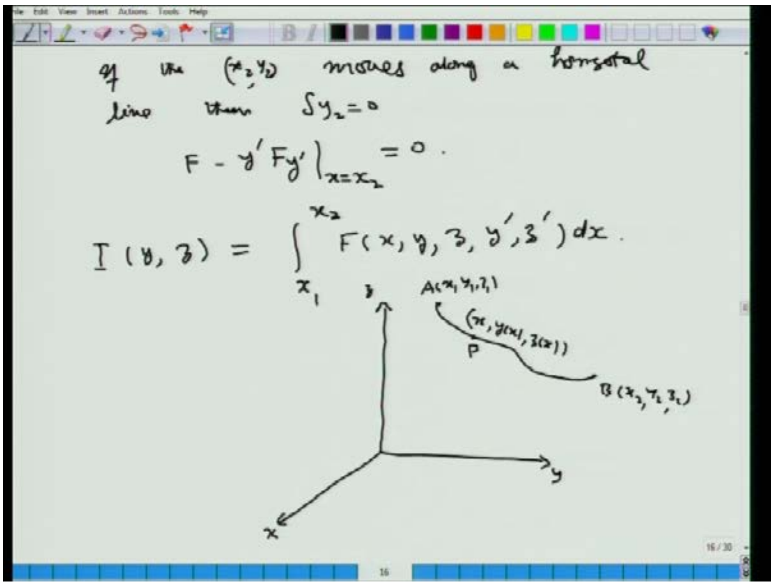

Here, now if the movement is along the horizontal line. If the movement; if x 2 y 2 moves along a horizontal line, then delta y 2 is equal to 0 and we get this again from this equation here. So, delta y 2 is 0. We get F minus y prime F y prime evaluated at x equal to x 2 times delta x 2 0. Since delta x 2 is arbitrary, we get the coefficient of this equal to 0 here. So, we get F minus y prime F y prime at x equal to x 2 equal to 0; that is what will be used in this case.

Next case we consider is the more general functional like this: I(y, z) equal to integral x 1 to x 2 F of x y z y prime z prime dx. As we have already seen, that the situation is like this; we have x y and z here. The point is in three dimension: this A, which is x 1 y 1 z 1 and this B, x 2 y 2 z 2; this is the extremal here, which is parameterized at x y at x and z at x. So, any point p here moving point on this curve will be parameterized here x is the parameter ; x itself. So, x y at x z at x here that is what earlier we had already explained this.

In this case, both A and B can move or we can take A fixed and B moving. Like this, in the earlier case also we had taken here.

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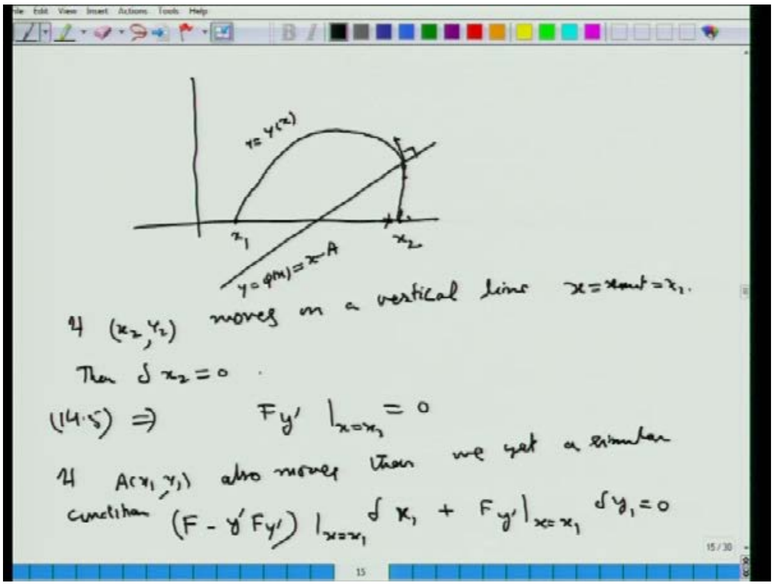

This A is fixed and B is moving. If A is also moving, we get a similar transversality condition at the point A also. If A x 1 y 1 also moves, then we get a similar condition, transversalty similar condition that is, F minus y prime F y prime evaluated at x equal to x 1 delta x 1 plus F y prime evaluated at x equal to x 1 delta y 1 equal to 0. Since delta x

1 and delta y 1 can move freely, then we get the coefficients 0 and if x 1 y 1 also moves along a curve, y equal to psi x; we get a transversality condition there at the point in a similar manner.

So, in the three dimension case also; here, A and B can move freely they can move along a curve and now, there is one more possibility that it can move along a surface also, because here we are in three dimension. We can have this point A or B or both can move freely either along a curve or along two different curves or along two different surfaces. So, we will consider those cases here separately.

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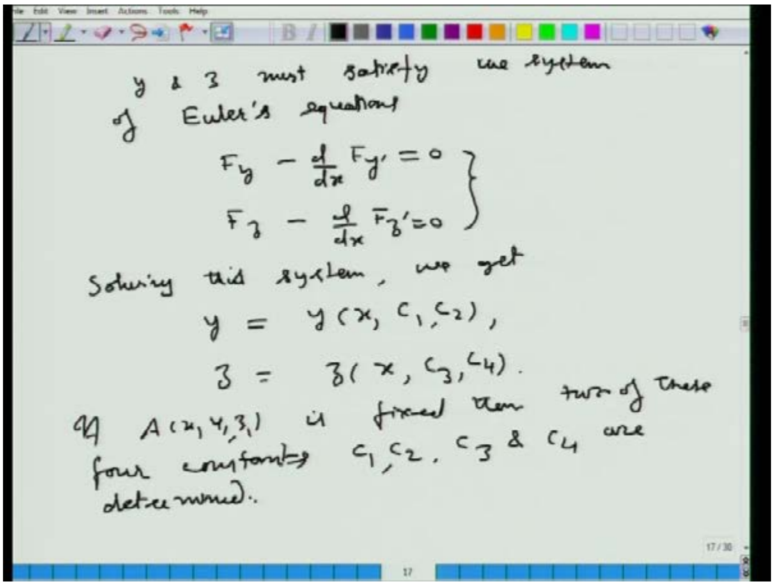

In this same manner this y and z must satisfy the system of Euler's equation F y minus d by dx F y prime equal to 0 and F z minus d by dx of F z prime equal to 0.

Whether the points A and B are moving or they are fixed, this necessary condition must be satisfied. So, we get y solving. The extremals are solving this system, we get y equal to y of x c 1 c 2 and z equal to z of x c 3 c 4. If A is fixed, then two of these four constants c 1, c 2, c 3 and c 4 are determined.

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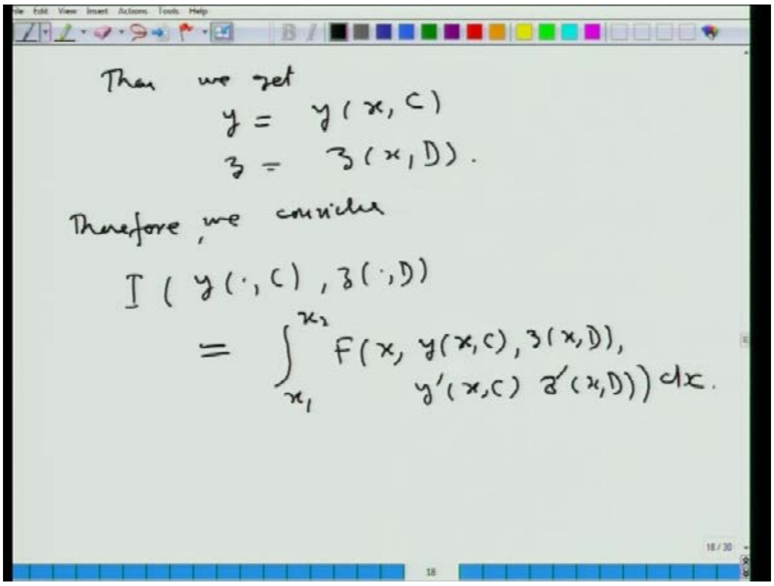

That means, then we get y equal to y of x let us say the parameter we write now C and z as z of x comma D. Therefore, we consider I on these sub families y of C and z D which is integral x 1 to x 2 F of x y x C z x D y prime x c z prime x D dx and the point B moves freely or moves in a curve or in a surface. So, that is what we consider here.

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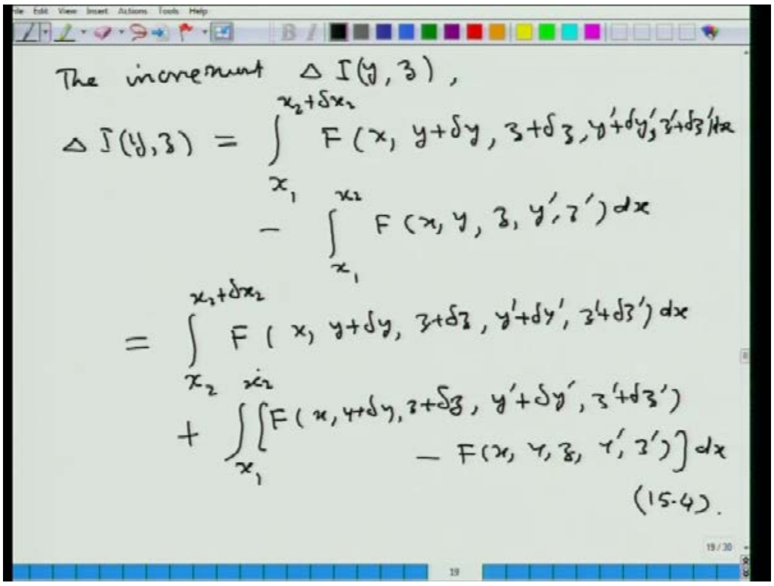

We consider this increment delta I(y). We will not write the dependents here, but it is understood that these y and z are the solutions of this system of Euler’s equation. So, that C and D dependents is always there; which we are not going to write it explicitly. It is assumed that we are taking y and z as solutions of. Give it the number. Here, this is 15.1 and this is 15.2 and this is 15.3.

We are taking these y and z as solutions as the functions in 15.3. We get this delta I y z here as integral x 1 to x 2 plus delta x 2 F of x y plus delta y z plus delta z y prime plus delta y prime z prime plus delta z prime dx minus x 1 to x 2 F of x y z y prime z prime dx. As before, we break this into two; the first term into two integrals that is x 2 to x 2 plus delta x 2 and then x 1 to x 2, which we will club with this second term. We get here F of x y plus delta y z plus delta z y prime plus delta y prime z prime plus delta z prime dx plus x 1 to x 2 F x y plus delta y z plus delta z y prime plus delta y prime z prime plus delta z prime minus F x y z y prime z prime.

This is 15.4. Now, here first term, we will use the Mean Value Theorem and with the second we will use Taylor's series expansion.

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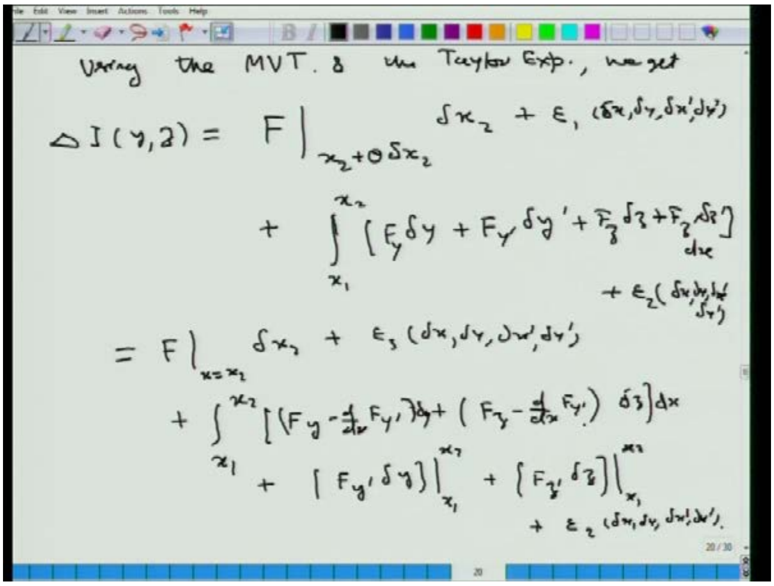

Using the Mean Value Theorem and the Taylor's expansion, we get this increment y delta I(y, z) as F evaluated at x 2 plus some theta delta x 2 times delta x 2 plus some epsilon 1. Here, it is delta x delta y delta x prime delta y prime and plus this integral you get, x 1 to x 2 F y delta y plus F y prime delta y prime plus F z delta z plus F z prime delta z prime dx plus epsilon 2 delta x delta y delta y x prime delta y prime.

And in the second term we will. Here, first term we use the continuity and this is F evaluated at x equal to x 2 delta x 2 plus let us say epsilon 3 delta x delta y delta x prime delta y prime. So, clubbing whatever remaining reminder here with this and plus we get x 1 x 2; shifting these derivatives, we get F y minus d by dx of F y prime plus F z minus d by dx of F z prime delta y here and delta z times dx and plus the boundary terms; that is F y prime delta y evaluated x 1 to x 2 plus F z prime delta z evaluated at x 1 to x 2 and plus this epsilon 2 there, whatever.

Now, since the functions y and z are solutions of Euler's equation, So, this term is 0 similarly this part of the integrand is 0.

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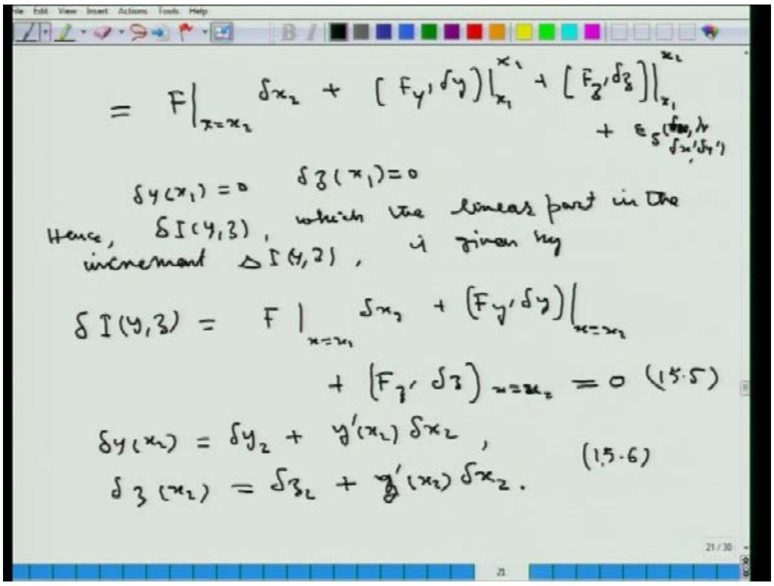

We get finally, this equal to F evaluated at x equal to x 2 delta x 2 and plus F y prime delta y evaluated at x equal to and also, here x 1 to x 2 plus F z prime delta z of this evaluated at x 1 x 2 plus; let us say, some epsilon 5 delta x delta y delta x prime delta y prime. Here, all these epsilons are functions of higher order terms in delta x delta y delta x prime delta y prime.

It is of little order o in those terms. Since A is fixed, delta y at x 1 is 0 delta z at x 1 is 0. Hence, the variation delta I y z which is the linear part in the increment, delta I y z is given by delta I y z equal to F x equal to x 2 delta x 2 plus f y prime delta y x equal to x 2 and plus F z prime delta z x equal to x 2 and this should be equal to 0.

Now, here as before this delta y at x 2 is not equal to delta y 2 and delta z at x 2 is not equal to delta z 2. And so, you can see that in the same manner we get delta y at x 2 approximately here we write it as y delta y 2 plus y prime at x 2 delta x 2. Similarly, delta z at x 2 is delta z 2 plus z prime at x 2 to delta x 2. So, let us say this is finally, we got this 15.5 and this is 15.6.

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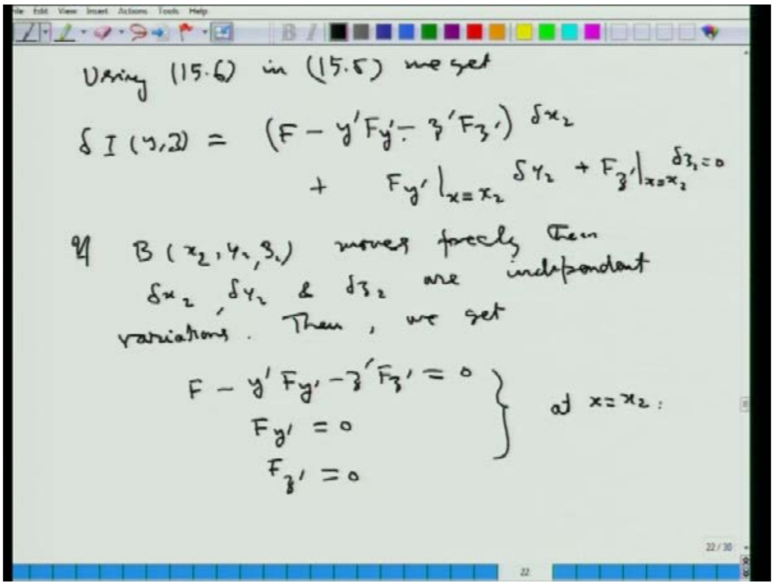

Using 15.6 in 15.5, we get this delta I at y z equal to F minus y prime F y prime evaluated at x equal to x 2 delta x 2 plus and the second term also, minus z prime F z prime times delta x 2.

So, these terms are coming from here. Substituting this here, we will have a prime y prime times delta x 2. Similarly, F z prime z prime times delta x. So, that is what is clubbed here and plus F y prime evaluated at x equal to x 2 delta y 2 plus F z prime evaluated at x equal to x 2 delta z 2 equal to 0. Now, if this B which is x 1 x 2 y 2 z 2 moves freely, then this delta x 2 delta y 2 and delta z 2 are independent variations. Then, we get these three conditions F minus y prime F y prime minus z prime F z prime equal to 0 F y prime equal to 0 F z prime equal to 0 at x equal to 0.

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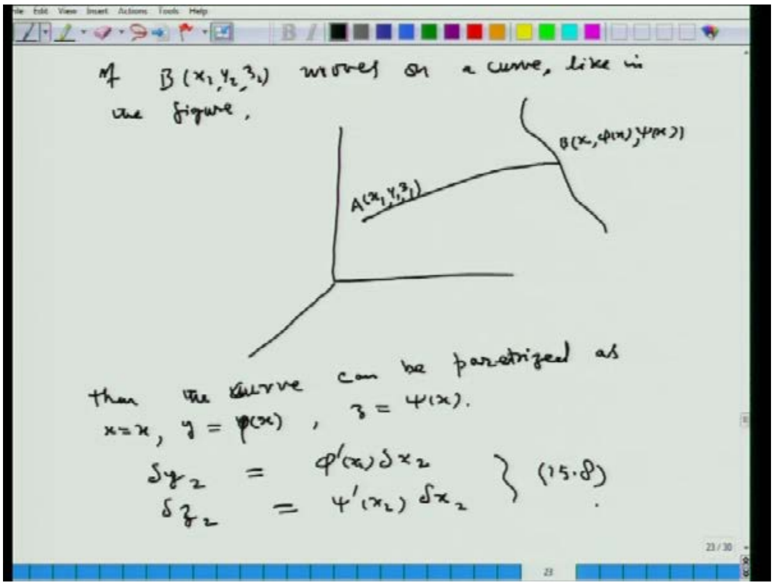

If this B x 2 y 2 z 2 moves on a curve, then here like in this figure. So, here A is like this; that is x 1 y 1 z 1 and this B is moving on this curve in three dimension that is parameterized by so like this b is parameterized here x phi of x and psi of x. Then, the curve can be parameterized as y. So, x equal to x; y equal to phi of x, and z equal to psi of x, then you have delta y at x 2 will be y prime at x 2 delta x 2. Similarly, delta z at x 2, this is phi prime at x 2 and this will be psi prime at x 2 delta x 2.

This delta y 2 and delta z 2; these variations will be then given by in terms of delta x 2 variations. Because the variations of this delta y and delta z cannot be independent, they will be given in terms of the variation of. If x moves, then these phi x and psi x also move. So, these delta y 2 and delta z 2 will be given by this. Let us say these as 15.7.and 15.8.

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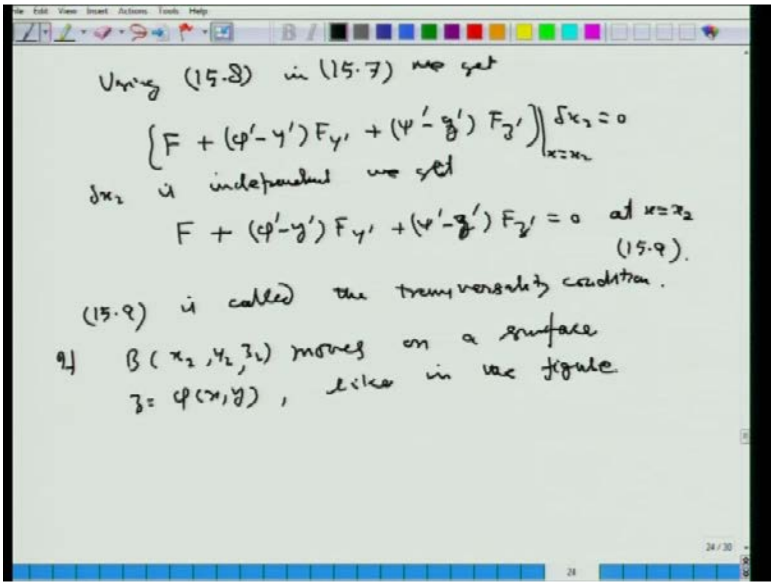

Using 15.8 in 15.7 we get, F minus F plus y prime minus y prime F y prime plus psi prime minus y prime of F z prime times delta x 2 equal to 0. So, that is what we will get here. Since delta x 2 is independent, we get F plus phi prime minus y prime F. This is evaluated at x equal to x 2. F y prime plus psi prime minus y. Sorry, it should be z prime. Here, F z prime equal to 0 at x equal to x 2.

So, this is the transversality condition. This is 15.9. 15.9 is called. As before, here this is the general case of the earlier one. Transversality condition; this will be evaluated at x equal to x 2. If point A is also moving we get and if it moves along the curve we get similar thing at x equal to x 1. So, we will have a transversality condition at that point. Now, if the other variation could be like this: B x 2 y 2 z 2 moves on a surface, this z equal to phi x y then.

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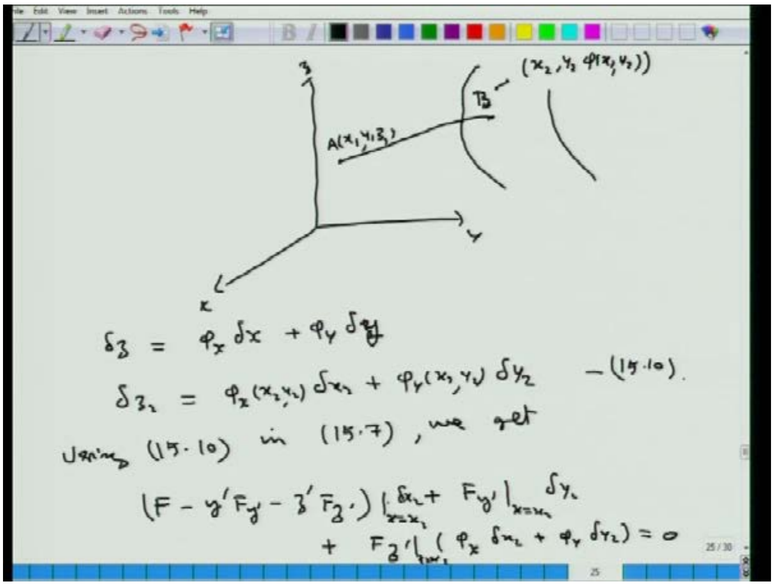

Like in the figure, you have this x y z . Here, A is fixed and this is the surface here; this B is moving on this. So, this is x y and z is phi of x y.

So, this will be moving on this; this will be x 2. Let me write this in terms of x 2; this B will be actually x 2 y 2 and phi of x 2 y 2. In this case we get this delta z as phi x delta x plus phi y delta y. Therefore, delta z 2 will be phi x evaluated at x 2 y 2 delta x 2 plus phi y x 2 y 2 to delta y 2. And so, substituting it here; this delta z 2 from this. Here this delta z 2, we substitute here and then collect the terms. So, we get the following ; let us say this is 15.10. Using 15.10 in 15.7, we get this F minus y prime F y prime plus phi x. This should be actually implicit. This one, when we substitute delta z 2 here in the last term, let us put that here. Here, we have F minus y prime F y prime minus z prime F z prime is evaluated at x equal to x 2 plus F y prime evaluated at x equal to x 2 delta y 2 and this F z prime and then here phi x delta x 2 plus phi y delta y 2. All these are evaluated at x equal to x 2. This is evaluated at x 2 and these are evaluated at x 2 y 2. So, this is equal to 0.

Now this will be collected with phi delta x 2. Here, this is delta x 2 ; this delta x 2 will be collected with this and this one will be collected with this.

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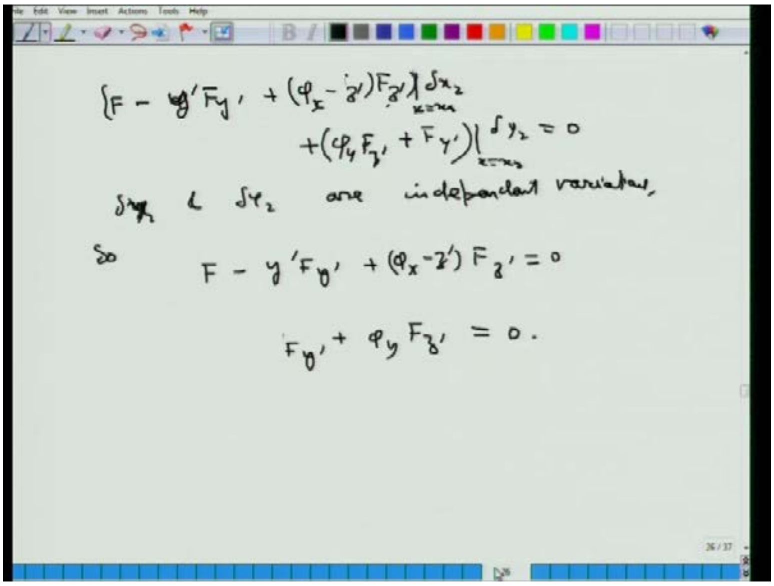

We will have F minus y prime plus phi y minus F y prime F … This F z prime with this; like this. F minus y prime F y prime plus phi x minus z prime of. So, this is (()) times delta x 2 plus this one ; phi x minus z prime. So, phi x minus z prime and similarly, this phi y plus F y prime; like this. So, plus phi y plus F y prime y prime to delta y 2.

All these are evaluated at x equal to x 2. This is also evaluated at x equal to x 2. Now, delta y 2 and delta x 2 and delta y 2 are independent variations. We get F minus y prime F y prime plus phi x minus z prime F z prime equal to 0 and phi y. This should have been phi y z prime; y prime here and phi y F z prime. This is F y plus phi y F z prime; this should be equal to 0, because here in this previous one, we have F z prime phi x which comes here in times delta x 2 with this. So, we get phi x with plus sign and minus z prime; that is phi x minus F z prime times delta x 2. And here, we have F y prime and then plus this phi y plus phi y times F z prime. So, that is what phi y F z prime. This is also evaluated at x equal to 0.

When we have this delta x 2 and delta y 2 independent variations we get the coefficients 0 here in this case because they are moving freely on the point B on the surface. So, delta x 2 and delta y 2 will be independent.

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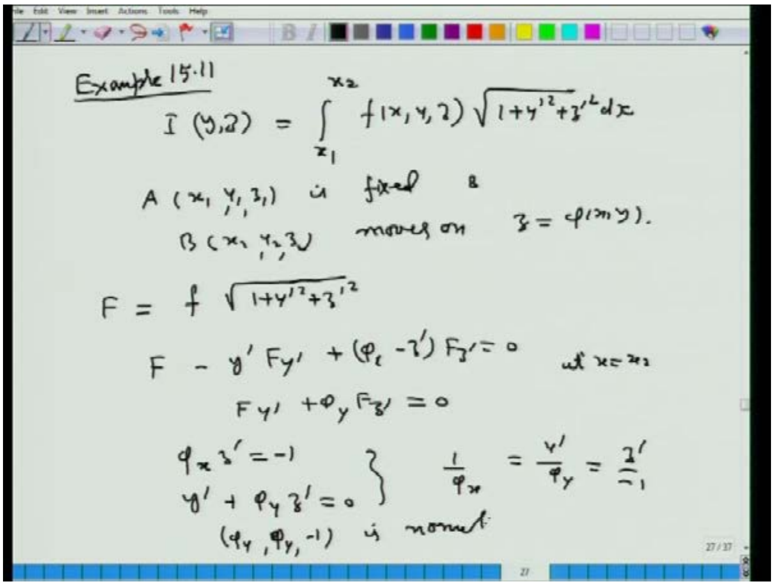

So, let us consider this example here. This one will be then number 15.11. So, I y z is the general case of the earlier one x 1 to x 2 f of x y z square root 1 plus y prime square plus z prime square dx. So, here A(x 1, y 1, z 1) is fixed and B(x 2, y 2, z 2) moves on the surface z equal to phi x y. Here in this case, we have F equal to little f times one plus y times square plus z prime square. Here, the condition F minus y prime F y prime plus phi x minus z prime F z prime equal to 0 and F y prime plus phi y F z prime equal to 0 at x equal to x 2 gets translated to this phi x z prime equal to minus one and y prime plus phi y z prime equal to 0. So, these two solving this we get 1 over phi x equal to y prime over phi y equal to z prime over minus 1. This, we see that this phi x phi y minus 1 is the normal to surface and this 1 y prime z prime is normal to the surface, z equal to phi x y and 1 y prime z prime is tangent to the extremal.

So, due to the lack of time we will be finishing in the next lecture. Thank you very much for viewing this.