**Calculus of Variations and Integral Equation**
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**Lecture No. # 14**

Welcome viewers to the NPTEL lecture series on the Calculus of Variations. This is the fourteenth lecture of the series. In this lecture, we will see that we will consider the functionals with moving boundary points. In the last few lectures, we have seen that we have considered various functionals, where the functions which are called extremals, which give us the optimal value of the functional, they are known as extremals, those extremals were subjected to pass through those points A and B; and these 2 points, A and B were fixed. Now, we will allow these points A and B to move freely in the xy-plane or freely on certain curves, and in high dimensions, these boundary points will be allowed to move either on a curve or on a surface. And, we will consider various such cases in this lecture and subsequent lectures.

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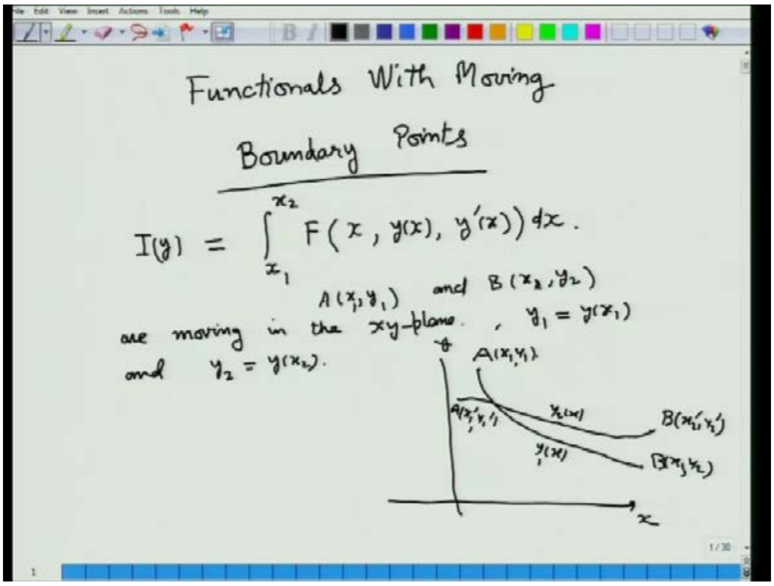


So here, recall that our functional I (y). Here, the simplest case which we considered earlier was like this: integral x 1 to x 2; the integrand function was dependent on 3 variables x y as a function of x and then its derivative y prime x dx. Here, this point A (x 1, y 1) and B (x 2, y 2) are moving in the xy-plane. Here, this y 1 is equal to y at x 1 and y 2 is y at x 2.

So, situation is the following. Here, we have this xy-plane and we have these 2 points: A which is (x 1, y 1) and B (x 2, y 2) and there is a curve this y x which pass through these 2 points. Now these 2 points are allowed to move. So, A can move here and B can move here, like this. So, this is A dash now, or A (x 1 dash, y 1 dash) and B is (x 2 dash, y 2 dash). Here, A and B can move in this xy-plane and this y. So, let us say this is y 1 and this is y 2 and so on. We can consider various such cases.

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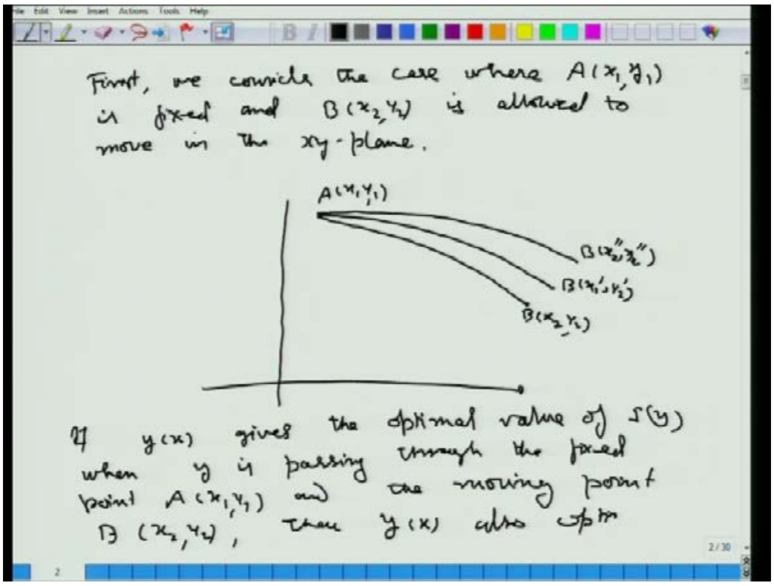

So, first we consider the case, where this A, which is (x 1, y 1) is fixed and this B (x 2, y 2) is allowed to move in the xy-plane. So here, we have the following picture now; here, let us say this A is this one and now this is fixed and this B is allowed to move. So, let us say this is one curve B (x 2, y 2) and then, this is moving somewhere here B (x 2 dash, y 2 dash) and likewise, it keeps on moving; B (x 2 double dash, y 2 double dash) like that. This B is allowed to move; it can move anywhere or it can move along a curve, where this will be then a constrained movement and we will consider these 2 cases separately.

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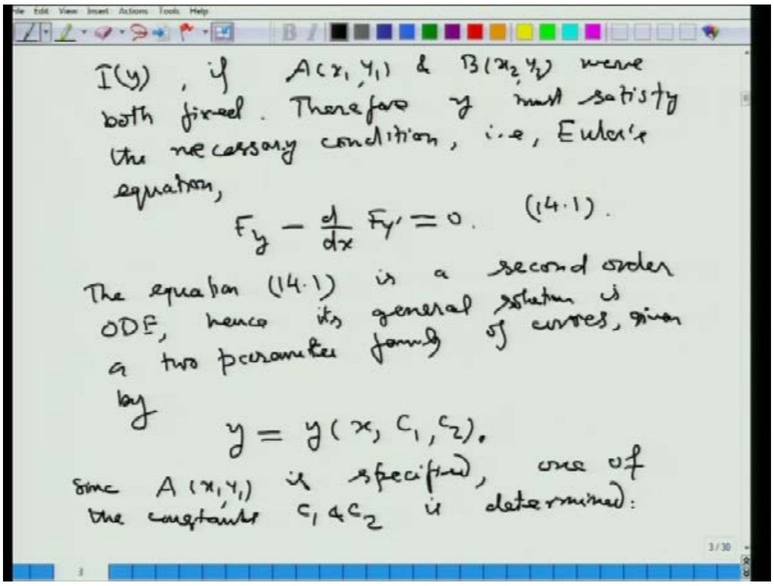


So here, this is called pencil of curves. And, if this y x optimizes, it gives the optimal value of I y, when y is passing through the fixed point A (x 1, y 1) and the moving point B (x 2, y 2). Then it also y x also optimizes I y if A (x 1, y 1) and B (x 2, y 2), both fixed. Therefore, y must satisfy the necessary condition that is Euler's equation F y minus d by dx F y prime equal to 0. Let us say this is 14.1.

The equation 14.1 is the second order ODE; hence, its general solution is a 2 parameter family of curves, given by y equal to y (x, C 1, C 2), where C 1 C 2 are arbitrary constants behaving as parameters here. Since, A (x 1, y 1) is specified, one of the constants C 1 and C 2 is determined.

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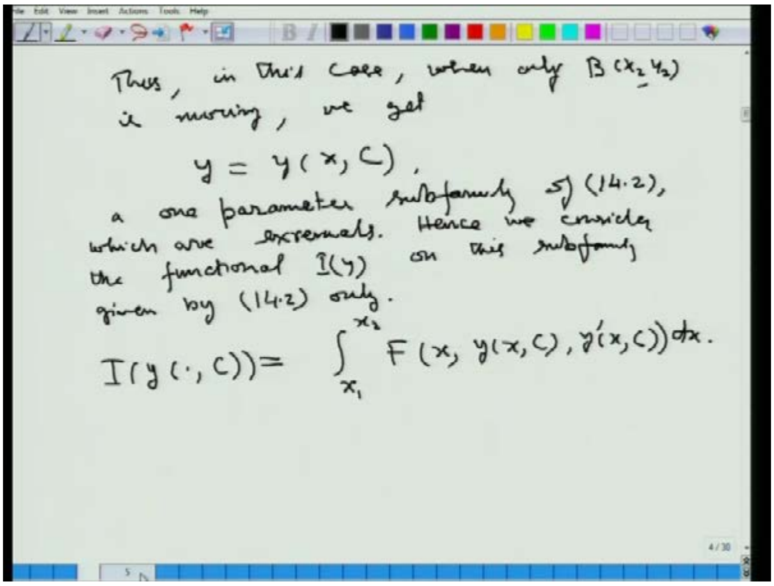


Thus, in this case, when only B (x 2, y 2) is moving ,we get y equal to y of x. One of the constants we denote as C, if one parameter sub family of 14.2 which are extremals, because they are the solution of an Euler's equation. And so, they are known as extremals. Hence, we consider the functional I y on this sub family, given by 14.2 only. So, we consider here I y, which is function of x and then c, like this; integral x 1 to x 2 F of x y x C and y prime x C dx. Now, we will consider the variation of this functional over this family of extremals and then derive the necessary condition.

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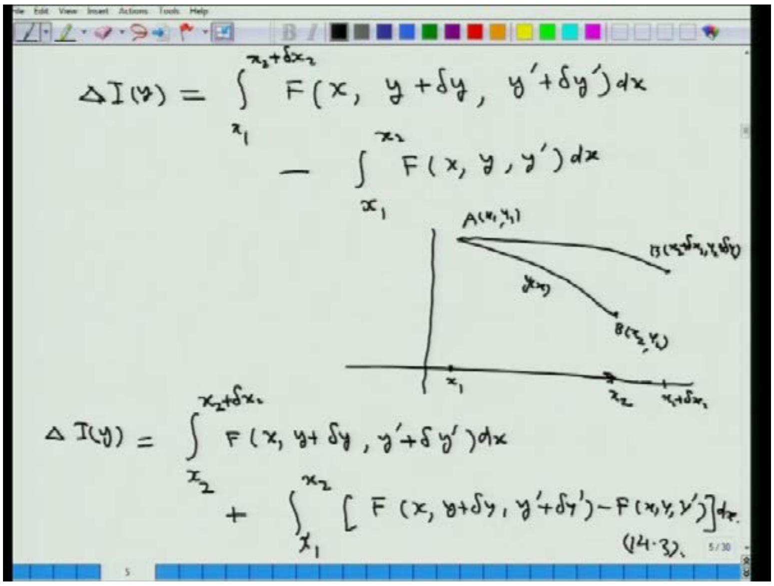

So, we consider here delta I y, which is the variation of; rather, first we considered the increment and then the linear part in the increment; so, this capital delta is denoting the increment and small delta will be denoting the variation. So, this will be x 1 to x 2 plus delta x 2 F of x. We will not denote the dependence on C; it is understood that wherever we are taking y is only extremal, the solutions of 14.2. So, y plus delta y they are functions of x and C, both and y prime plus delta y prime dx minus x 1 to x 2 F of x y y prime dx. Here, picture is like this: A is fixed and now B, let us say one is y x here, B (x 2, y 2) and now this point moves to somewhere here. So, this point is x 1 and this is x 2. So, x 2 gets incremented by delta x 2, so, this is x 2 plus delta x 2 here and then so this is, B (x 2 plus delta x 2 , y 2 plus delta y 2).

Here, now we have this difference: the capital delta I, given by this. So, this delta I y can be written as; we will break this x 1 to x 2 and x 2 to x 2 plus delta 2. So, first term we will write as, x 2 to x 2 plus delta x 2 F of x y plus delta y y prime plus delta y prime dx and then, plus x 1 to x 2 and one term will come from here. So, collectively we get F of x y plus delta y y prime plus delta y prime minus F of x y y prime dx. This is, let us say as 14.3.

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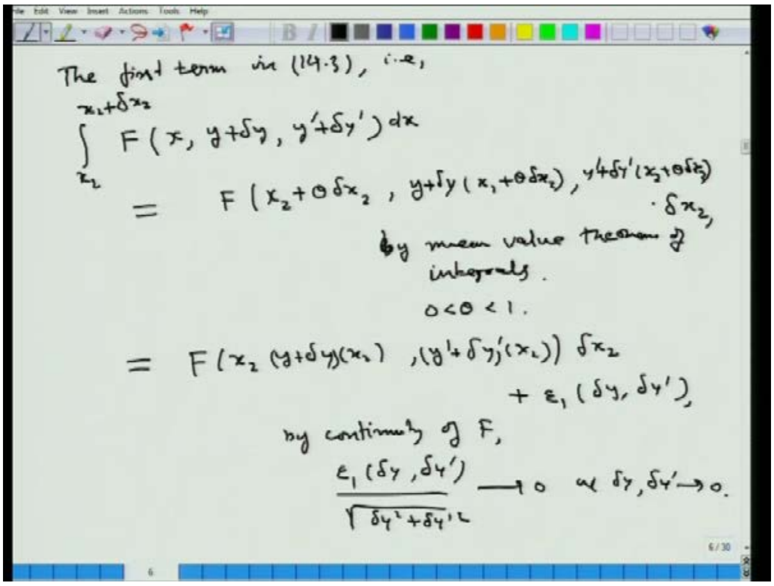


Now, the first term in 14.3, that is x 2 to x 2 plus delta x 2 F of x y plus delta y and y prime plus delta y prime dx. Here, we apply the Mean Value Theorem and write it as F at x 2 plus theta delta x 2 and y plus delta y evaluated at x 2, plus theta delta x 2 and y prime plus delta y prime evaluated at x 2, plus theta delta x 2 times delta x 2; that is, the length of the interval x 2 to x 2 plus delta x 2. So, this is by Mean Value Theorem of integrals, where theta lies between 0 and 1. And, this can also be written by continuity that F evaluated at x 2 and y plus delta y evaluated at x 2. Here, these are all dependent on C also; y prime plus delta y prime at x 2 delta x 2 plus some epsilon 1 which is function of delta y and delta y prime; this is by continuity of F, we are assuming that F is continuous in all arguments and this epsilon 1 delta y delta y prime divided by squared root delta y squared plus delta y prime squared goes to 0 as delta y delta y prime tend to 0. That is the result of the first term.

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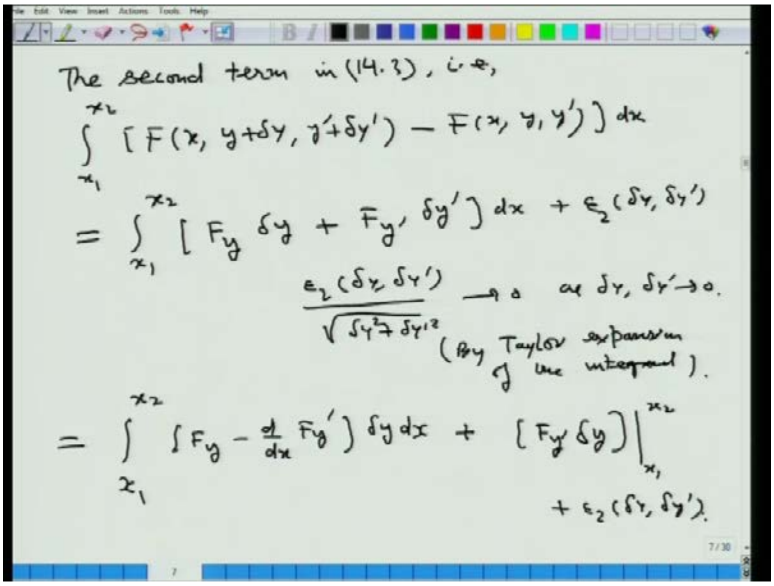


The second term in 14.3, that is x 1 to x 2 F of x y plus delta y y prime plus delta y prime minus F of x y y prime dx, is treated using the Taylor theorem. This can be written as, integral x 1 to x 2, F of partial derivative with respect to y delta y plus F of y prime delta y prime times dx plus some epsilon 2, which is again function of delta y and delta y prime; and, epsilon 2 delta y delta y prime divided by y squared delta y prime squared; this tends to 0 as delta y delta y prime tend to 0. That means, this epsilon 1 and epsilon 2 are of higher order in delta y and delta y prime.

So, this is by Taylor series expansion of the integrand; we expand it by Taylor series, and take only these linear terms and non-linear terms are put here, which are satisfying this property. Here, we can then shift this derivative as we had been doing earlier. So, this can be then written as integral x 1 to x 2 F y minus d by dx of F y prime delta y dx plus the boundary term, that is coming from here; F y prime into delta y, evaluated at x 1 to x 2 plus epsilon 2.

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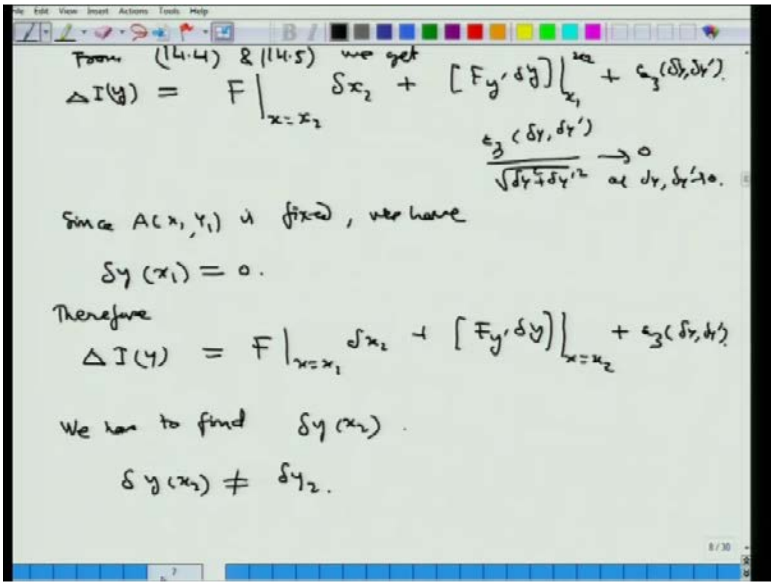


So, then collectively we get this delta I y. Let us put some number here: 14.4 and this is 14.5. From 14 .4 and 14.5, we get this one as F; we simply write this as, all those arguments evaluated at x equal to x 2 and this is delta x 2 plus we get F y prime delta y, evaluated at x 1 x 2 plus some epsilon 3. Since, F y minus d by dx of F y prime is 0 here, that is what is used here; the first term here, or let us write it here itself. This gives us F y prime delta y, evaluated at x 1 to x 2 plus this epsilon 2 delta y delta y prime. Put it here up; epsilon 2 delta y delta y prime and this is equal to this, as this F y minus d by y of F y prime equal to 0, because y being extremals, it is a solution of the… We can remove this from here, which is put there already. So, delta y delta y prime, where epsilon 3 is then satisfying the same; 0 as delta y delta y prime tend to 0.

Now, since A which is (x 1, y 1) is fixed, we have delta y at x 1 equal to 0 as before and therefore, this delta I y is F at x equal to x 2 delta x 2 plus F y prime delta y, evaluated at x equal to x 2 plus this epsilon 3, which is delta y delta y prime.

Here, we observe that we have to find delta y at x 2 which is actually, this delta y at x 2 is not equal to delta y 2. See that in the figure, this delta y 2 is the increment; this is y 2 and it has gone upto here. So, this is delta y 2; this distance is delta y 2. But this is not equal to delta y at x 2, which is actually equal to… We will see that at y 2 delta y at x 2 would be this distance completely. That is not equal to delta y at x 2. So, let us find out this delta y at x 2.

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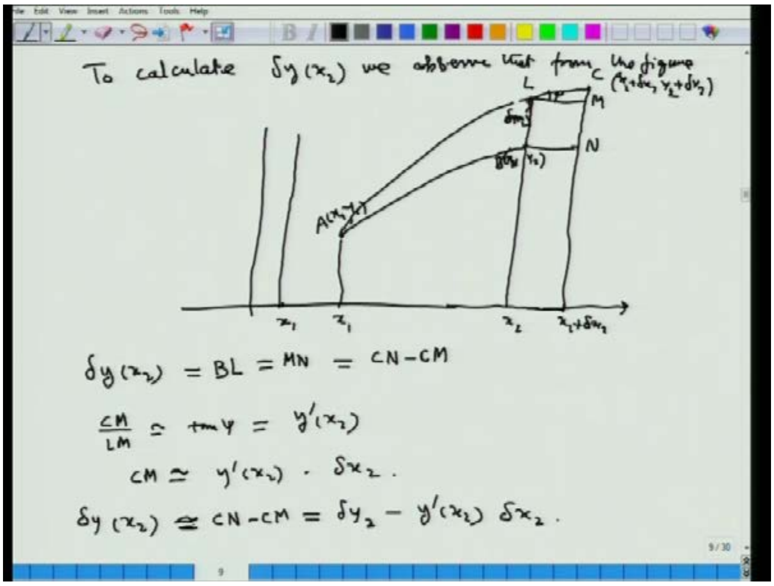


To calculate this delta y at x 2, we observe that from the figure that A is here and this is (x 1, x 2) and this is B (x 2, y 2) and now, it has gone here, B x 2 x 2 plus B is moved here (x 2 plus delta x 2, y 2 plus delta y 2). This is x 2; this is x 1 here. This says x 2 plus delta x 2; and, here we will have something like this; this is x 2. So, this one is delta y at x 2. Let us write it slightly different, because here the figure is not coming properly.

So, let us write it in a better way. Let us take A, here down so that x 1 y 1 and B, up here. And, now it has moved to here this B (x 2 plus delta x 2, y 2 plus delta y 2). This is x 1, this is x 2; extend it here. So, this is the rectangle. Let us give the names here. This is A, this is B; put it up there; let us say B (x 2, y 2). And then, let us give a different name, rather C. So, B is moved to C and this is let us say, L M and N, here.

We see that this delta y at x 2 is BL. Look at x 2; this is the increment that is what, delta y at x 2. Because delta y is the increment at each point x here; so, when x moves to x 2, we get delta y at x 2 here. And, this is actually equal to MN; BL is equal to MN and it can be written as, CN minus CM. So, delta y at x 2, BL equal to MN and this is CN minus CM; that is what is MN? Now, this CM over LM is approximately the tangent of this angle, let us say this is some psi here; so, tangent of psi that is y prime at x 2. So, here this is approximately y prime at x 2 and we get CM equal to y prime at x 2 into LM. LM is the increment that is delta x 2. This is approximately like this. And therefore, this CN is nothing but delta y 2. Therefore, delta y at x 2 is CN minus CM, which is delta y 2 minus y prime at x 2 into LM, we have written as delta x 2. Of course, this is approximately here. This is what we use in this delta y at x 2.

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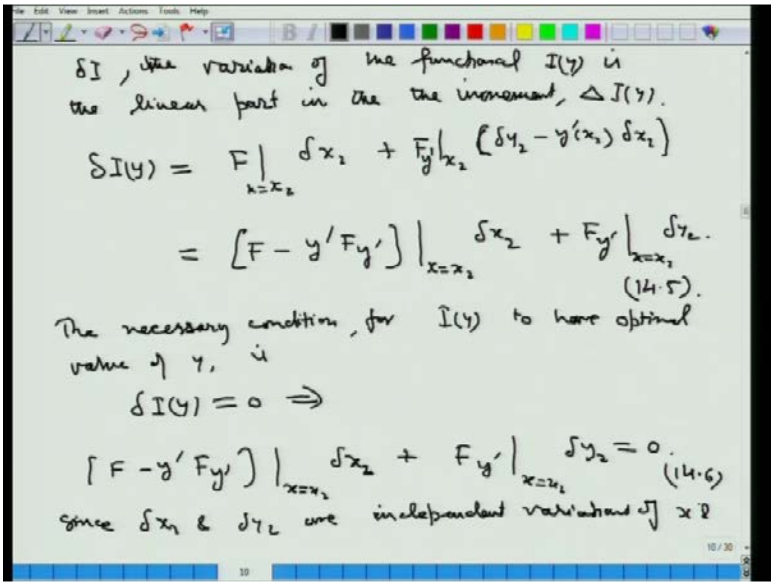


We know that this delta I, which is the variation of the functional I y is the linear part in the increment, delta I y. Here, we see that the linear part of this and this is the non-linear part; so, we drop that non-linear part here. So, delta I y is equal to F evaluated at x equal to x2 and delta x 2 plus here, this F y prime and delta y evaluated at x 2. We get F evaluated at x 2 and now delta y, evaluated at x 2 is written from here, like this: delta y 2 minus y prime at x 2 into delta x 2 times here. So, that is what we get here and this can then be simplified, F evaluated at x equal to x 2; taking this with this common; we take F minus y prime F; this was F y prime; here, it was F y prime. So, y prime into F y prime, this evaluated at x equal to x 2 into delta x 2. So, we combine this term with this first one and plus F y prime, evaluated x equal to x 2 into delta y 2. So, this is let us say, 14.5.

The necessary condition for I y to have optimal value at y is that delta y delta I at y must be equal to 0. So, this implies that F minus y prime F y prime (x) equal to x 2 delta x 2 plus F y prime, evaluated at x equal to x 2 delta y 2 equal to 0. We call these delta x 2 and delta y 2 were increments in this point, x 2 y 2. They are arbitrary; we can take delta x 2, any number here and delta y 2 any number. So, these are arbitrary and since delta x 2; we call it as 14.6; delta x 2 and delta y 2 are independent variations of x and y; this x 2 and y 2.

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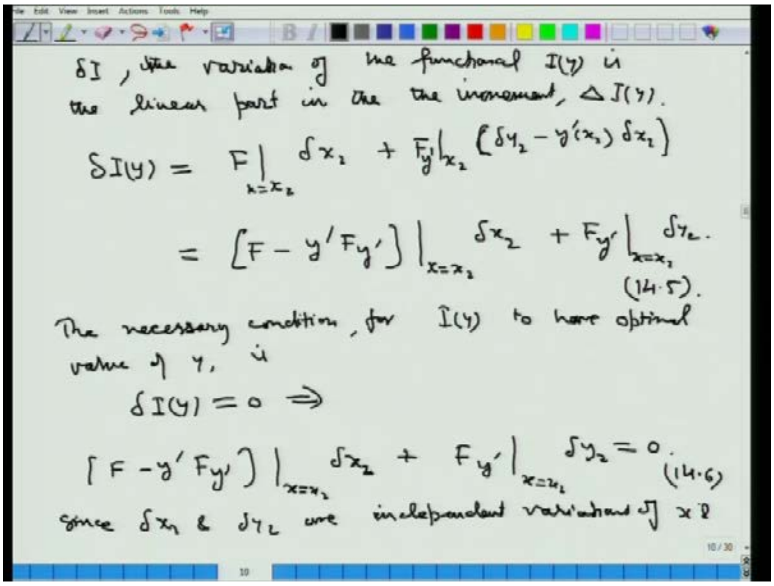


Here, we can take delta y 2 0 and delta x 2 equal to 1; then, we see that this coefficient must be 0. Similarly, the other round will tell us this coefficient must be 0. F minus y prime F y prime at x equal to x 2 must be 0 and F y prime at x equal to x 2 must be 0. These are the necessary conditions for this functional to have optimal value. So that, we have to choose that point x 2, for which these conditions are satisfied. Now, if the point x 2 y 2 moves on the curve, y equal to phi x; then we have this delta y at x 2 will then be phi prime at x 2 delta x 2. Hence, we get F minus y prime F y prime at; so, this collectively for both; at x equal to x 2 delta x 2 plus F y prime delta y 2 x equal to x 2. Here, delta y 2 rather 0, reduces to F minus y prime and then, you can take this delta y 2 here x 2 delta x 2 and substituting it here, you get plus phi prime x 2 F y prime at x 2 evaluated at x equal to x 2 into delta x 2 equal to 0. So, here delta x, we can take common. F plus phi prime minus y prime of F y prime, evaluated at x equal to x 2; since delta x 2 is arbitrary, the coefficient must be 0. This is known as transversality condition. Let us call it as 14.7.

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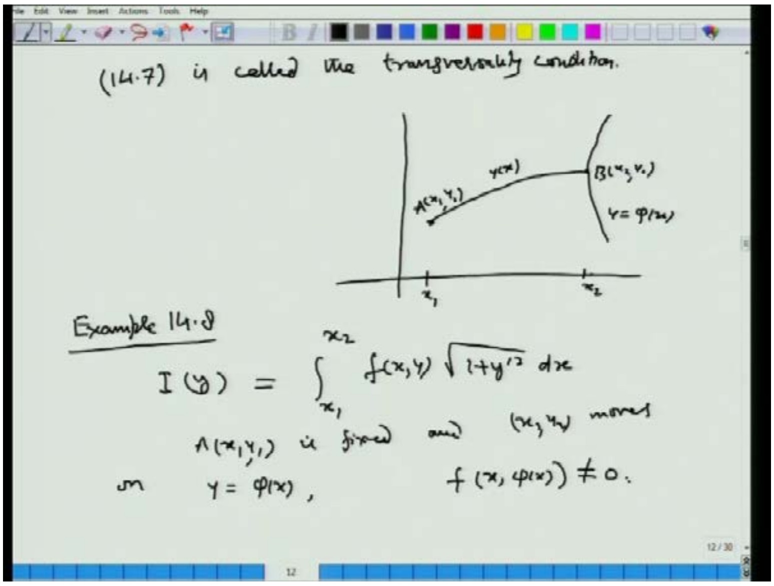


14.7 is called the transversality condition. Here, A is (x 1, y 1) and then this B is moving along this curve y equal to phi x; this is B that is (x 2, y 2) here. So, we have to select that x 2 here ; x is moving here this is x 1 is fixed and x 2 is moving here; so we have to select that x 2 , so that x 2 y 2 is on this curve and this optimal extremal y x is giving us the optimal value of I. So, that is what is required. Here, we will find the condition that is what is known as the transversality condition. So, this condition must be satisfied at x 2 in order this extremal y x, which joins this fixed point (x 1, y 1) and this moving point B, that is (x 2, y 2). So that, we have to select that point here on this curve for which this transversality condition is satisfied.

So, let us see in this example; let us consider this functional x 1 to x 2 on function f x y squared root 1 plus y prime squared dx. Here, f x y is assumed not equal to 0 for all x y or rather the moving point; this is not 0. Here this point, A (x 1, y 1) is fixed and this x 2 y 2 moves on y equal to phi x and here, this f of x 2 y 2, such that this is not 0. That means, this f x y is not 0 on phi x. Here, we will put it because y 2 we do not know; we put it like this: f x phi x, this is not equal to 0.

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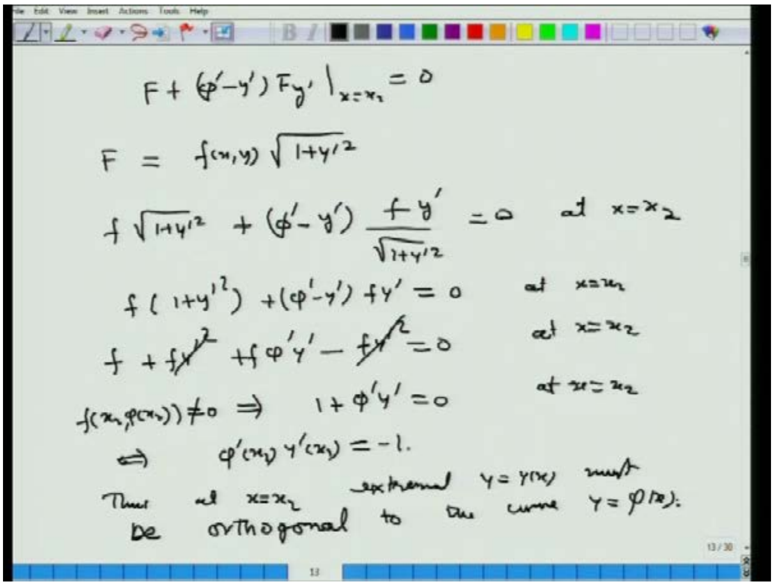


So, here we see that this F plus phi prime minus y prime F y prime, this x equal to x 2, this should be equal to 0; so that is the transversality condition here. And now, F is f(x, y) squared root 1 plus y prime squared. So, putting it here; so, we get f root 1 plus 1 y prime squared plus phi prime minus y prime into this f over squared root 1 plus y prime squared into y prime; this is at x equal to x 2 or, let us put it this equal to 0 at x equal to x 2. So, simplifying this, we get f into 1 plus y prime squared plus phi prime minus y prime to f y prime equal to 0, or f plus f y prime squared plus phi prime f phi prime y prime minus f y prime squared equal to 0, so, this cancels. So, we get, taking f here, all this at x equal to x 2 at x equal to x 2.

So, we had assumed that here, this f since x 2 y 2 are moving on f, and we have f at x 2 and phi x 2 will not be 0. So, this implies that 1 plus taking f out, so 1 plus phi prime y prime equal to 0 or at x equal to x 2. So, this implies that phi prime at x 2 y prime at x 2 equal to minus 1; so that means, this curve y prime is the tangent here and phi prime is tangent on this curve. So, here at this point they should be orthogonal. So, at this point here the tangent here and tangent to this curve, they are orthogonal at x equal to x 2 extremal, y equal to y x must be orthogonal to the curve, y equal to phi x. So, the transversality condition is the orthogonality condition of these extremal and the given function, on which a given function phi on which the point x 2 y 2 move.

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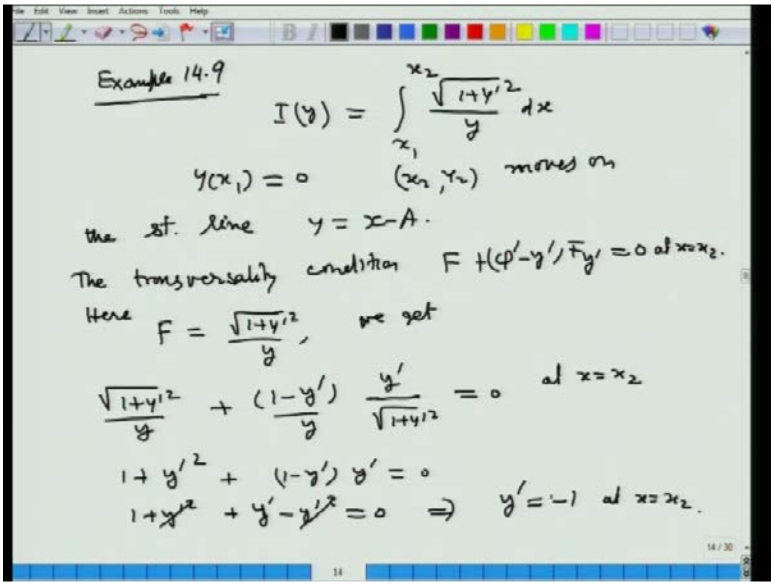


Now, the next example; this is 14.8 sorry, 14.9, we have to take. So, in this I y is x 1 to x 2 root 1 plus y prime squared over y dx and y at x 1 is 0 and x 2 y 2 moves on the line, the straight line y equal to x minus A. So, here the transversality condition F plus phi prime minus y prime F y prime equal to 0 at x equal to x 2, so that is what we get here in this case. So, here phi x is this x minus A, so phi prime is 1 and here, F is root 1 plus y prime squared over y. So, we get root 1 plus y prime squared over y plus this y prime is 1 here, so, 1 minus y prime did F y prime means, this y will come here, 1 over root 1 plus y prime squared, and then 1 over 2 and then that 2 y prime. So, this gives us y prime.

This must be equal to 0 at x equal to x 2. Simplifying this, we get that 1 plus y prime squared plus 1 minus y prime y prime equal to 0, or 1 plus y prime squared plus 1 minus y prime squared equal to 0; this is y prime minus. So, this cancels this one and so, we get y prime equal to minus 1 at x equal to x 2. So, here y prime should have this value, minus 1.

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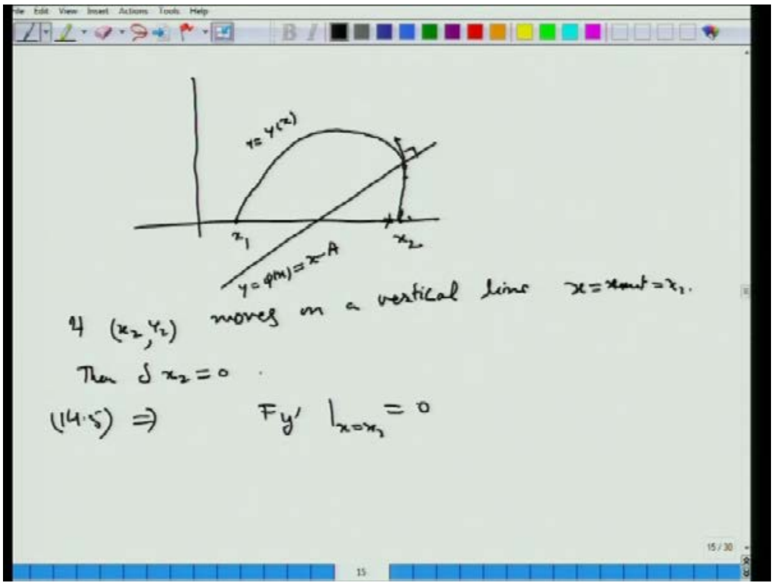


So, let us see in this case, what we have is the following: here, this is x 1 here and x 2 is moving here and this is the line y equal to x minus A, and so x 2 is moving here; so that x 2 we have to choose, so that this curve hits this one orthogonally here this should be... So, this... So, what is actually x 2 here, so that this is 90 degree in this case. So, this is the extremal, y x y equal to y x and this is y equal to phi x which is x minus A, given here. So, let us remove it from here.

So, this is. So, at this point x 2, we can see that here we get 90 degree angle, because y prime equal to minus 1 here. So, transversality condition again is the orthogonality condition in this case. And now, we can take other cases like this line is vertical if x 2 if x 2 y 2 moves on a vertical line that x equal to x 2 x equal to constant x 2 x equal to constant that is equal to x 2; x 2 is fixed, only this one is moving, then we see that then delta x 2 is 0 and therefore, the first one here this delta x 2 0, so, we get only this second term in 14.5. 14.5. then implies that F at F y prime at x equal to x 2; this must be equal to 0, because delta y 2, there is… See this delta x 2 is 0 in this, so this term is gone and you get F y prime at x equal to x 2 times delta y 2 equal to 0; delta y 2 is arbitrary and so, we get this. Thank you very much for viewing this.