Calculus Of Variations and Integral Equation

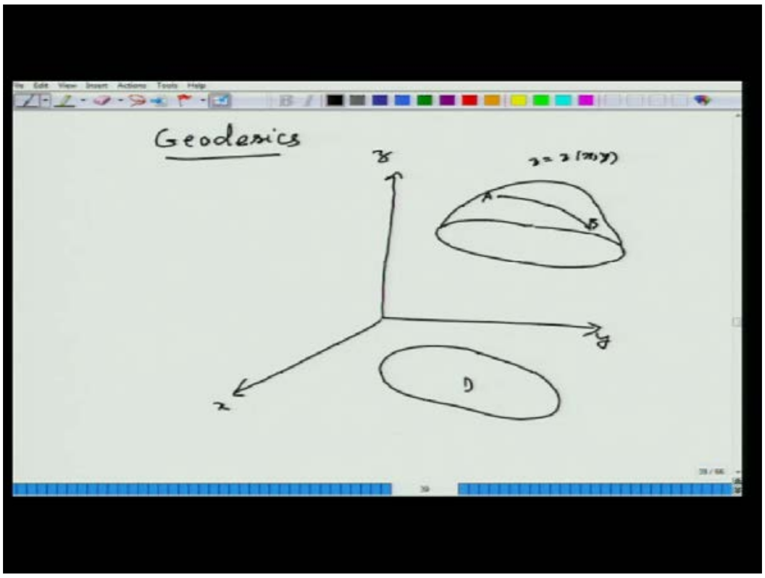
Prof. Dhirendra Bahuguna
Prof. Malay Banerjee
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur

Module #01
Lecture #13

Welcome viewers to the NPTEL lecture series on the Calculus of Variations this is the 13th lecture in this series. In the last lecture, we had discussed isoperimetric problems, now we will discuss in this lecture, the problem of finding geodesics.

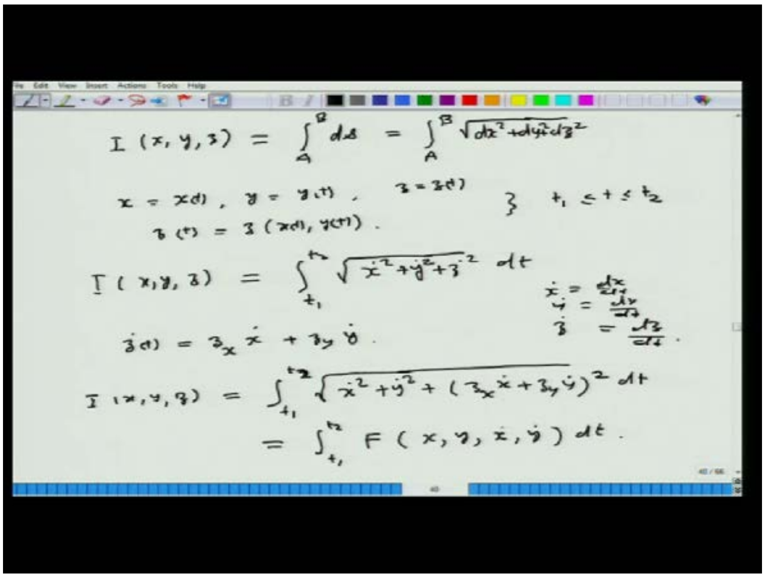
Geodesics are defined as the straight lines on certain surfaces that means, when you stretch those surfaces as flat surfaces, then these will reduce to straight lines. So, they have the minimum length that means, in the neighborhood of these curves, if you consider any other curve joining two given points, then that curve is having the least length that is what is called geodesics.

(Refer Slide Time: 01:09)



Now, this can be described geometrically like this, that you have a domain D in x y plane and this is the surface, this is described by like this z of function of x y, so here there are two points given a and b and a curve joining this, these two points a and b lying on this surface. Now, we want to find that curve on this surface joining these two points, which will have minimum length, so that is what will be called geodesic on this curve, this surface given by z equal to z of x y.

(Refer Slide Time: 02:00)

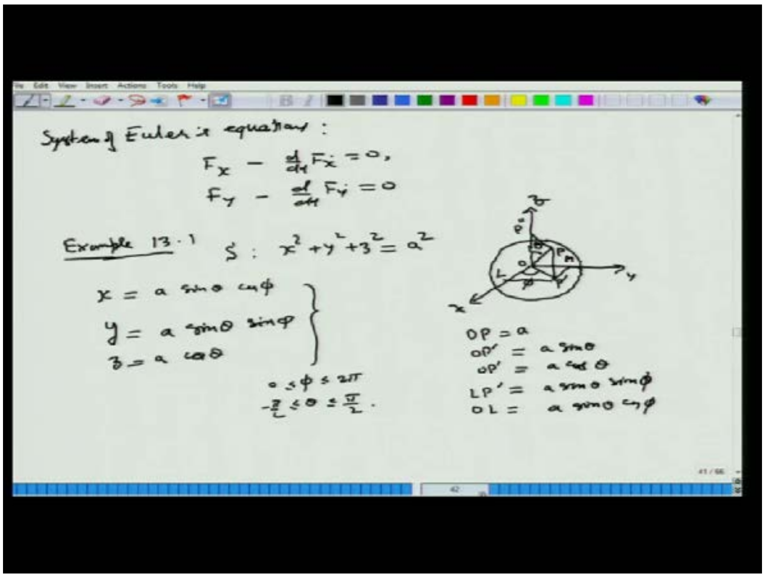


So, here we will take the example here like this, so you will have this functional I y I of x y z and here this will be joining two points, let us say a and b and then, this arc length on this surface is given by d s. And this will be, integral it to up to a to b and square root of d x square plus d y square plus d z square. So, if you parameterize this x as x of t, y as y of t, and z as z of t such that, on this surface z t is given as function of x t y t like this.

So, here you can see that, then here is I x y z, then will be given, here t have this ranges t 1 less than equal to t less than equal to t 2, t 1 to 2 square root x dot square plus y dot square plus z dot square and d t. Here, x dot is d x by d t, y dot is d y by d t and z dot is d z by d t. And here, this x y and z are satisfying this relation.

So, that is what we will have more generally, we can have the following situation here, or this z t z dot t will actually be then z x x dot plus z y y dot. And so, I of x y z will be given by t 1 to t 2 square root x dot square plus y dot square plus z x x dot plus z y y dot square d t. So, this is functional of this type t 1 to t 2 where F, here t does not appear explicitly, only x y, and because this z x and z y will be function of x and y. So, and x dot y dot d t, here t does not appear explicitly, so it is of this type.

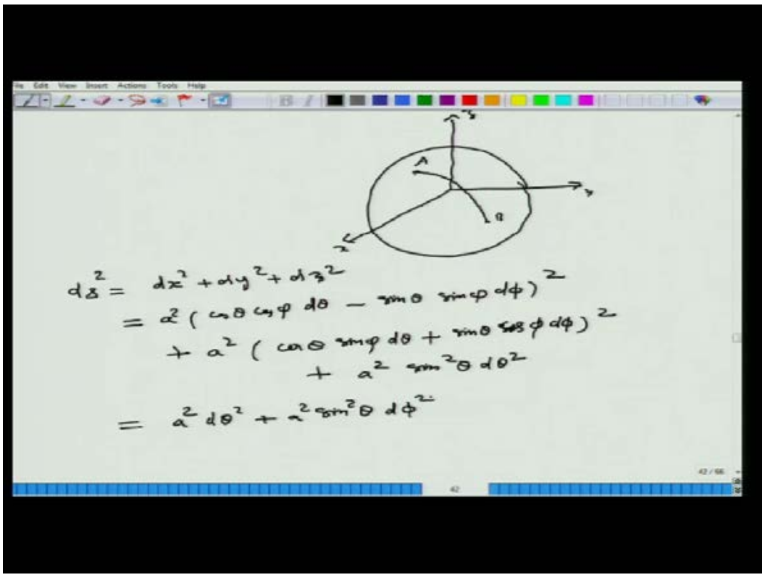
(Refer Slide Time: 05:16)



And so, you can see that here you have Euler's, the system of Euler's equation equation given by F x minus d by d t of F x dot equal to 0, F y minus d by d t of F y dot equal to 0, here because you have only x and y are appearing in this and x dot y dot. So, let us take this example, he knows the answer of this, let us take this sphere. So, our surface as here is given by x square plus y square plus z square equal to a square, so this sphere here. And so, we parameterize this x as a sin theta cos phi, here any point p on this sphere, you can see that this vertical angle is theta, this angle is theta and this horizontal angle is phi. So, x y z are given like this, a sin theta sin phi, and z equal to a cos theta, because here p is o p o p is equal to a, and when we projected here.

So, let us say this is p dash, so o p dash is a sin theta. And so, let say this 1 is e double dash and o p double dash that is the jet coordinate that is a cos theta. And so, this x coordinate will then be let us say, this is L and this is M then l p dash that is the y coordinate which is a sin theta sin phi. And projection of o p dash on y axis is similarly o l, equal to a sin theta, this is phi sin theta cos phi. So, that is what the parametric representation here is, and here phi is ranging between 0 to 2 pi and theta is ranging between minus pi by 2 to plus pi by 2.

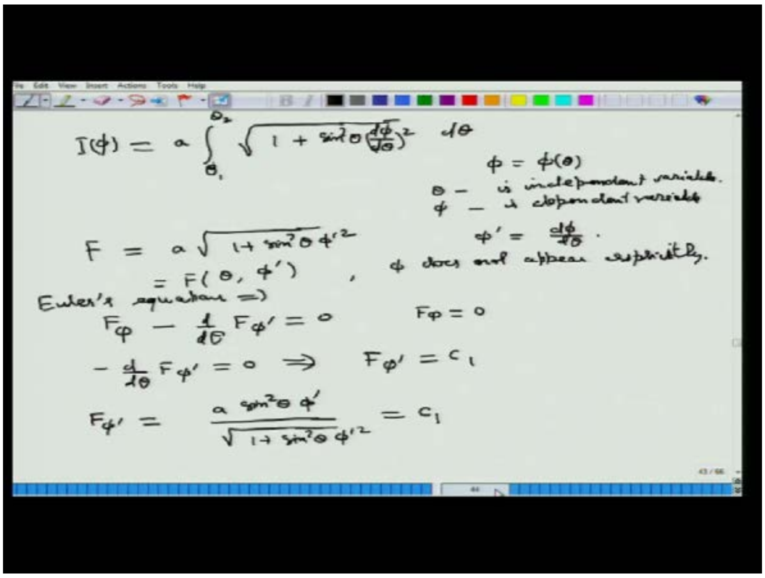
(Refer Slide Time: 09:33)



So, here on this surface, we have the situation like this and given any two points here, A and B. So, this will be the kind of curve on this surface and we want to find minimum length curve on this surface.

So, here we see that d s is d s square is d x square plus d y square plus d z square. And this is equal to a square cos theta cos phi d theta minus sin theta sin phi d phi square plus a square cos theta sin phi d theta plus sin theta cos phi d phi square plus a square sin square theta d theta square. Squaring and summing it up gives us the following, that is a square d theta square plus a square sin square theta d phi square.

(Refer Slide Time: 11:24)

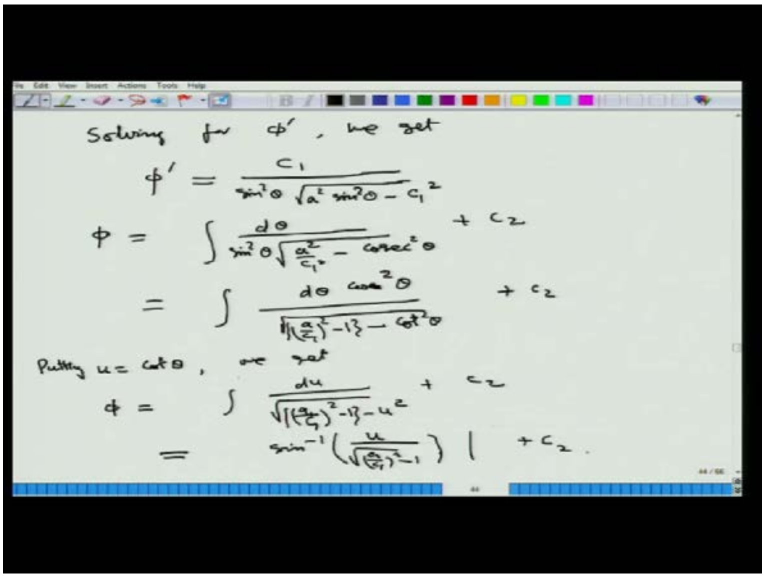


Hence this, the length here and x, so here, it will be then function of theta and phi now only. So, here I of, so the length of, length of this, that is the dependence will show later. Here, this is some independent variable, here we will take as theta as function of phi. And so, we will have square root 1 plus a will come out, and sin square theta and you will have d phi over d theta square and d theta, here we treat that phi as function of theta. Because here, on this, this surface is described by these parameters theta and phi, so a curve on this surface curve on this surface will be function of one of the variables, either phi as a function of theta or theta as a function of phi.

So, we can take anyone as independent variable. So, here theta is independent and phi is dependent variable dependent variable. So, this is the functional we have here, which gives you the length of, so here, it will be function of this phi. And so, that is what we will write it as I phi and this is line between two angles; here theta 1 and theta 2, integration over theta have line between theta 1 theta 2, where they will be describing theta 1 will be for this a and theta 2 will be for d. So, that is what we have here.

So, for this F is now a square root 1 plus sin square theta phi dash square, where phi dash means d phi by d theta. So, here this is function of theta and phi prime, phi does not appear explicitly. So, here you will have the Euler's equation. So, Euler's equation implies F phi minus d by d theta, in place of x we have theta, F phi prime equal to 0. And here, F phi is 0, so we get minus d by d theta of F phi prime equal to 0; this implies that F phi prime equal to c 1. So, for centrigral is readily available here, and that is F phi prime equal to c 1. Now, F phi prime in this case is a sin square theta phi prime over square root 1 plus sin square theta phi prime square and this equal to c 1.

(Refer Slide Time: 16:22)

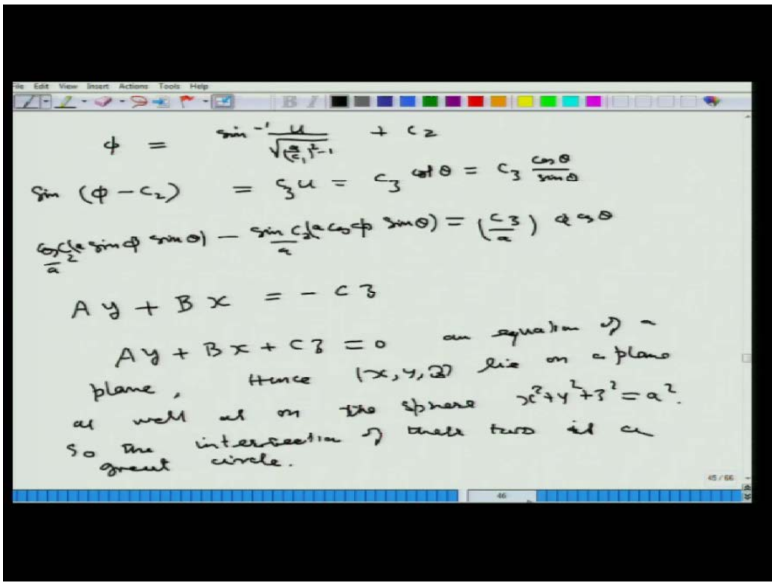


So, simplifying this squaring and solving it for phi prime, gives you solving for phi prime, we get phi prime equal to c 1 over sin square theta square root a square sin square theta minus c 1 square.

So, here we we integrate it here, so like this, so phi equal to, so this can also be written like this, d theta over sin square theta square root a square over c 1 square, this c dividing by c 1 square will comes, I mean dividing by c 1 here will give us this, and taking out this sin square phi will give us the following, minus cosec square theta plus some c 2 here. So, this can be simplified and d theta cosec, we can take this up there cosec square theta over square root a by c 1 whole square minus 1, and then writing it as 1 plus cot square theta, so that is what we will have.

So, then putting u equal to cot theta, gives us put this, we get this as, so phi s d u over square root a a by c 1 whole square minus 1 minus u square plus c 2. So, that is sin inverse u by square root a by c 1 whole square minus 1 evaluated at theta 1 to theta 2 here, so plus c2.

(Refer Slide Time: 19:43)

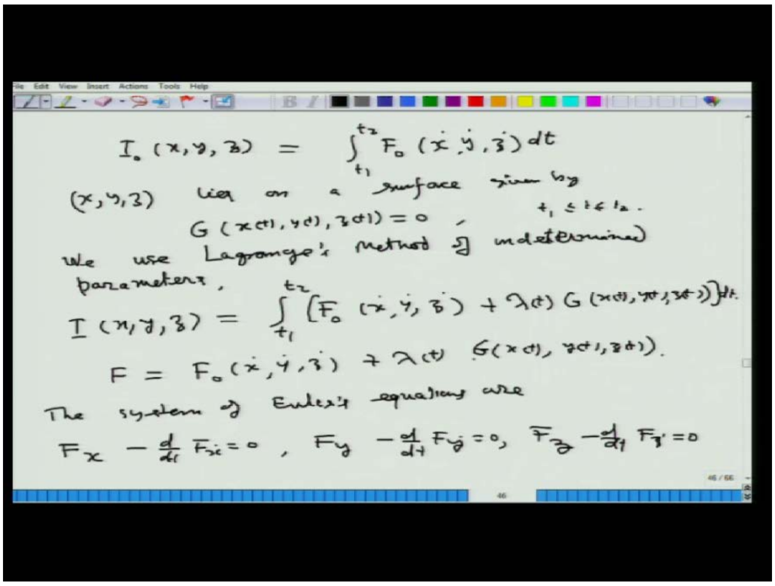


So, here then substituting back we get the following, so phi will be then, sin inverse u over a by c 1 square minus 1 plus c 2, so taking this c 2 here. So, phi minus c 2 and taking sin of this equal to u by something.

So, let us say this is c 3 u, which is c 3 than cot theta which we can write as c 3 cos theta over sin theta. Simplifying this here, and taking sin theta here, so sin phi sin theta and this cos c 2 minus sin c 2 cos phi sin theta equal to c 3 by a. Here also, we can take by a, a like this, at a a cos theta. Now, substituting back we get here some constant A y plus this also can write it as B x and this minus c z, and so we get A y plus B x plus C z equal to 0. And equation of a plane, that is hence x y z lie on a plane as well as on this sphere x square plus y square plus z square equal to A square.

So, the intersection of these two is great circle, so we get the answer as great circle. So, joining these two points, the least length would be achieved when the curve is a part of great circle. So, that is what we get here in this case.

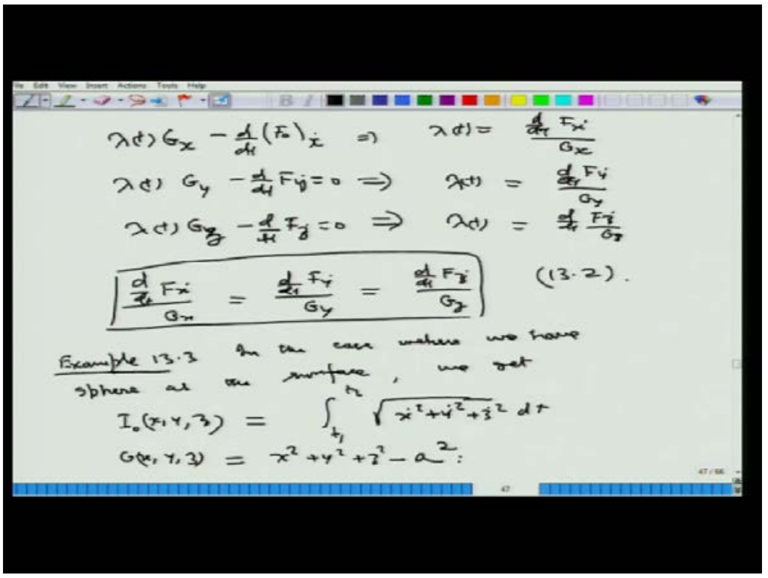
(Refer Slide Time: 23:12)



Now, let us generalize this, because here we had a very special case and in general we have situation like this, that we have some functional I 0 as x y z, which is t 1 to t 2, some functional F, integrant of the functional is function of x dot y dot and z dot only, and these points x y z lies on surface given by some this, G x t y t z t equal to 0, here t 1 less then t2 less than t 3. So, we apply the Lagrange's method, we use the Lagrange's method of undetermined parameter. So, we consider this I x y z as t 1 to t 2 F 0 x dot y dot z dot plus some lambda, here lambda will be a function of t, in general x t y t z t. So, here integrant is F, which is F 0 x dot y dot z dot plus lambda t G of x t y t z t.

And so, we get the system of Euler's equations or F x minus d by d t of F x dot equal to 0 F y minus d by d t of f y dot and F z minus d by d t of F z dot equal 0. So, here these give us here F F x will give us G x here lambda times G x, so that is what we get.

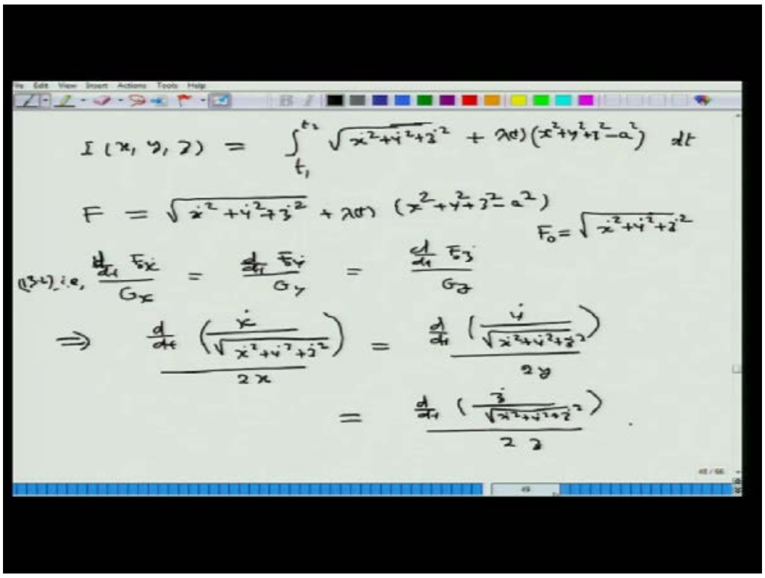
(Refer Slide Time: 26:50)



lambda t G x and minus d by d t of F x dot equal to 0 lambda t, so solving it for lambda t, we get from here lambda t equal to d by d t of F x dot over G x. Similarly, here G y minus d by G t of F y dot will give lambda t by d t of F y dot over G y and third equation, F z dot equal to 0 implies the lambda t equal to d by d t of F z dot of g z. So, we get d by d t of F x dot over G x equal to d by d t of F y dot over g y F z dot, here G z.

So, this is what we get in this case, in general case and let us call it 13.2, so in particular in this case, so if we consider the earlier example of sphere. So, we have in the case, and we have sphere as surface, we get this I 0 as I 0 of x y z as t 1 t 2 square root x dot square plus y dot square plus z dot square d t and G x y z as x square plus y square plus z square minus r or a square.

(Refer Slide time: 29:54)

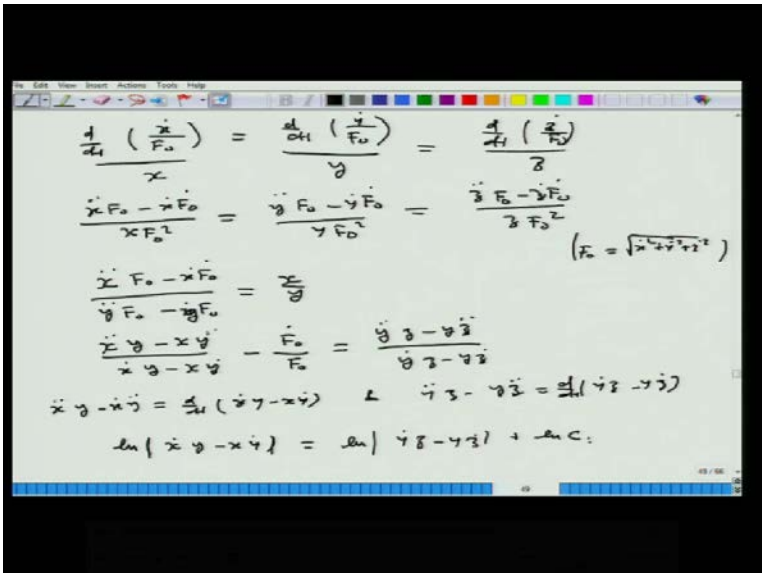


So, here we get this I of x y z as t 1to t 2 square root x dot square plus y dot square plus z dot square plus lambda t x square plus y square plus z square minus a square. So, here we have f as x dot square plus y dot square plus z dot square plus lambda t x square plus y square z square minus a square, so here we get F.

So, the question we have, d by d t of F x dot over G x equal to d by d t of F y dot over g y d by d t of f z dot over G z. So, that what this will imply, that is the number we had given 13.2. So, this is 13.2, that is this implies that d by d t of f x dot gives you us this 2 x one over one over root to this thing, and then we get 2 x and 2 cancels here. So, x dot over square root, this is F 0 not F actually this should be because F 0 because only x 0 involves only F 0 involves x dot y dot z dot here. And so, we have in this this should be d by d t of F 0, here hence this will anyway give us, this equation will give us as this where F 0 dot here, so that is what we have F 0 F 0.

So, F 0 here is square root x dot square plus y dot square plus z dot square, sorry this should not have been like this, forget about this, this should be square root x dot square plus y dot square plus z dot square plus lambda t of this square root is only up to here, that is x square plus y square plus z square minus a square this, so this is what we have here. And so, whether we write F 0 or not it does not really matter, we will have the the same result here, because this x dot y dot z dot are involved only in this F 0 part and that is why we get F 0 here. So, we get this x dot over this x dot square plus y dot square plus z dot square d by d t of this and this over we have this G x is this 1 that is lambda t, lambda t is gone anyway only G x is this one, so we get 2 x and then d by d t of y dot over a square root x dot square plus y dot square plus z dot square over G y means 2 y, this is equal to d by d t of z dot over square root x dot square y dot square z dot square over 2 z. So, simplifying this will give us the following.

(Refer Slide Time: 36:06)

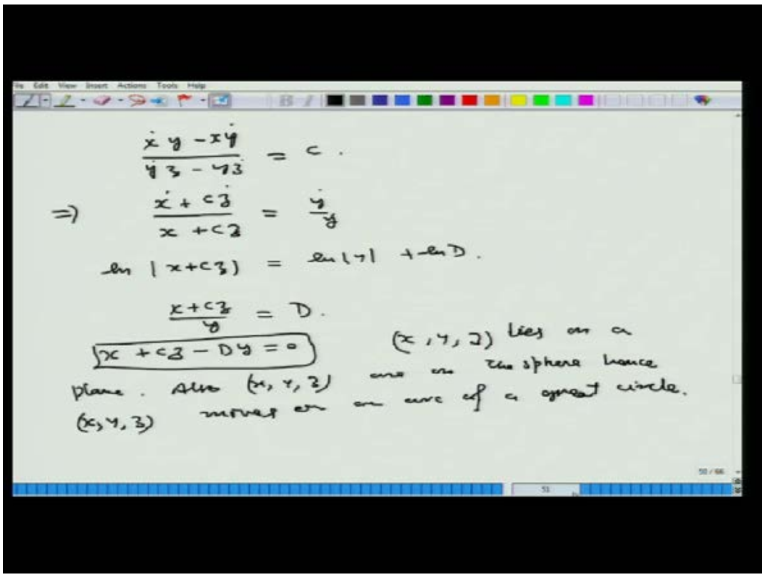


So, here we have, so we get in this case, d by d t of x dot over F 0 and this is d by d t over x and d by d t of y dot over F 0 over y d by d t of z dot over F 0 over z. And so, opening this we get x double dot F 0 minus x dot F 0 dot over x F 0 square y double dot F 0 minus plus y dot F0 dot over y F 0 square z double dot f 0 minus z dot F 0 dot over z f 0 square.

So, simplifying this, solving it for x over y, we get x double dot over F 0 minus x dot F 0 dot, here this over this y double dot F 0 minus x dot sorry y dot F 0 is x over y. And now, cross multiplying and then solving it for F 0 dot over F 0, we get x double dot times y minus x dot y double dot, here and then over x dot y minus x y dot equal to F0 over F 0 dot over F 0, here F 0 is square root x dot square plus y dot square plus z dot square..

And so, this is also solving it for the other two, y double dot z minus y z double dot over y dot z minus y z dot. Now, the numerator is the derivative of this, since x double dot y minus x dot y double dot is d by d t of x dot y minus x y dot and y double dot z minus y z double dot equal to d by d t of y dot z minus y z dot. So, integration gives you l n of this x dot y minus x y dot, exclude value of this and l n of y dot, now y dot z minus y z dot plus some l n c.

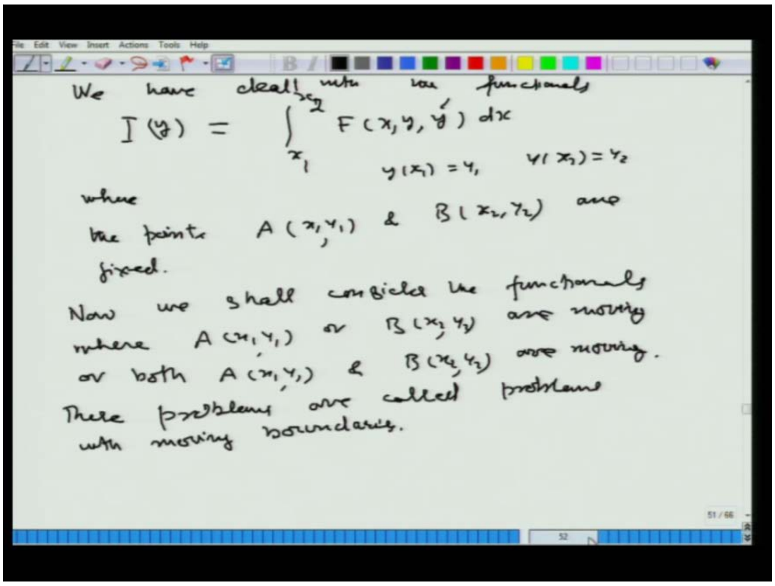
(Refer Slide Time: 39:41)



And so, rising it to exponential we get, x dot y minus x y dot over y dot z minus y z dot equal to c. Now here, so again, solving cross multiplying and solving it for let us say, we can write it here in this manner that x dot plus c z dot over x plus c z equal to y dot, this implies, so again integrating here, now this gives l n mode x plus c z equal to l n mode y plus l n D. So, we get here, x plus c z over y equal to d and so, x plus c z minus d y equal to 0. So, this gives you equation of a plane.

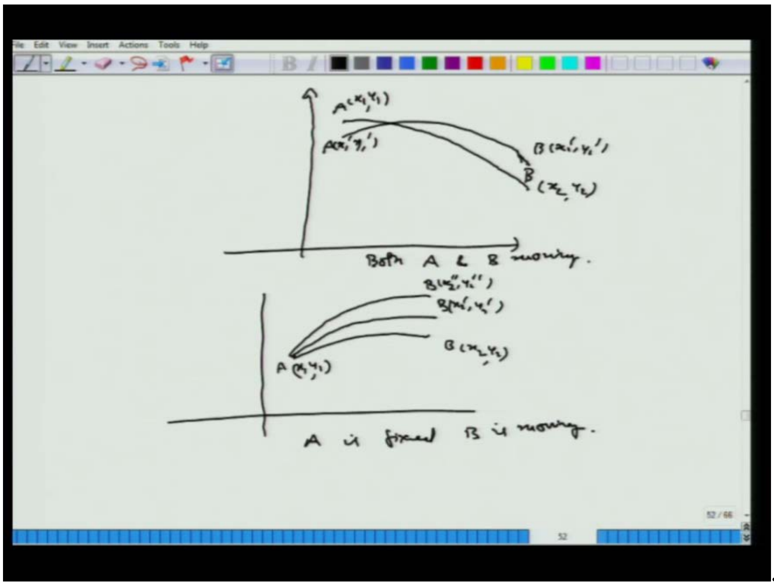
So, x y z lie this point x y z lies on a plane, also x y z are on this sphere, hence, x y z move moves on an arc of a great circle, so we get the same result, but in a different manner. So, here like this, we can have this equation solved, now we move on to certain problems were boundary points will be move moving.

(Refer Slide Time: 42:11)



So, what we have so far, that a functional like this I y equal to x 0 to x 1 F x y y prime y prime x. So, d x where this rather we took x 1 to x 2 where y at x 1 is y 1 and y at x 2 is y 2, where the points A which is x 1 y 1 and B, x 2 y 2, we have delta with the functional like this, where this the points are fixed. Now, we shall consider the functional here, let say we should consider the functional where A x 1 y 1 or B x 2 y 2 are moving or both A x 1 y 1 and B x 2 y 2 are moving. So, these problems are called moving boundary value problems. These problems are called problems with moving boundaries now.

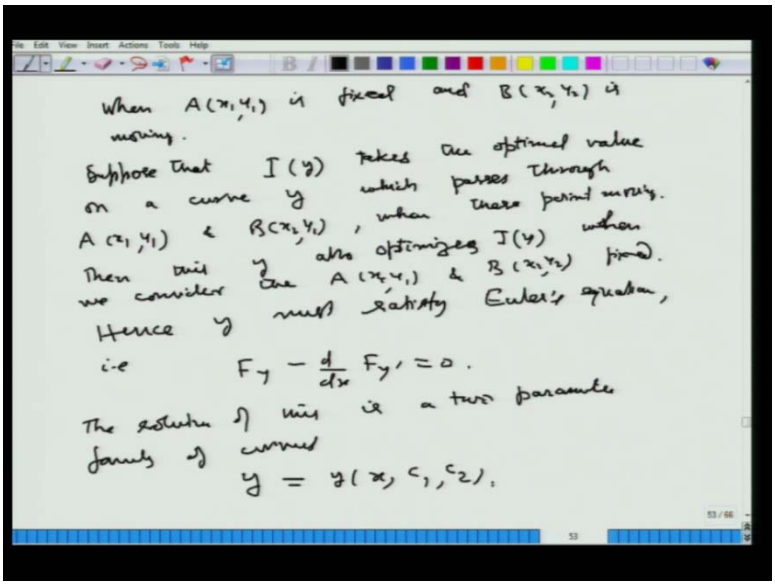
(Refer Slide Time: 45:01)


.

So, here now situation is like this, you have this point A and B, and let say this the curve joining them this x 1 y 1 and this is b x 2 y 2 and these points are moving. So, next one curve could be like this, other could curve could be like this. So, A moved here, so that is A x 1 dash y 1 dash and B is moved here, B x 2 dash y 2 dash or situation could be like this, that only A is fixed and here both A and B moving, a and b moving here A is fixed B is moving. So, A is x 1 y 1, B is x 2 y 2 and now B has moved here. So, B x 2 dash y 2 dash and it can move like this could be x 2 double dash y 2 double dash. So, B is moving here freely or it may move along the curve.

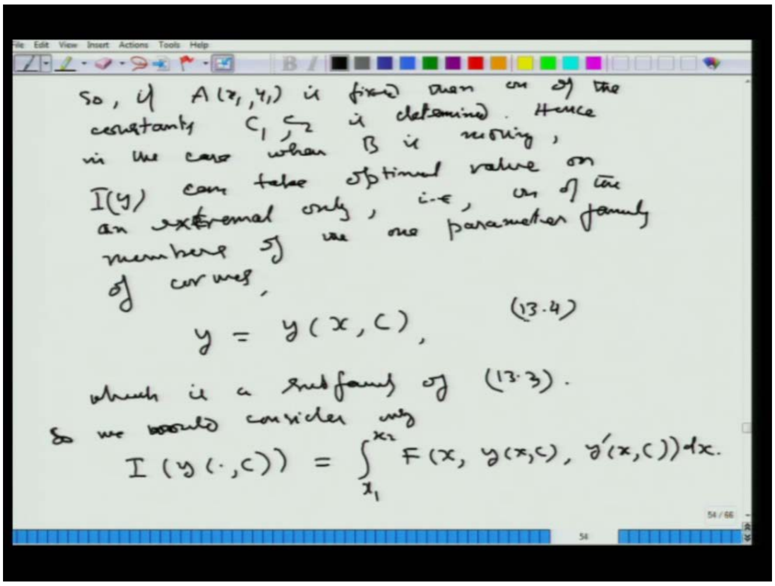
So, it may be constrain movement or it may be a free movement, here also A can move on a curve are on a surface, similarly B can move if it is a high dimensional problem then it can move on a surface also, here in two dimensional case they may move freely in the plane or they never move along the certain given curves. So, those will be constraint movements and we will be constraining these types of functional where either either the point, points are moving or one point is moving and we will be dealing with various kind of functional where such movements will take place.

(Refer Slide Time: 47:37)



So, that is what we are going to have here and so, let us see the simple case first we start when A that is x 1 y 1 is fixed. So, that is what will take here when we x 1 y 1 is fixed and B x 2 y 2 is moving. Now whether both A and B are moving, then suppose that this I y takes. So, this is what we had here I y, so, I y suppose that I y takes the optimal value on the curve y, which passes through A x 1 y 1 and B x 2 y 2, when these points are moving, then this y also optimizes by y, when we consider these k x 1 y 1 and v x 2 y 2 fixed. Hence, y must satisfy Euler's equation, which is a necessary condition that is F y minus d by d x of F y prime it could be 0. So, here the solution of this is a two parameter family of curves y as y x c 1 c 2

(Refer Slide Time: 51:00)



So, if A is x 1 y 1 is fixed, then one of the constants c 1 c 2 is determined, hence in the case when B is moving I y, I y can take optimal value on an extremely only, that is one of the members of the one parameter family of curves y equal to y x c, which is a subfamily of, so let us say, here we had given numbers.

So, let us say this is 13.3 and this is 13.4, a subfamily of 13.3, because one of the constraints there from c 1 c 2 is determined that we will denote as c and so, we get this family from this. So, we need to consider, so we will, so we would consider only I y which is functional and it will be parameter, function of parameters c also like this.

So you have x 1 x 2 F of x y x c and y prime x c d x. So, we will find the optimal value of this over the family of, so we will be in all the calculation here, over the family of this one parameter curves given by 13.4, which are the solutions of this Euler's equation. So, that is what we are going to consider, and here we will be considering the variation of this functional, and we will restrict our analysis, our calculations wherever this y appears, even though you may not denoting this dependences on c. But wherever we consider this I, subsequently this y will be actually the solution of the Euler's equation. So, that is what we are going to consider in the next lecture. Thank you very much for giving this.