Calculus Of Variations and Integral Equation

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Module No. # 01

Lecture No. # 12

Welcome viewers to the NPTEL lecture series on the Calculus of Variations. This is the 12th lecture of the series. Recall that, in the last lecture, we had considered functions of several independent variables.

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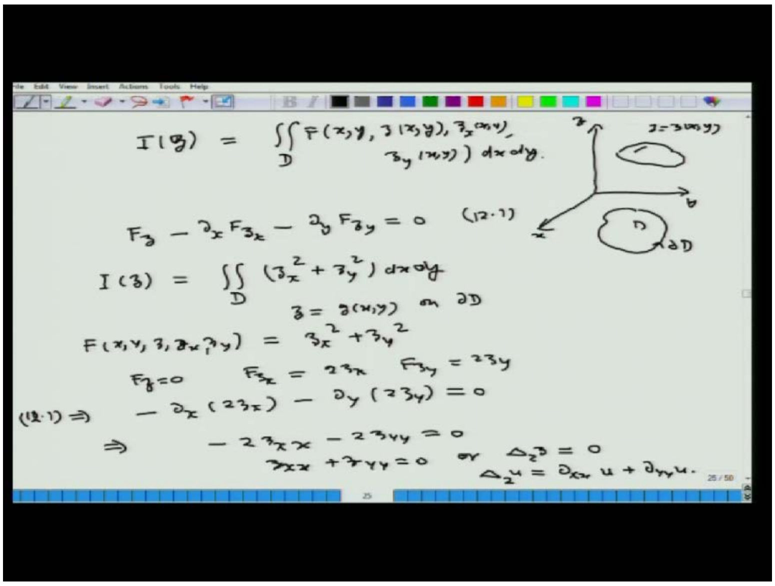

So, there we considered this functional I of z, which is double integral over certain domain D in r 2, x y plane, and here on this surface, z is defined here, so this is z as a function of x y. And here, this functional was considered I of z as function of x y independent variables and z as function of x y and then z x x y and z y of x y d x d y. So here, we had seen that the necessary condition is the following equation that is F z minus del x of F z x minus del y of F z y equal to 0. So, here we had considered in the last lecture the example that, I of z as double integral over D z x square plus z y square d x d y, and z equal to sum g x y on delta D, that is, this is delta D, the boundary of domain D.

So here, this case F is which is a function of x, y, z, zx, z, y, in this case, it is z x square plus z y square and so, here F z is 0 and F z x is 2 z x, F z y is 2 z y and so, we get minus del x of 2 z x minus del y of 2 z y equal to 0. So, this let us say this is 12.1, earlier it was something, so that give it 12.1 numbers and so here, this 12.1 implies the, this condition. And so, this gives us minus 2 z x x minus 2 z y y equal to 0. So, multiplying by minus, we get and dividing by 2, we get z x x plus z y y equal to 0, or this is Laplacian of two dimension, Laplacian of this equal to 0, where this Laplace operator in two dimension is del x x plus, so u plus del y y of u.

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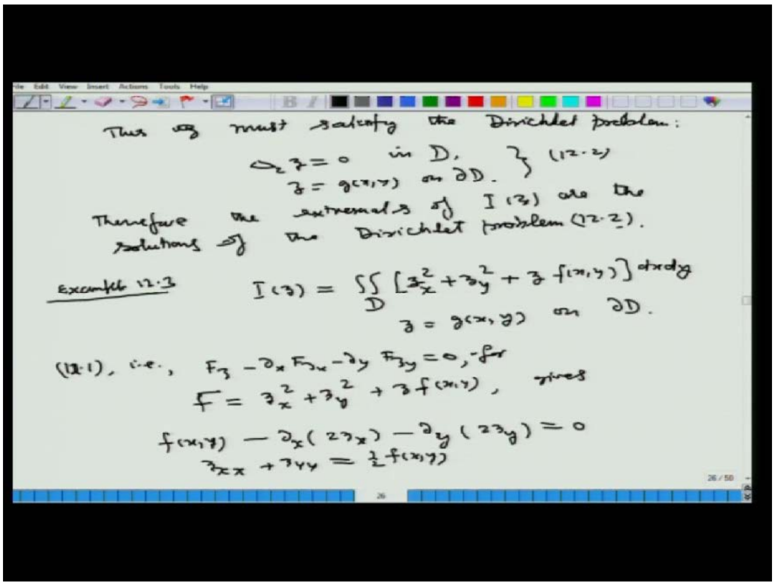

And so we get, thus z must satisfy the Dirichlet problem, that is Laplacian of this two dimension z equal to 0 in D and z equal to g x y on the boundary of D. So, what it says that, the solutions of this Dirichlet problem optimize the functional I z. Therefore, the extremals of I z are the solutions of this 12.2, the Dirichlet problem 12.2.

Here, in the next example, so we consider here this example. Let us give this name here, this example 12.2 of course, we considered this earlier also. So, example 12.3, here we considered I z as double integral over D of z x square plus z y square plus z, some function f x y here, and d x d y and z equal to g x y on delta D. So here, this 11 point Sorry 12.1 that is F z minus del x F z x minus del y F z y equal to 0, implies that f, so F z derivative here, here F is for this F here, which is z x square plus z y square plus z f x y gives f x y because F z gives you this small f, and then minus Del x, so of 2 z x minus Del y of 2 z y equal to 0. Simplifying this, we get z x x plus z y y equal to 1 by 2 f x y.

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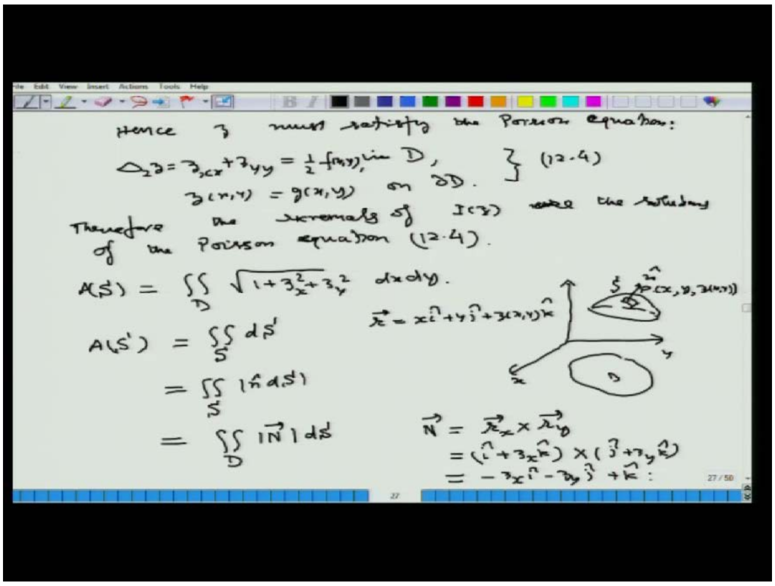

Thus, hence z must satisfy the Poisson equation, namely that two dimension Laplacian here of z equal to 0 in D and sorry half of, so this is, that is that is z x x plus z y y equal to 1 by 2 f function of x y in D, and z x y equal to g x y on delta D. So, this is the generalization of the Dirichlet problem that is the Poisson equation here, the right hand side is non zero, so it is a non homogeneous problem.

Thus therefore, the externals of I z must, this are the solutions of Poisson equation which says 12.4, the solutions of the Poisson equation 12.4. Now, we consider here the example which we had mentioned in first few lectures that, surface area of this is given by square root of 1 plus z x square plus z y square d x d y, just recall that.

Here, we know that surface area of S is given by, so here the situation is like this, and here this domain D on this, we have this surface S here and so this element area d s and this is n cap here and so this is over S, this d s is surface element, surface area element. And this we have seen that, this can be written as over S, absolute value of n cap d s, which is if we use the parameters u and v, we know that now. So, here any point p here is given, so p has the coordinates. So, position vector of p, let us say r as the position vector p that is then x i plus y j plus z as a function of x y k here. So, here point p has the coordinates x y and z as a function of x y, and so here, the parameters are x and y.

And therefore, this will be equal to, now it will be projected onto this. So, projection is D, projection of S on x y plane that is D u, u v plane here is now x y plane, and so the projection of the surface is D here. And so, we get this N, absolute value of this N d s, d s is positive, so it comes out of the absolute value. Now here, we know that this N is r x cross r y and this is r x is, i plus z x k cross product j plus z y k, and so you can see that this comes out to be minus z x i minus z y j plus k.

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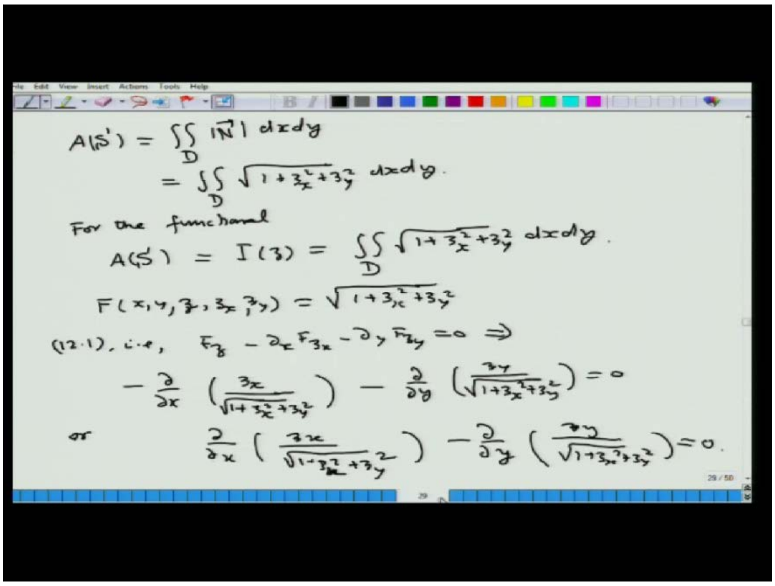

And so, this A of S, area of S which is double integral over D now of this absolute value of N, now d x d y, this should be, d s is now d x d y. And so, you get this absolute value of N as square root 1 plus z x square plus z y square and d x. So, that is what we had obtained earlier also, and so in this case, we have so thus, for the functional, here this area of now S can be, so here, we have I of z which is integration over D square root of 1 plus z x square plus z y square d x d y.

And so here, F is F of x y z x sorry z z x z y in this case is square root 1 plus z x square plus z y square. And so, this 12.1 which is that is F z minus del x F z x minus del y F z y equal to 0 implies that, minus del by del x of they seem 1 over. So, here you will have 2 will cancel, so z x over square root 1 plus z x square plus z y square. So, you will have 1 by 2 root that, and then twice of z x, so that 2 will cancel here, minus del over del y of z y over root 1 plus z x square plus z y square equal to 0.

So, we can multiply by minus or del by del x of z x over root 1 plus z x square plus z y square minus del by del y of z y over root 1 plus z x square plus z y square equal to 0. So, that is the equation we get which can be simplified like this. So here, you will have, so using that u upon v derivative formula.

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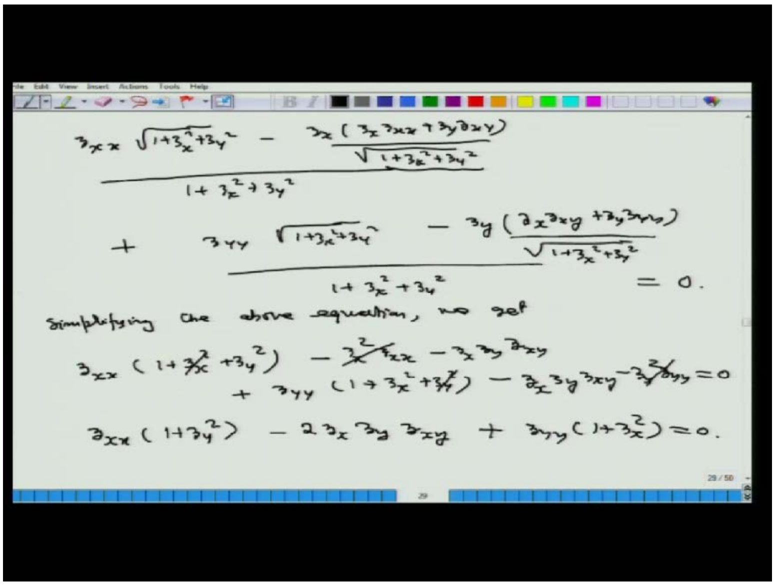

So, you get z x x square root 1 plus z x square plus z y square minus z x into z x z x x plus z y z x y over square root 1 plus z x square plus z y square over 1 plus z x square plus z y square, that is the first term plus z y y 1 plus z x square plus z y square minus z y into z x z x y plus z y to z y y over z y square upon 1 plus z x square plus z y square, this is equal to 0.

Simplifying this, we get the above equation, we get z x x to 1 plus z x square plus z y square minus z x square z x x minus z x z y z x y plus z y y 1 plus z x square plus z y square minus z x z y z x y minus z y square z y y upon, so that this term can be taken on the right hand side. So, we get this equal to 0 and we can see that, this z x x z y square will, so here, this thing we can cancel here, this z x z x square z x x will cancel with this, and similarly this z y z y y will cancel with this. And so we are left with z x x into 1 plus z y square minus these two terms, will give us twice z x z y z x y plus z y y into 1 plus z x square. So, that is what we get finally, z has to satisfy this necessary condition.

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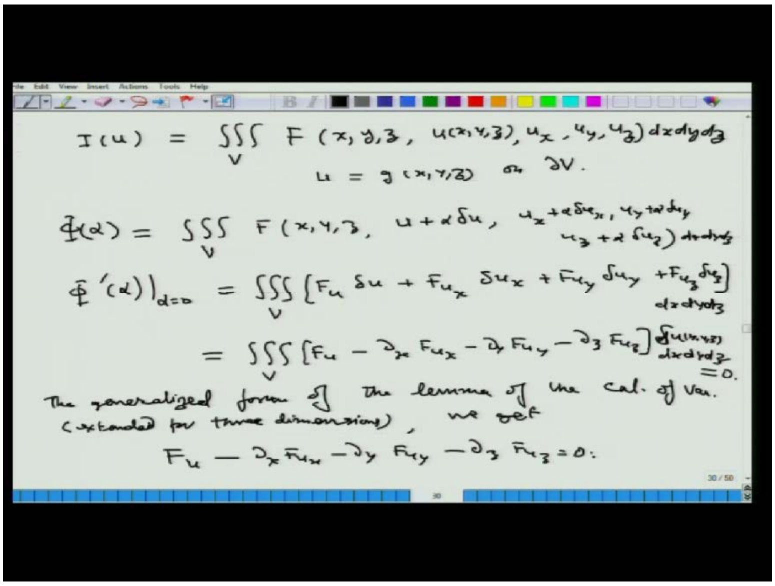

Now, going to higher order functional, now let us consider three independent variables, so I u, now use a function of x y z. So, we have now let us say, triple integral over certain volume v F of x y z. Now, these are free independent variables and use a function of x y z and then you have u x, all these are functions of x y z d x d y d z and u is to satisfy equal to sum g x y z on the boundary of this volume v. So, v is a bounded domain in three dimension x y z and its boundary will be the surface delta v like that.

So, here we can see that in this case, we can proceed in the same manner, we will have phi alpha defined as over v F of x y z, and then u plus alpha delta u, and then u x plus alpha delta u x, u y plus alpha delta u y, and u z plus alpha delta u z d x d y d z. And then phi prime alpha equating at alpha equal to 0, will have this F u and then delta u plus F u x delta u x plus F u y delta u y plus F u z delta u z d x d y d z. And then shifting these derivatives here, having more general integration by parts formula for three dimension, we will have F u minus del x of F u x minus del y of f u y minus del z of F u z and delta u, which is function of x y z and then d x d y d z.

And this is equated to 0, and then having the generalized form of the calculus, generalized form of the lemma of the calculus of variations for three dimensions, we see that here. So, the generalized form of the lemma of the calculus of variation extended for three dimension, we get F u minus del x of F u x del y of F u y minus del z of f u z equal to 0, so that is what we have.

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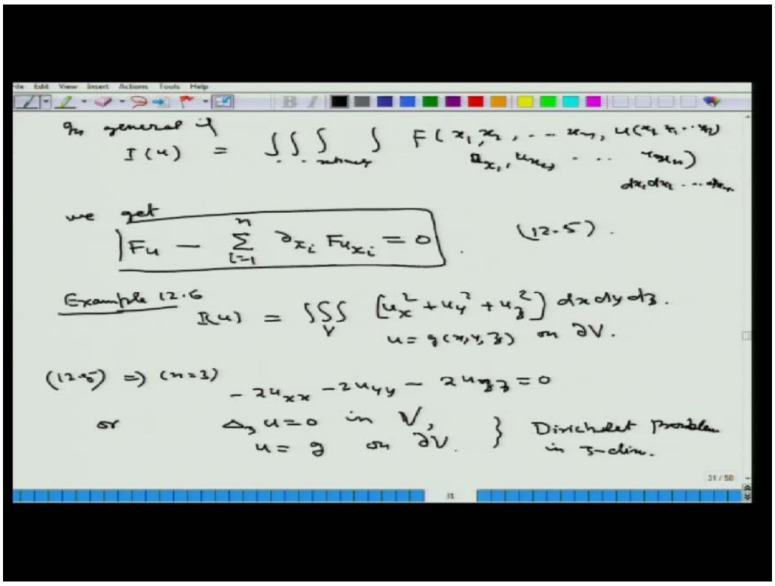

And so in general, if you have n variables, if I of u is is enfold n times, and you have F of x 1 x 2, x n and u as a function of x 1 x 2 x n, and then you have del u x 1 u x 2 and so on, u x n d x 1 d x 2 so on d x, we get F u minus sigma i equal to 1 to n del x i of f u x i.

So, this is we get here with the general case, so let us put it as 12.5. Now here, this example we consider, i u as triple integral over v u x square plus u y square plus u z square d x d y d z and u equal to g x y z on the boundary of this volume v. So, this 12.5 in this case for n equal to 3, we get minus 2 u x x minus 2 u y y minus 2 u z z equal to 0. And so, or this three dimension Laplacian of u equal to 0 in d in v, and u equal to g on delta v. So, that is the Dirichlet problem for three dimensions in three dimension.

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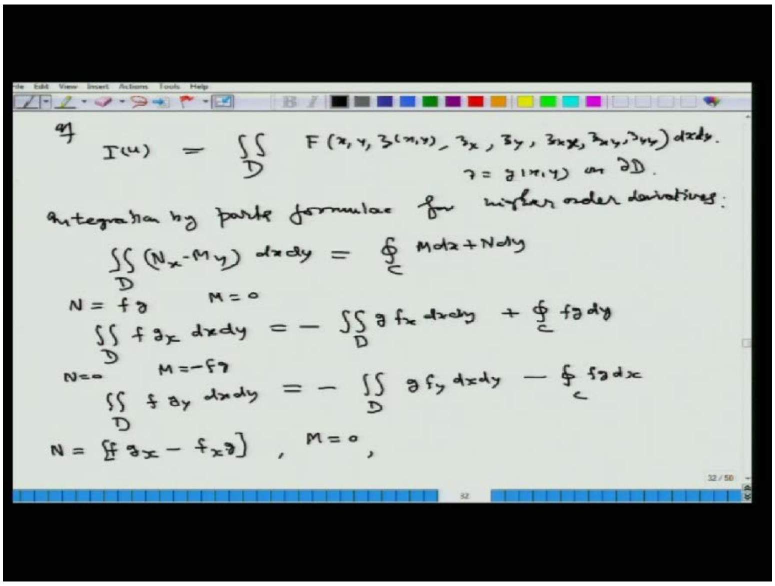

Now, if you have mix kind of, for higher order derivatives like this, if I u is of the form that triple integral over v F x y z, let us do it for two dimensional then, can be extended plus d and z as a function of x, y, z x, z y and z x y z sorry z x, x z x y and z y y.

So, here we have higher order derivatives of the dependant variable appearing and z equal to g x y on the boundary of D. So, here how do we tackle for the higher order derivatives, we have to take now, here the integration by parts for higher order derivatives which we do, so integration by parts for for higher order derivatives. So, recall that, we had this double integral over D N x minus M y d x d y to us equal to M d x plus N d y.

So, if we take, if we take N equal to first, if we take n equal to let us say f g, and M equal to 0, we see that this f g x plus f x g that will take on the other side equal to minus g f x d x d y and we get here plus. So, n is this f g d y, so it is like the derivative of this on g is shifted onto f, and so we get minus sign here. And similarly, if we take N equal to 0 and M equal to f g, then we get y derivative f g y d x d y is minus g f y d x d y plus rather minus here, because we have this minus sign is to be shifted on to the right and so you have minus phi f g d x. So, we take instead of this, we take minus here minus f g, so that is what will come here.

Now, if we take here, N equal to f g x minus f x g and M equal to 0, we get so, here N is this. So, here we will have double derivatives coming, so f f g x x and then f x g x, so that we will cancel with f x g x here.

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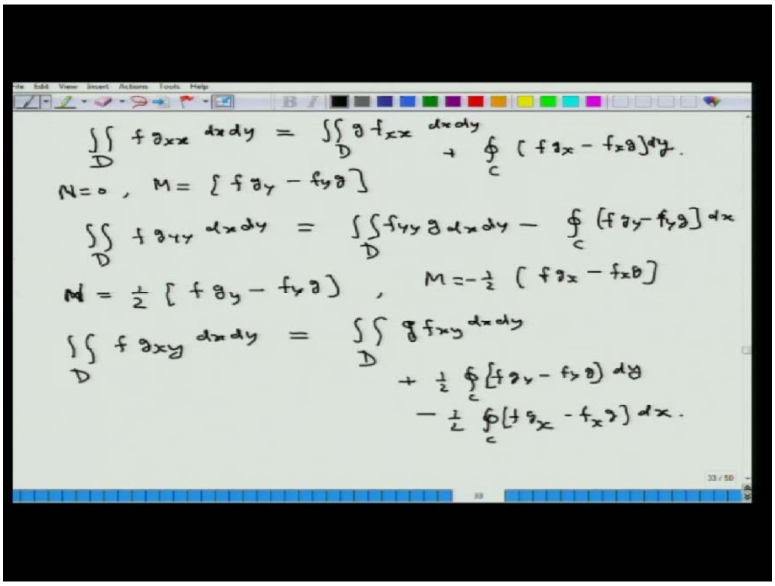

So, we will have only terms like this, f g x x d x d y and we will have here in this, we will we will have now f x x g and with minus sign which will come on the right with plus sign. So, we will have this g f x x over D d x d y and then, here n we have taken like this, so n d y plus, so this f g x minus f x g here d y.

So, this is the second order integration by parts formula, where this this x x derivative shifted onto f here, and similarly if we take N equal to 0 and M equal to f g y minus f y g, we get f g y y d x d y is g f y y and then minus over this f g y minus f y g d x, also if we take now, here since an N inverse x derivative. So, if we take here y derivatives type of this then we will have mixed kind of a thing, so that is what we will take.

So, M equal to let us say, half of rather N half of, now here, we need to take y once f g y minus f y g, and M equal minus half of f g x minus f x g, so we get now over D, so N x will give you f g x y and the terms f x g y and here f y g x, and they will also come from here they will cancel. So, we will have an half of this, and half of this make it 1 here. So, f g x y d x d y will give you over D g f of x y d x d y and then plus half of f g y minus f y g d y minus half f g x minus f x g.

All this boundary integrals are over c, so these are the higher order integration by parts formally for this, these can be used here in this where you have mixed kind of derivatives x y and this x x and y y derivatives will be shifted.

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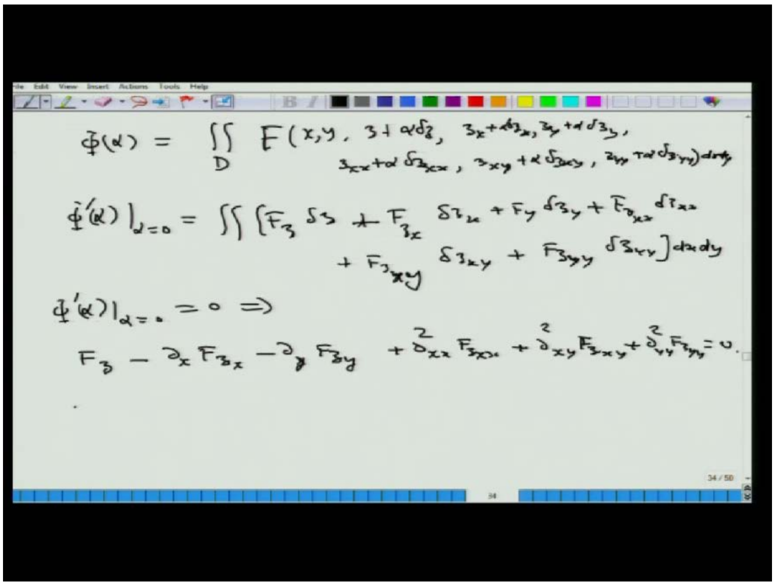

So here, now we consider same way phi alpha like this. So, that is double integral over D and so, F of x y and z plus alpha delta z, and then z x plus alpha delta z x, and then z y plus alpha delta z y, and then higher order derivatives z x x plus alpha delta z x x, this is alpha delta z x plus alpha delta z x x and z x y plus alpha delta z x y and z y y is alpha delta d x d y.

So, this phi prime alpha, at alpha equal to 0 will give us this F z delta z plus F z x delta z x plus F y delta z y plus F z x x delta z x x plus F z y y z x y delta z x y plus f z y y delta z y y d x d y. Now, these first order derivatives will be shifted in the same manner as we have done earlier and this x x derivative x y derivative and y y derivative of z will be shifted onto F and will give us this.

So, here this phi prime alpha, alpha equal to 0 equal to 0 will imply this, keep those steps here, just using those integration by parts formula will give us F z minus del x of z x F z x minus del of del y of F z y and plus del 2 x x F z x x plus del 2 of x y F z x y plus del 2 of y y of F z y y equal to 0. So, that is the equation we will get in this case.

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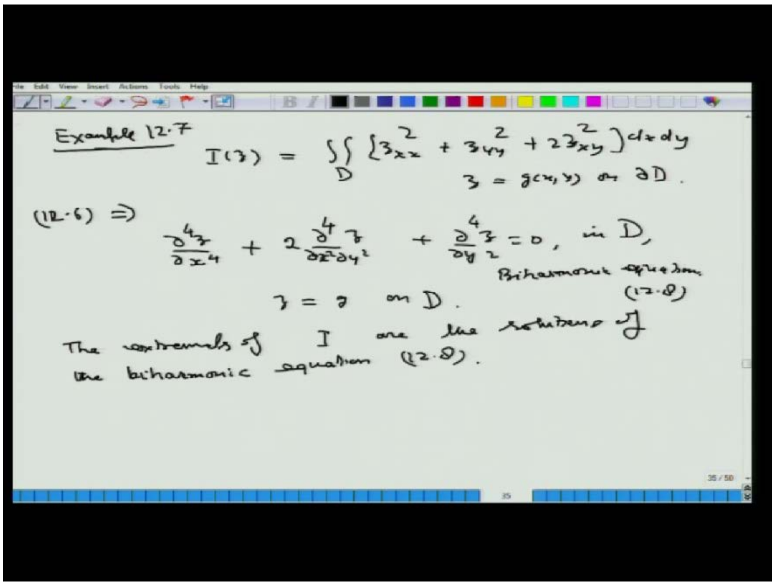

So, we take an example here, so this is 12.6 and 12.7. So here, I z is z x x square plus z y y square plus 2 z x y square d x d y and z equal to g x y on delta D. And so here, this 12.6, so 12.6 implies that del 4 of z del x 4 plus 2 del 4 of del x square del y square of z plus del 4 of del y square of z equal to 0, which is known as Biharmonic equation, and here z equal to g, this is to be satisfied in D, and z equal to g on D. So, the extreme of I are the solutions of, let us say this is 12.8, the Biharmonic equation 12.8.

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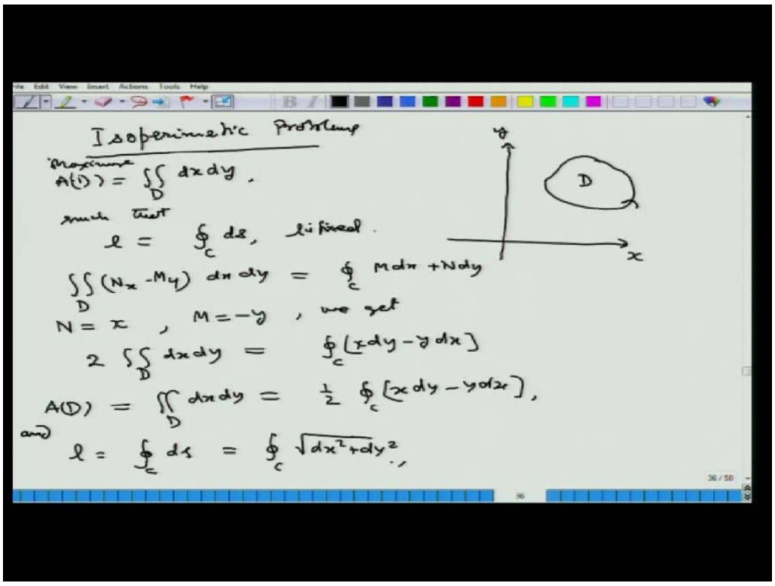

So now, we consider isoperimetric problems which were discussed in the first few lectures. So here, we have the following picture in x y plane, there is certain curve c is given of fixed length and it is enclosing the area D here. So, we know that this area of D is this double integral d x d y, subject to the condition. So, we have to optimize this or maximize rather. So, maximize this such that this length which is d s, s is small, is the arch length is fixed is like given is fixed.

So, l is fixed here, so that is what is, we have to do here. So, area this is to be maximized subject to the condition that this length l is given fixed number. So, just recall that here we have this double integral over D is N x minus M y d x d y equal to this M d x plus N d y as the Green's theorem. And you take N equal to x and M equal to minus y, we get twice double integral D d x d y equal to, so here x d y minus y d x like this, so area of D is d x d y. And so, this is half of this x d y minus y d x and this arch length s. So, here this l is an integral over c d s which is square root d x square plus d y square.

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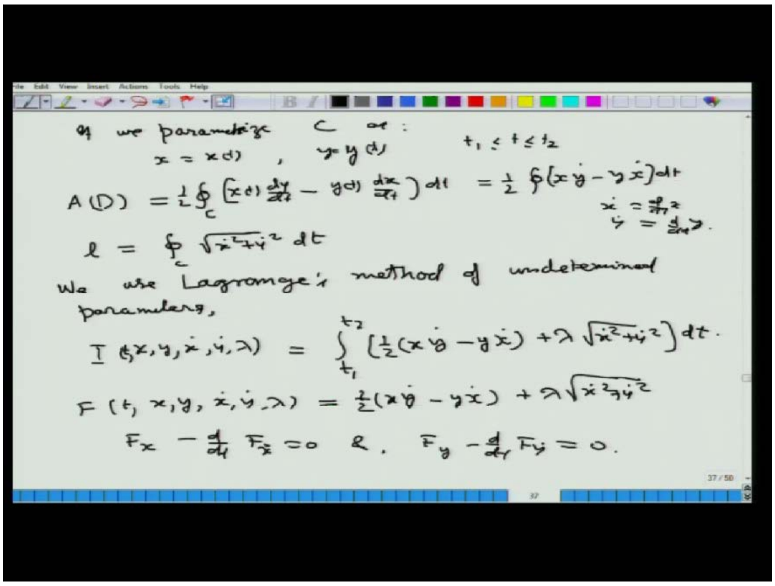

So, using this if we parameterize, c as x equal to x t, y equal to y t and t lying between t 1 to t 2, then we get this area of D as integral over c half of x, which is function of t then d y by d t minus y t d x by d t into d t. So, in short we will write it like this, x y dot minus y x dot d t, here x dot is d by d t of x and y dot is d by d t of y, and this l is then will be integral over c x dot square plus y dot square into d t.

So, we use the Lagrange's method of we use Lagrange's method of undetermined for undetermined parameter parameters. So, we consider I of x sorry t x y x dot y dot and of course, lambda is also there. So, that is t 1 to t 2 half of x y dot minus y x dot plus lambda square root x dot square plus y dot square and then d t.

So, here f is t is the independent variable, x and y are functions of t and x dot y dot and lambda is also there, so half of x y dot minus y x dot plus lambda square root x dot square plus y dot square. And so, here we get these equations F x minus d by d t of F of x dot equal to 0, and F y minus d by d t of F y dot equal to 0, because there are two dependant variables x and y.

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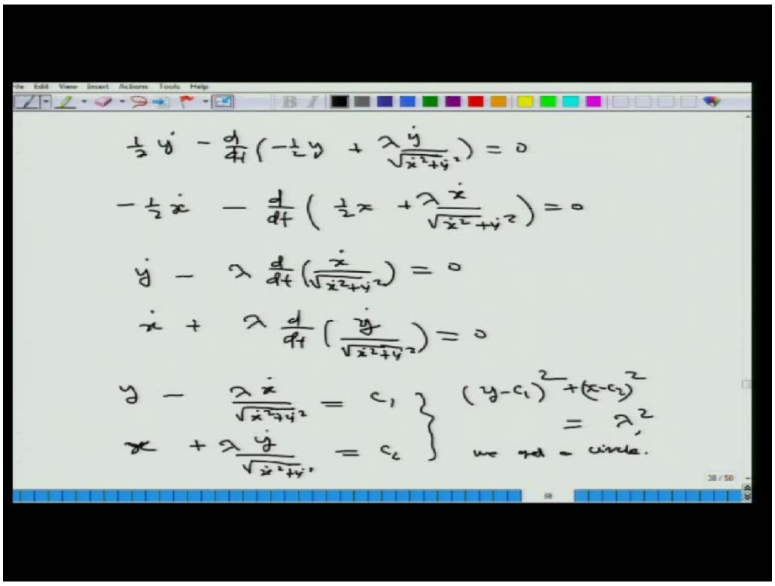

So, these two equations give us half y dot minus d by d t of minus half y plus lambda y dot over square root x dot square plus y dot square equal to zero.

Similarly, we get minus half x dot minus d by d t of half x plus lambda x dot over x dot square plus y dot square equal to 0. So, here we get, simplifying this we get y dot is lambda d by d t of x dot upon square root x dot square plus y dot square equal to 0. And x dot plus lambda d by d t y dot upon x dot square plus y dot square equal to 0, these can be integrated to get y minus lambda x dot over x dot square plus y dot square equal to c 1, and x plus lambda y dot upon square root x dot square plus y dot square equal to c 2.

So, we do not need to integrate it, you can see that y minus c 1 square and x minus c 2 square will give us lambda square, so we see that we get a circle. So, here; obviously, as expected that in this case, circle is the one which optimizes this area when the length of the circle is fixed here, and next to 1, we will be considering the geodesic here, and in this case we will have, so what is geodesic case? Just recall that.

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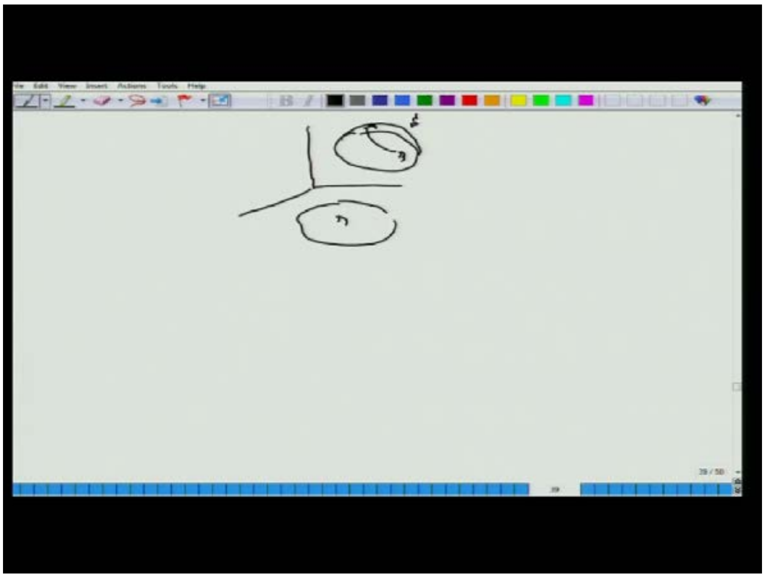

We have certain domain D here, and let us say this surface here, on the surface we have two points, a and b. And so, here this we need to find the minimum distance between these points, a and b such that these points are moving, these points are there on the surface s. So, such curves are called minimum length on surfaces are called geodesic, so that we will constrain the next lecture. Thank you very much.