Calculus Of Variations and Integral Equation

Prof. Dhirendra Bahuguna
Prof. Malay Banerjee
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur

Module No. # 1

Lecture No. # 11

Welcome viewers to the NPTEL lecture series on the Calculus of Variation. This is the 11th lecture in the series so far, we have considered various cases of the functional of the type.

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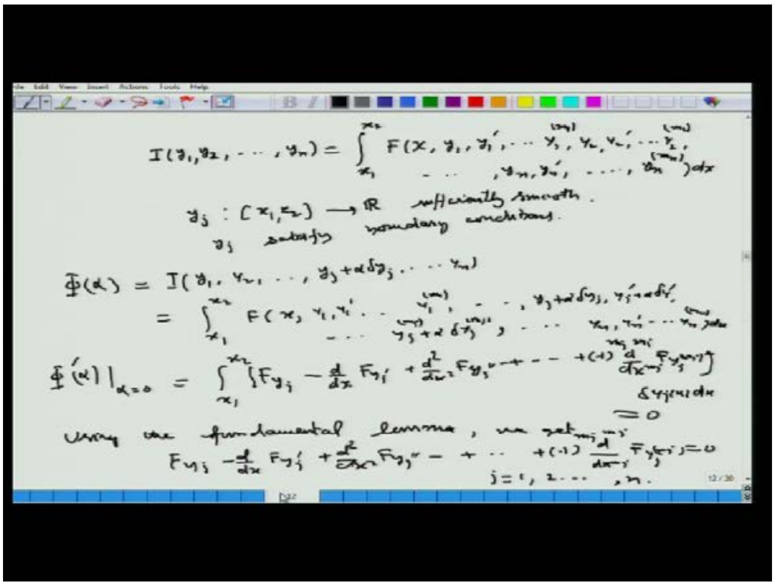

I, which is function of several function y 1, y 2 and so on, y n in the following manner integral x 1 to x 2 f of x, y 1, y 1 prime and so on, up to y 1 to the m 1 at derivative and then y 2, y 2 prime and so on, y 2 m 2 th derivative and so on, like this to y n y n prime and so on, up to y n m n n th derivative d x. Here, we have x as independent variable and y 1, y 2, y n has functions of x define on the interval x 1 to x 2, so all these y j's are functions from x 1 to x 2 into r sufficiently smooth, so that, all these derivatives exists and this integral make sense.

Earlier in the first case, we have taken only one function one dependent function y 1, so our integral verse our functional I of y 1 was x 1 to x 2 f x y 1 y 1 prime, then we considered I of y 1, y 2 y and f of x y 1 y 1 prime y 2 prime and up to y and prime and like that, we went to a higher order derivatives also and, so all those cases were particular cases of the most general functional of this type.

And so here of course, these y j's are satisfying certain boundary conditions, so these were the cases, which we had considered and we can here again consider these five alpha in the form that, I of y 1, we just change only one of them and then we consider like this, y 2 and so on, and y j plus alpha delta y j and keep others fixed and so you will have this x 1 to x 2 f of x y 1 y 1 prime and so on, to y n m on nth and these are fixed. So, only this y j plus alpha delta y j and y j plus alpha y j prime plus alpha delta y j prime and so on, up to y j plus alpha delta this is upto m jth derivative delta y j and jth derivative and so remaining ones are y n y n prime and upto y and m n derivative.

So, we vary only one of them and then use the same techniques here and so, we put f this 5 prime alpha at alpha equal to 0 equal to you will have x 1 to x 2 f y j minus d by d x of f y j prime plus d 2 by d x 2 f j f of y j double prime and so on, minus plus and plus minus 1 to the power m j d of m j or d x m j of f of y j m j and then, we will have delta y j x d x is put to 0 and so, for this jth one we will have, we will invoke the fundamental lemma of the integral calculus for fundamental lemma of the calculus of variation, the fundamental lemma we get, this f of y j minus d by d x of f of y j prime plus d 2 or d x 2 of of d j double prime minus plus and so on, plus minus 1 to the power and j d and j over d x and j of f of y j and jth derivative is equal to 0.

Here then, j goes from 1 to 2 n, so this is the system we get, for in this most general case were y 1 y 2 y n and their derivatives of any order or appearing, so various cases of this we had already considered earlier; so we will just consider one example in this, so lets say this is a 11.1.

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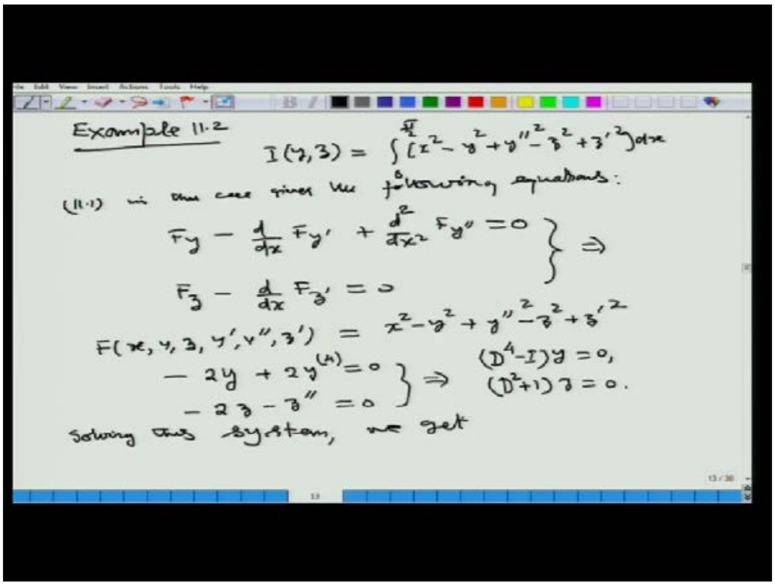

So, example is 11.2 is the, now it take an equal to 2 here, so I is function of y and jth like this and we consider here 0 to phi by 2 x square minus y square plus y double prime square minus z square plus z prime square d x. So, here this y is going up to second order derivative were z is going only up to first derivative, so this 11.1 in this case gives two equation with following namely, this f y minus d by d x of f y prime plus d 2 by d x 2 of f y double prime equal to 0, that is for the first dependent variable y and second dependent variable z, we get another equation that is, f z minus d by d x of f z prime equal to 0.

So this is the system here, in this case we have f, which is the function of x y z and y prime y double prime and z prime, which is x square minus y square plus y double prime square minus z square plus z prime square, so these two equations will reduce to the following that is minus 2 y plus 2 y fourth derivatives equal to 0 and minus 2 z minus z double prime equal to 0. so here, these can be written in this manner that, D 4 minus I of y equal to 0 and D square plus 1 of z equal to 0.

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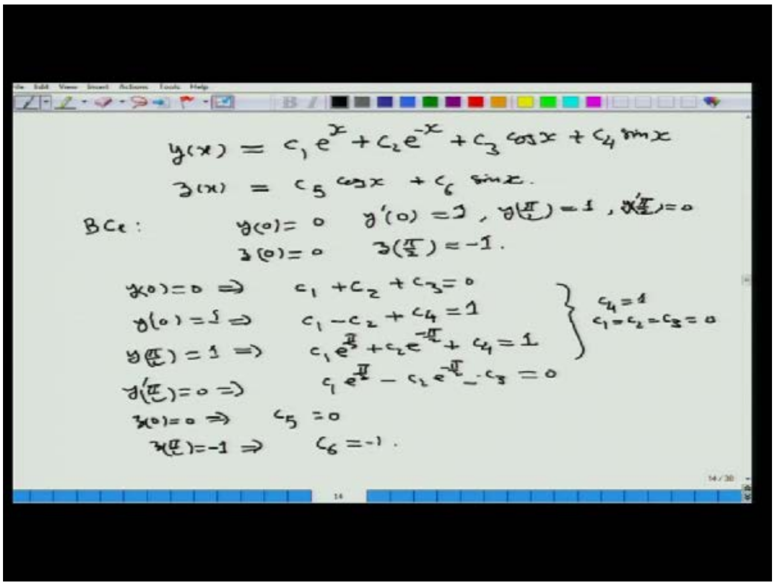

Solving this. Solving the system we get, y of x as c 1 e to the power x plus c 2 e to the power minus x plus c 3 cos x plus c 4 sin x and z x is c 5 cos x plus c 6 sin x the boundary conditions, we use the following, that y 0 equal to 0 y prime 0 equal to 1 and y at y phi by 2 equal to 1 and y prime at phi by 2 equal to 0 and on z, you will have z 0 equal to 0 and z at 5 by 2 equal to minus 1.

So, invoking these boundary conditions we get, so y 0 equal to 0 implies, that c 1 plus c 2 plus c 3 equal to 0 y prime 0 equal to 1 implies c 1 minus c 2 plus c 4 whereas, this will become cos x and and 0 it will be 1, so c 4 equal to 1 and y at phi by 2 equal to 1 implies that c 1 e to the power phi by 2 plus c 2 e to the power minus phi by 2 plus c 4, this is also equal to 1. This is at phi by 2 this term will be 0 also will get this and y at phi by 2 is this and we get finally, y prime at phi by 2 equal to 0 implies that c 1 e to the power phi by 2 minus c 2 e to the power minus phi by 2 minus c 3 equal to 0 and here z 0 equal to 0 implies that c 6 c 5 must be 0 and z at phi by 2 equal to 0 implies that z at phi by 2 equal to minus 1 implies that c 6 into minus 1.

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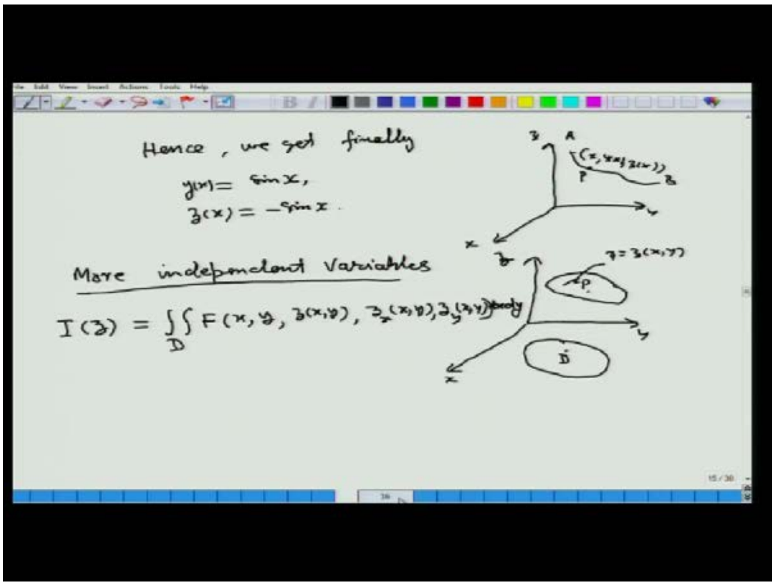

Now, here we see that, there is c c 4 equal to 1 and c 1 equal to c 2 equal to c 3 equal to 0 satisfy these equations and c 5 c 6 are there, hence we get finally, y equal to y of x equal to sin x and z of x equal to minus sin x, so this is the solution here and we can the the cur[ve] this will give us curve in three dimension space parameterized by x and so, y as a function of x and z as a function of x, so that is the situation here; so the[se] these are two points a and b and this curve will be parameterized by x y x and z x, so any points will have quardinates like this.

Next we consider now, a more independent variables, so here we will assume, that the simplest situation, we will consider is x y z and there is a domain here is D in x y plain bounded domain with sufficiently smooth boundary and on this this surface, here any point p on this it will be denoted as z as function of x y, so each point x y, here this is the image, that is z of x y.

So, here we will have the functional of the form it will be a function of z and double integral over D F x y or now independent variables z as a function of x y and z x that is also a function of x y and z y function of x y d x d y, so that is the functional you would be using here and so what we do here, we take now the following situation.

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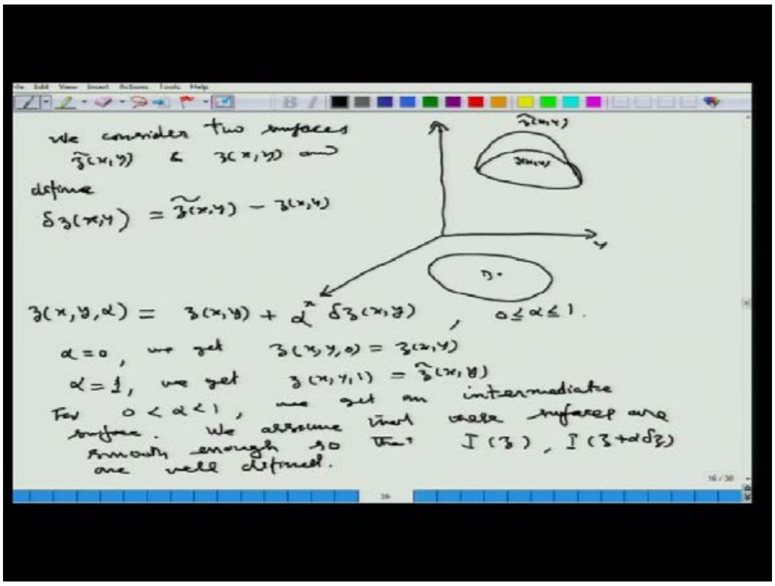

Let us slightly enlarge this, so here on this domain D, we have this surface, so let us say and then there is a neighboring surface, so this is z x y and this is z tilde x y, so these are two surfaces; now, we consider two surfaces z tilde x y and z x y and define delta z as which will be x y and alpha like this that z x y tilde x y minus z x y.

So, here difference between that (( )) in the z direction vertical distance will be defined as at any point x y here and we will write this sorry this we will define delta z x y like this and then define this z x y alpha as z x y plus alpha delta z x y, so that is what will be defined as the family of surfaces, were this alpha lies between 0 and 1 for alpha equal to 0 we get, z x y 0 as z x y and alpha equal to 1 we get for alpha equal to 1 we get z x y 1 as z tilde x 1 and in between we get for alpha strictly between 0 less than 1 we get an intermediate surface; so we assume that, these surfaces are smooth enough, so that, this I z I z plus alpha delta z or well define, so this i z similarly you will have this.

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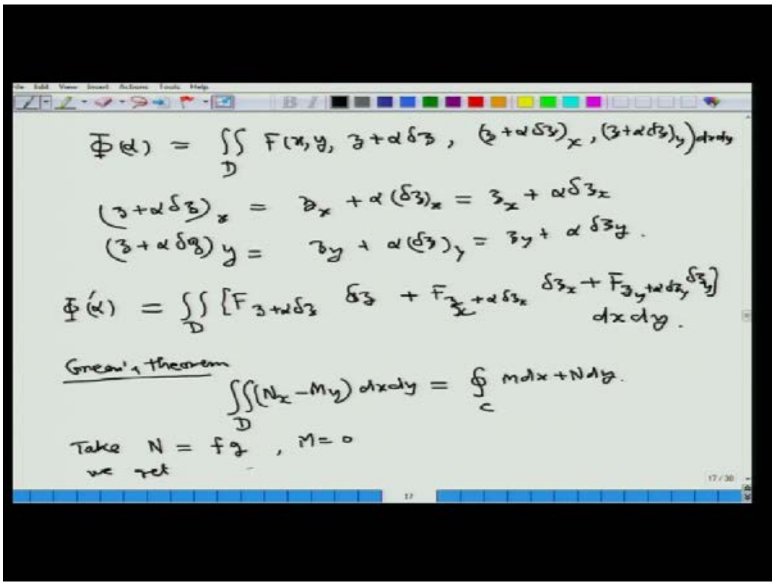

So, we define phi alpha now is integral over d f of x y z plus alpha delta z and then you have z plus alpha delta z x partial derivatives z plus alpha delta z partial derivative y and d x d y. We observe that, this z plus alpha delta z x is same thing as z x plus alpha is a constant comes out delta z x and which is also equal to z x plus alpha delta z x. So, the variation of the partial derivative of the variation delta z is same thing as the variation of partial derivative of z with respective x.

Similarly, this y will be z y plus alpha delta z y it will be z y plus alpha delta z y, so now, we differentiate with phi with the respective alpha as we have been doing in the case of one independent variable and so we get here F z plus alpha delta z plus into and then, this is gives you the delta z plus F of z prime sorry z x rather plus alpha delta z x and then z delta z x plus F of z y plus alpha delta z y delta z y d x now here, as we have been doing in the single variable case we shifted this derivatives onto F like here, so since this is a add dimensional case we will invoke Green's theorem here, so just recall Green’s theorem which stated like this, that over d if you have N x minus M y d x d y integral over the boundary of M d x plus N d y.

So, if we take here N equal to, lets say f into g two functions partial derivatives are continues in D and and on the boundary they are continues, so he will have M m is equal to 0, we get here, now here N x N has two function f and g. So, here n x will be f g x plus f x g and so we will have this f g x d x d y and this term will take on the other side, this term and so it will be minus g f x g x g y plus the boundary term here, because m we have taken 0 n as f g.

So, we get integral over this f g d y n is there, so g d y this on the boundary here; so, this is a integration by parts formula for hard dimension here, x derivative is there on g it is shifted on to now f, so you gain minus sign here and you get boundary integrals here similarly, if you take now earlier we took N equal to f g and now, will take M here and N 0. So N equal to 0 and M equal to let us say h g then, we get or we can take minus h g whatever you want, so we will have this, then multiplying by minus we get h g y d x d y and you will have this integral now g h y d x d y and minus this over this g h d x.

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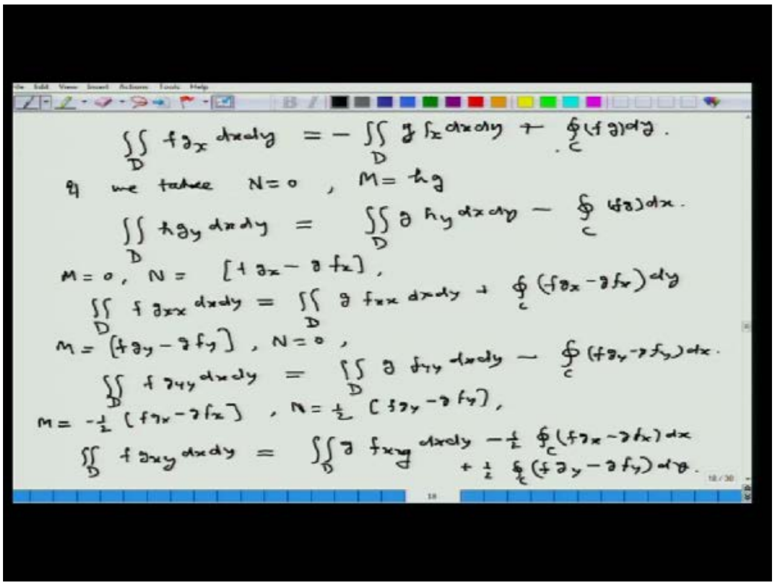

So, that is one thing, then we may need higher order integration by parts formulae also which we had already introduced earlier, if we take this M equal to 0 and N equal to f g x minus g f x then we get d f g x x d x d y equal to g f x x double shifting gives you plus sign here and plus this boundary terms, here f g x minus g f x g y and similarly, you take M equal to f g y minus g f y and N equal to 0 we get g f y y d x d y minus f g y minus g f y and for mixed kind of thing, we can take M equal to minus of f g x minus g f x and N equal to plus of f g y minus g f y you get these kind of this double integral over d of f g x y d x d y is now integral is shifted on to f.

Now, so, g f x y d x d y plus minus 1 by 2 c f g x minus g f x d x and plus half here f g y minus g f y d y here, so like this, all this here we have now, so here N x gives you those terms and because, here N has derivatives with respective y. So, N x will give you terms like f of g x y minus g f f x f x y and other once will cancel here, with each other and, so you will have and ultimately we have used in this M here and N here and with.

So, these are the integration over domain d whereas, this integration over the boundary of this domain d, so that is what we get here in this case; so we use some of these integration by parts formally to get the result here, so we had arrived at this point here. So, let us call it 11. 11.2 you have used, so let us call this as let us call as 11.3 and this as 11.4, so in a 11.4, we use these shifting of derivatives like this f g x d x d y equal to minus g f x d x d y plus this integral of f g d y over the boundary term, so that is what we will use here.

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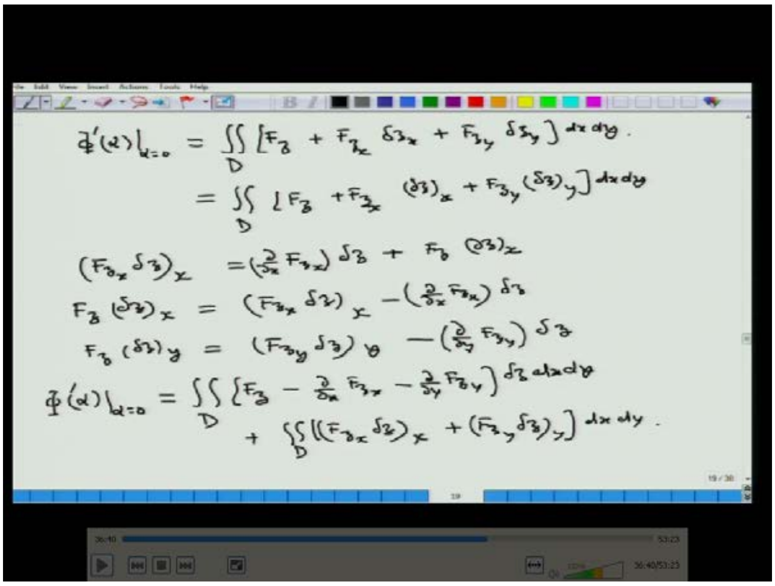

So, have now this phi prime alpha at alpha equal to 0 is taken as F z plus F z x delta z x plus F z y delta z y d x d y, now x derivative is also as we have already explained that, this can be written as delta z x d x d y.

Now, we will shift this x derivative here using the greens theorem here. So, if we look at this F z x delta z partial derivative of this with respective x is F, so that is del by del x of f z x x delta z plus f z delta z x, so that is what we have here. So, we will put this f z delta z x as F z x delta z partial derivative of this minus this del by del x of F z x delta z similarly for this we have F z delta z y equal to F z y delta z y minus delta d x F z y delta z.

So finally, we get this phi prime alpha at alpha equal to 0 like this D F z minus del by del x of F z x minus del by del y of F z y delta z d x d y and then you have these terms F z x delta z plus F z y delta z y d x d y, now this second term here, so we take in the second term.

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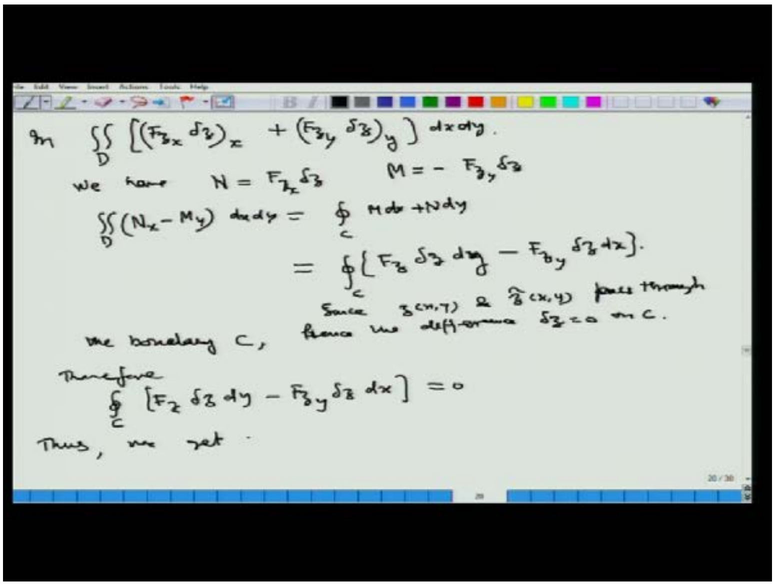

Is that is this F z x delta z x and this plus F z y delta z y d x d y here in this you take we have we have let us say N equal to F z x delta z and M equal to minus F z y delta z, so by this were D, so we have this is N x minus M y. Since, this is equal to integral over the boundary M d x plus N d y, so here in this case, it comes out to the F z delta z d x minus F z y delta z this should be d y and this should be d x. So, that is what we have here, now this is since here delta z equal to 0 on c because, here both z since z x y and z tilde x y pass through the boundary c. Hence, the difference delta z equal to 0 on, so let us write it in this way since these both z and z tilde pass through the boundary c we have the difference delta z equal to 0 and therefore, this integral over c F z delta z d y minus F put bracket here F z y delta z d x equal to 0, since this delta z here is 0 on c.

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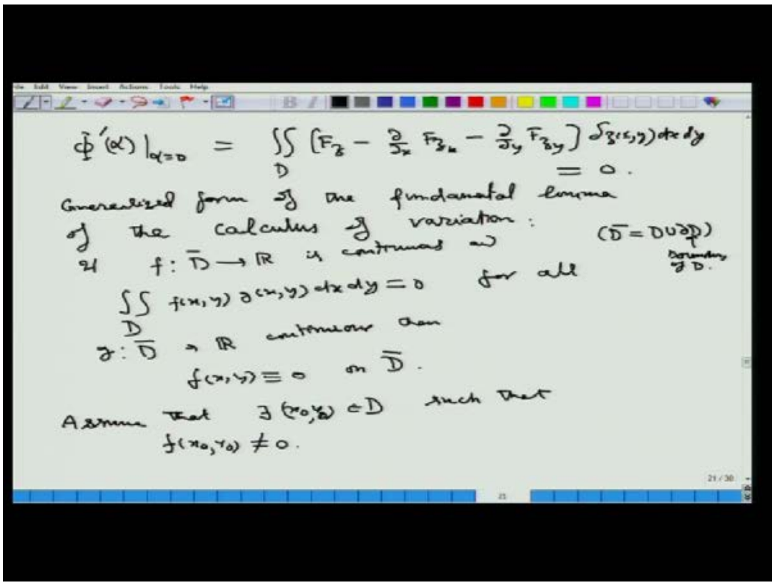

So therefore, ultimately thus finally this we get, this phi prime alpha at alpha equal to 0 as F z minus del by del x F z x minus del by del y F z y delta z function of x y and d x d y and this is now should be equated to 0, so as we have in the case of one variable, that fundamental lemma of the calculus of variations similarly we have here the generalized junglezed form of the fundamental lemma of the calculus of variation that say that if you have if f is from D to R closure into r is continuous and this integral effects f x y g x y d x d y it could be 0 for all g D closure to R here here D closure means D union its boundary delta d this is the boundary of of are continuous then this f x y is identically 0 on D closer.

Assume that assume contrary that, there exist x 0 y 0 and d we need to take we need to establish that this is 0 on d then; obviously, the continuity it will be 0 on D closure, so here will say. So, assume that here you have an interior point this x 0 y 0, such that f of x 0 y 0 is not equal to 0.

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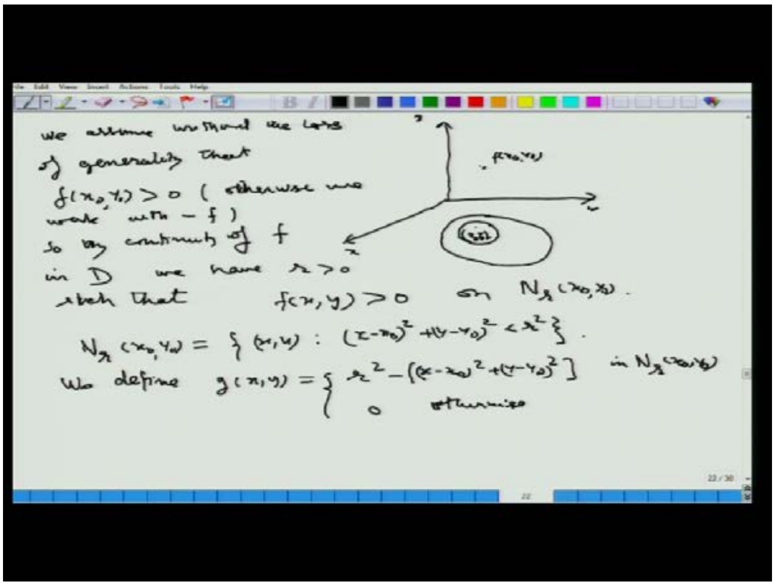

So, we have the situation here, and this the point x 0 y 0, such that f is here, so we can assume that we assume without loss of generality that f of x 0 comma y 0 is positive otherwise, we can multiply by minus sign otherwise we work with minus f. So, here this is f of x 0 y 0 and so, by continuity, so there is a (( )) around this, so by continuity of f in D we have r greater than 0 such that f x y is positive on N r of x 0 y 0 which is here N r x 0 y 0 is neighborhood set of all x y, such that x minus x 0 square plus y minus y 0 square is less than r square. So, it is open desk of radius small r likes this, now we define g x y equal to r square minus x minus x 0 square x y minus y 0 square m n r x 0 y 0 and 0 otherwise.

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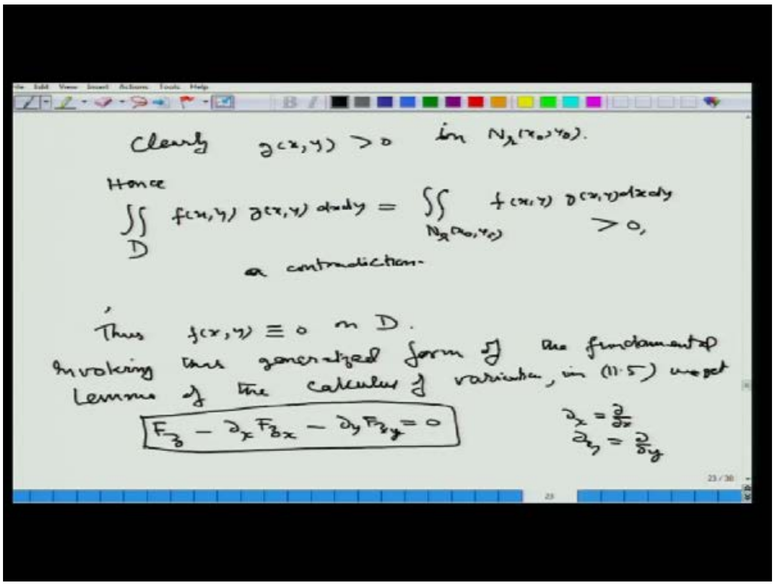

Clearly, if g x y is greater than 0 on, in rather is open N r x 0 y 0, hence this integral over d f x y g x y d x d y because outside D to 0 outside, it is outside N r x 0 y 0 it is 0, so this integral reduces to this and so it is strictly because this is positive f is positive g is positive on this and therefore, we get this a contradiction.

So here therefore, there cannot be any point x 0 y 0 in D, so we have f of x 0 y 0 equal to 0, but and so, this f of x y is identically 0 on D, because x 0 y 0 was any point, here we assumed that this is positive, here that cannot be possible and so it should be 0 on this let us not write thus thus we cannot have such a situation for any x 0 y 0 in D f of x 0 y 0 equal to 0 and therefore, this has to be 0 in this. So, invoking this this generalized form of fundamental lemma the calculus of variation we get this f z minus del by here we invoke this in this this, let us call this as 11.5 and so in working this fundamental in 11.5.

We get f z del x of f z x minus del y of f z y equal to 0, here short notation del x is del by del x and del z del y is del y del y, so this is what we get here, this is known as (( )) equation the more general form of Euler equation, here so now, we take various cases, various examples of this.

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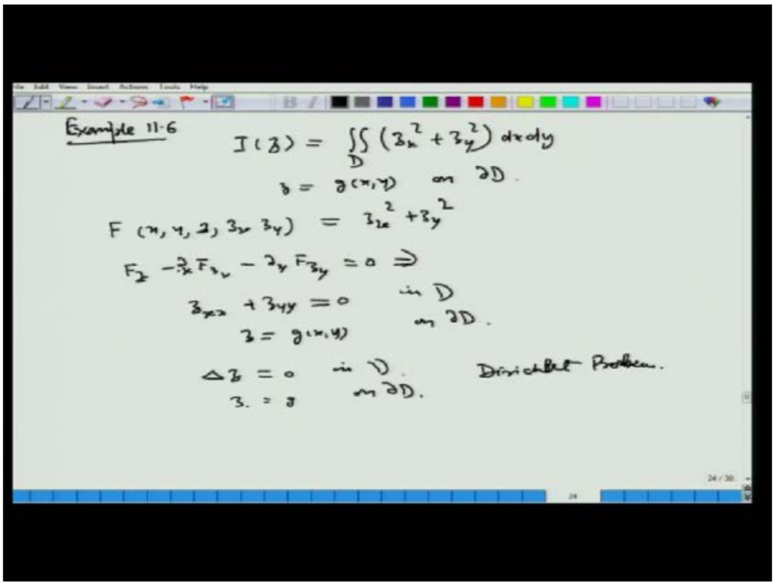

Now the first example is, so this we consider I z as this z x square plus z y square d x d y and for the boundary condition is z equal to g x y on the boundary del d here you have F x y z z x z y as z x is square plus z y square and. So, we f z minus del by del x of F z x minus del y of F z y equal to 0, implies that z x x plus z y y equal to 0, so this is in D and z equal to g x y on D. So, this is alpha and z equal to 0 g on on delta D is known as Dirichlet problem, so what is says that if Dirichlet problem has a solution than it minimize it optimizes the this functional, so this will be the general feature which will consider in the next lecture, thank you very much for…