Calculus of Variations and Integral Equations

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Statics Calculus of Variations and Integral Equations

Welcome viewers to the NP-TEL lecture series on the calculus of variations. This is the tenth lecture of the series in this we will consider more general functional recall that in the last lecture we had functional of the type I of y equal to integral x 1 to x 2 F of x, y, y prime into d x and y is supposed to satisfy these boundary conditions y of x 1 equal to y 1 and y of x 2 equal to y 2.

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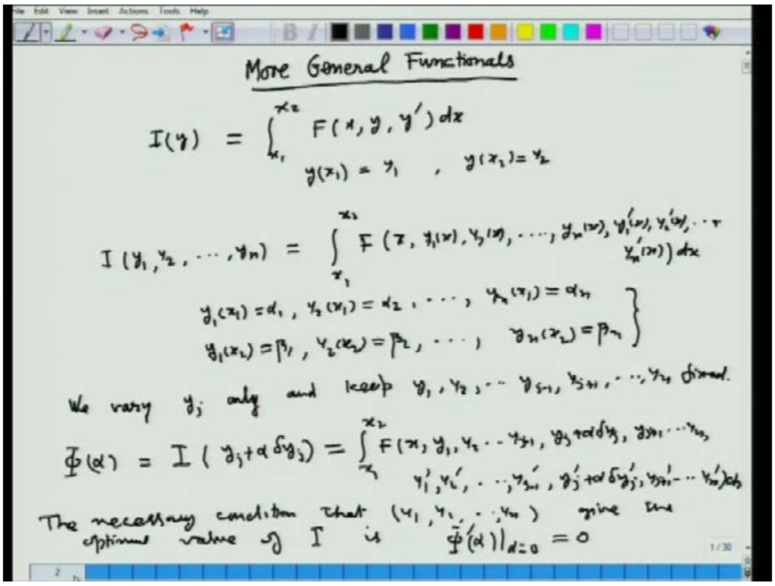

Now, we shall consider more general function of the type I of y 1, y 2…. y n. So, there are n functions admissible functions such that this I of y is of the type integral x 1 to x 2 F of x, y 1 of x, y 2 of x… y n of x and then there are derivatives are appearing y 1 prime of x, y 2 prime of x and so on y n prime of x into d x.

And then these y I's are to satisfy the boundary conditions that y 1 of x 1 equal to alpha 1 y 2 of x 1 equal to alpha 2 and so on y n of x 1 equal to alpha n and y 1 of x 2 is equal to beta 1 y 2 of x 2 is equal beta 2 and so on y n of x 2 is equal to beta n.

So, these are the boundary conditions for each of these y I's are to be imposed to satisfy and. So, this I is function of all these and functions y 1, y 2 ,y n. Here x is independent variable and these y 1,y 2,y n are functions of x and the derivatives are appearing here y 1 prime of x, y 2 prime of x and so on to y n prime of x.

So, here these entire y I's are assumed to satisfy certain (( )) properties. So, that this integral is well defined.

Now, here we would like to have this set of all these y 1, y 2, y n to satisfy certain equation so, that will be the kind of a necessary condition as we had got Euler's equation in the case where I of y was F of integral x 1 to x 2 F of x, y prime into d x. So, similar kind of equation we would like to obtain which will be a necessary condition to be satisfied by these n functions in order this functional I to have optimal value.

So, we proceed in the same manner what we do here we vary y j only and keep y 1, y 2, so on y j, minus 1 y j plus 1, y n fixed.

So, we change only y j. And So, we get this pi of alpha is equal to function I of y j plus alpha into small delta y j equal to integral x 1 to x 2 F of x ,y 1 ,y 2. y j minus 1,y j plus alpha into small delta, y j and then y j plus 1 and y n and then there are derivatives y 1 prime, y 2 prime, so on y j minus 1 prime, y j prime plus alpha into small delta, y j prime, and then y j plus 1 prime, and so on to y n prime into d x.

So, the necessary condition that this y j. So, y 1, y 2 , y n give the optimal value of I is pi prime of alpha divided by alpha equal to 0. Must be equal to 0 in this case this particular case and.

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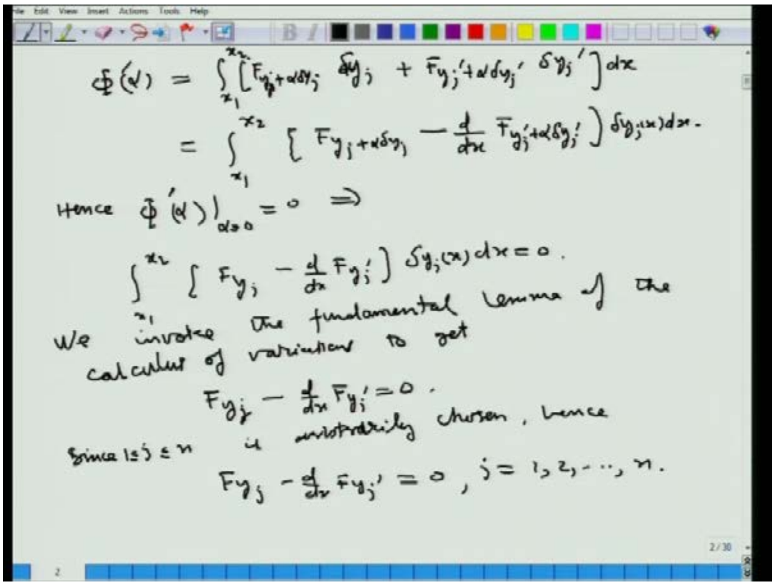

So, we get pi prime of alpha is equal to integral x 1 to x 2 F of y 1 plus alpha into small delta y 1 here only we have y j plus times delta y j here plus F of y j prime plus alpha delta y j prime and delta y j prime into d x.

And now as before we shift this derivative here and So, this is equal to x 1 to x 2 F of y j plus alpha into small delta into y j minus d divided by d x of F of y j prime plus alpha into small delta into y j prime into small delta y j of x into d x.

And So, hence this pi prime of alpha alpha equal to 0 is equal to 0 implies that integral x 1 to x 2 F y j minus d divided by d x into F of y j prime into small delta y j of x into d x.

We invoke the fundamental lemma of the calculus of variations to get F y j minus d divided by d x of F y j prime equal to 0. So, this is what we get this since j lies between 1 to n is arbitrarily chosen. Hence F of y j minus d by d x F of y j prime is 0 for j equal to 1,2 and so on up to n.

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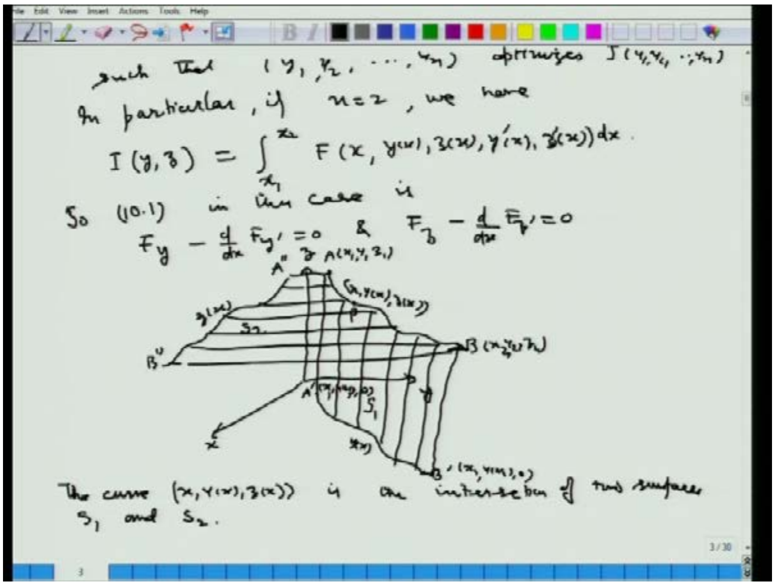

So, we get the system of the equations for each y j we get one equation like Euler's equation. So, the system is of second ordered differential equations forms the necessary condition such that this y 1, y 2, y n optimizes I of y 1, y 2, y n and there will be; obviously, constants appearing in the solutions and those constants are to be determined by the given conditions.

So, will take some examples here. So, in particular if n equal to 2 we get I of y, z is equal to integral x 1 to x 2 F of x, y of x, z of x ,y prime of x, z prime of x here. Let us give this is 10.1. So, 10.1 in this case is F y minus d divided by d x into F y prime equal to 0 and F z minus d divided by d x into F z prime is equal to 0 .Here, we have situation like this x axis, y axis, this z axis. So, here in a 3D space let say these are the 2 points A and B.

So, this is like you have x 1 y at x 1 will call at y 1 z at x 1 a z 1 and here x 2, y 2, z 2. So, these are 2 points here. So, this will be the curve which will be parameterized as x, y of x and z of x.

So, any point here typical point p will have this coordinates is x, y of x, z of x. Here say this is the projection of this here like this in the x y plane. So, this is A dash, B dash. So, this is x, y of x, 0. So, this is x 1 y at x 1 and 0 this is x 2 y at x 2 and z and if you project this on to this like this you have let say A double dash here and B double dash here.

So, this is the curve z of x curve which is in the x z plane and this is the curve y of x curve. So, here we will have in this case x and y of x are there and z is free. So, it is going to be this plane like this hyper surface rather and here this only x and z of x are here in this and y is free.

So, it will give us a surface like this. And so, here this curve is the intersection of 2 surfaces the curve x ,y of x and z of x is the intersection of 2 surfaces. That is here you have x and z of x given and y is free. So, it goes in this direction and here in this case you have in this surface you have x, y of x moving along this curve and z is free there.

So, let us say this is surface S 1 this is surface S 2. So, that is a situation here.

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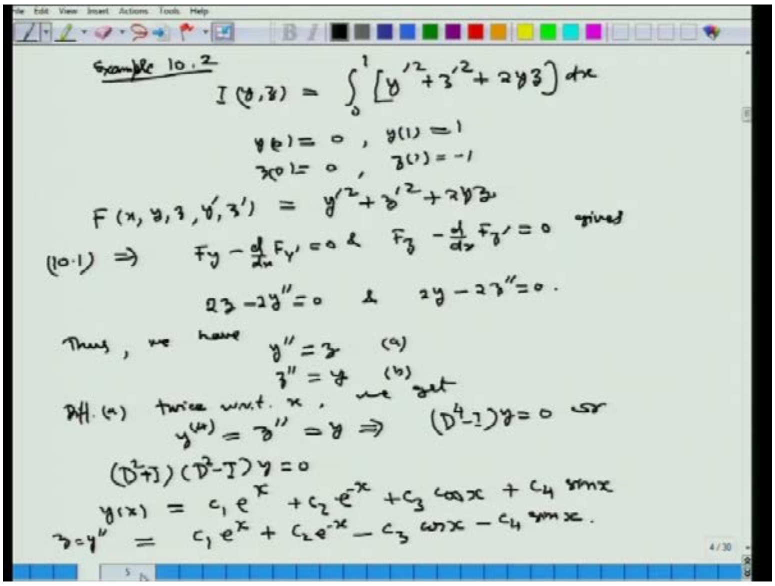

So, let us take this example 10.1. Let us call it 10.2 now. So, here I of y, z is equal to 0 to 1 y prime square plus z prime square plus 2 into y into z into d x and the conditions are y of 0 equal to 0. y of 1 equal to 1 and z of 0 equal to 0 and z of 1 equal to minus 1

So, in this case F of x, y, z, y prime, z prime is equal to y prime square plus z prime square plus 2 into y into z. And so, the system 10.1 implies that F y minus d divided by d x into F y prime equal to 0 and F z minus d divided by d x into f z prime equal to 0 .Here 2 into z minus you get y double prime equal to 0 and 2 into y minus 2 into z double prime equal to 0.

So, we have thus y double prime equal to z and z double prime equal to y. So, differentiating first let say this is A and this is B a twice with respect to x. We get y Fourth derivative equal to z double prime, but z double prime equal to y and. So, we get this implies that D power of 4 minus I of y equal to 0 or D square plus 1 into A square plus I into D square minus I of y equal to 0.

And. So, the auxiliary equation is n square plus 1 into 2 into n square minus 1 equal to 0 and it has roots plus 1,minus 1 and plus I, minus I. So, we get y of x as c 1 into e to the power x plus c 2 into e to the power minus x plus c 3 into cos x plus c 4 into sin x. And so, z equal to y double prime and so, this is equal to c 1 into e to the power x plus c 2 into e to the power minus x and here twice derivative gives you again plus sign here twice derivatives will give minus sign minus c 3 into cos x minus c 4 into sin x.

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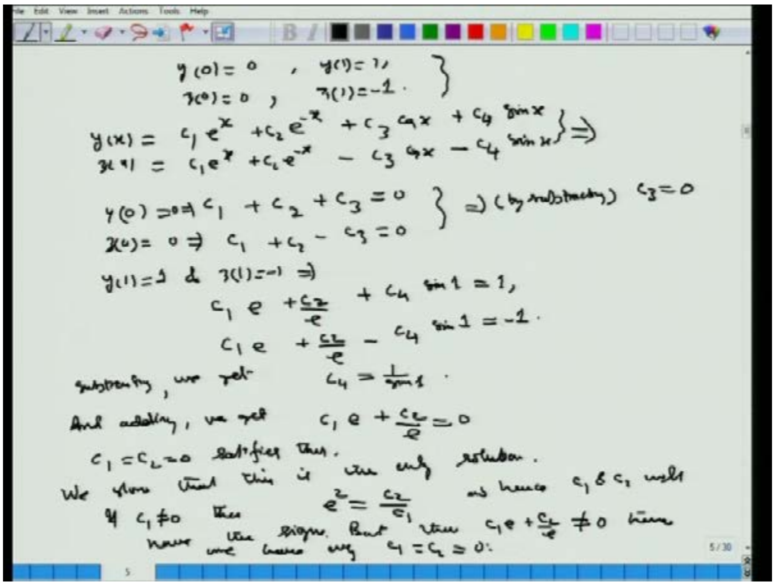

Now, this c 1, c 2, c 3, c 4 will have to be determined by the given condition. So, that we use here now since y of 0 equal to 0, y of 1 equal to 1, z of 0 equal to 0 and z of 1 equal to minus 1.

So, we have these conditions. So, this implies. So, first using this since now y of x is equal to c 1 into e to the power x plus c 2 into e to the power minus x plus c 3 into cos x plus c 4 into sin x and z of x is equal to c 1 into e to the power x plus c 2 into e to the power minus x minus c 3 into cos x minus c 4 into sin x. So, both this imply now that y of 0 equal to. So, we get c 1 plus c 2 plus c 3 equal to 0 and z of 0 equal to 0 implies z 0 equal to 0 this also implies that c 1 plus c 2 minus c 3 equal to 0 and. So, these 2 will imply simply subtracting these 2 by subtracting this gives us c 3 is equal to 0 and now y of 1 equal to 1 and z of 1 equal to minus 1 it imply that c 1 into e plus c 2 divided by e plus c 4 into c 3 is equal to 0. So, that is gone sin 1 equal to 1 and c 1 into e plus c 2 divided by e minus c 4 into sin 1 equal minus 1.

So, subtracting we get c 4 twice of this equal to 2. So, we get c 4 equal to 1 over sin 1.

Now, we see that and adding we get c 1 into e plus c 2 divided by e equal to 0. Now 1 solution is clearly we can get c 1 equal to c 2 equal to 0 satisfies this. We have to ensure that is the only solution to ensure that we show that this is a only solution.

Since if c 1 is not equal 0 then clearly then we get e square equal to c 2 divided by c 1 and hence c 1 and c 2 will have the same sign and therefore; but c 1 into e plus c 2 divided by e cannot be equal to 0 hence this is not possible. Hence we have only c 1 equal to c 2 equal to 0.

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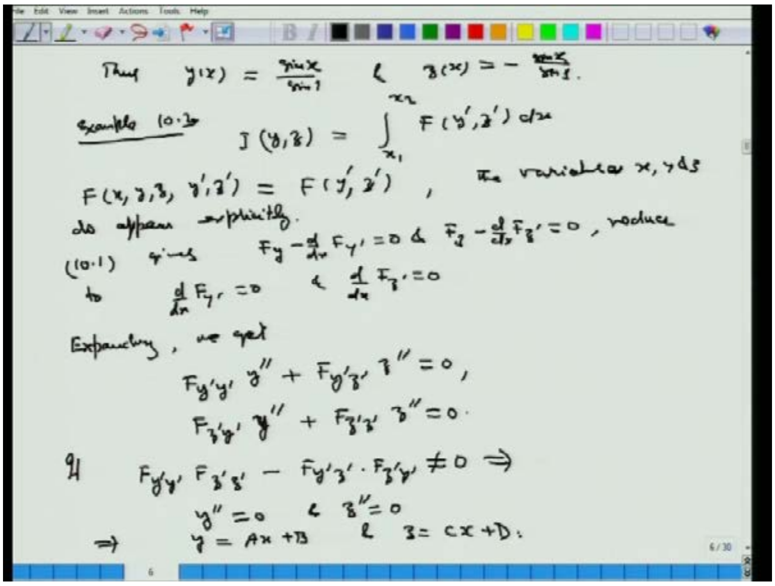

So, we get finally, this y of x equal to sin x divided by sin 1 and z of x equal to minus sin x divided b y sin y.

So, that is the case in this example. In the next example here 10.3, so I of y, z is it here integral of x 1 to x 2 F of y prime, z prime into dx. Here the variable x is missing and so, here F of x, y, z, y prime, z prime equal to F of y prime, z prime, the variables x, y and z do not appear explicitly.

So, it was in the earlier case function of y prime only in the special case which we had here like this and similarly here we only since there now there are 2 dependent variables and F is function of y prime and z prime only. So, it is similar to that case are generalization of that 1 and. So, here this system 10.1 gives that which is actually F y minus d divided by d x into F y prime is equal to 0 and F z minus d divided by d x into F z prime equal to 0 reduces to d divided by d x into F y prime equal to 0 and d divided by d x into F z prime equal to 0.

So, we now this is function of y prime and z primes and similarly F of z prime is also function of y prime and z prime.

Since y is… So, and therefore, opening this expanding we get F y prime y prime y double prime plus F of y prime z prime z double prime equal to 0 and F of z prime y prime y double prime plus F of z prime z prime z double prime equal to 0.

Now, here we would like to solve the system for y and z and so, if the coefficients a satisfies this if this into this minus this into this is not equal to 0.

So, y prime y prime into this F of z prime z prime minus F of y prime z prime into F of z prime y prime if this is not equal 0.

So, then we know that the only trivial solutions should hold and so, y double prime equal to 0 and z double prime equal to 0 and this implies that y equal to A into x plus B and z equal to C into x plus D.

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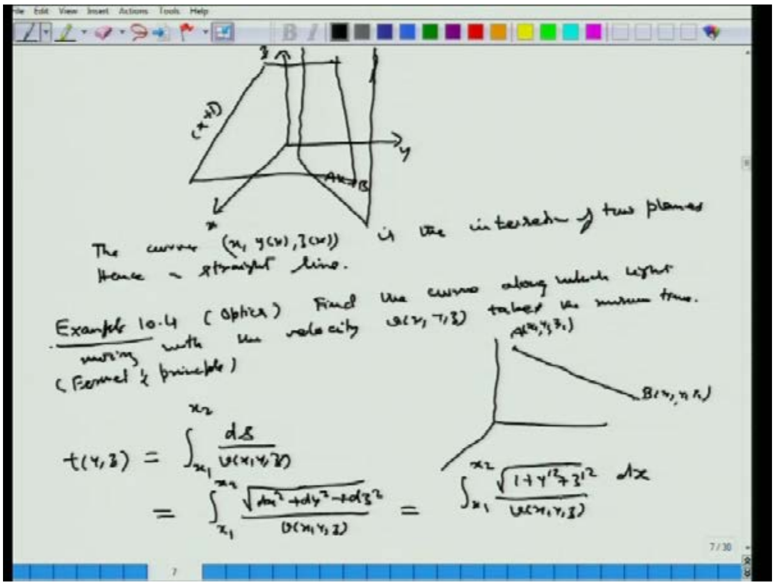

So, in this case you have situation like here and this is straight line like this is and similarly here. So, this is A into x plus B and this is C into x plus D and so, here z is free. So, this is this vertical plane like this and here.

So, here the intersection. So, the curve (x, y of x, z of x) is the intersection of 2 planes hence the straight line. So, in this case we get straight line

Now, this next example which is this is from optics here it is ask to find the curve along which light moving with the velocity v which is function of (x,y,z) takes the minimum time.

So, this is formal's principle what it says that the light passing through 2 points here A that is (x 1,y 1,z 1and B of (x 2,y 2,z 2) moving with velocity this takes.

So, the least time it takes that path. So, that here the time taken by this light rate which is moving with the velocity v which is the function of (x,y,z) takes least time. So, we know that the time taken by this here then it'll be function of (y,z) and it will be integral form x 1 to x 2 d s divided by the velocity this v of (x,y,z). And now d s we know that in 3 dimensional it is square root d x square plus d y square plus d z square divided by you have v of (x,y,z) .Now if we take x has parameter and y has a function of x and z has function of x then we see that this is integral from x 1 to x 2 square root plus 1 y prime square z prime square into d x divided by v of (x,y,z)..

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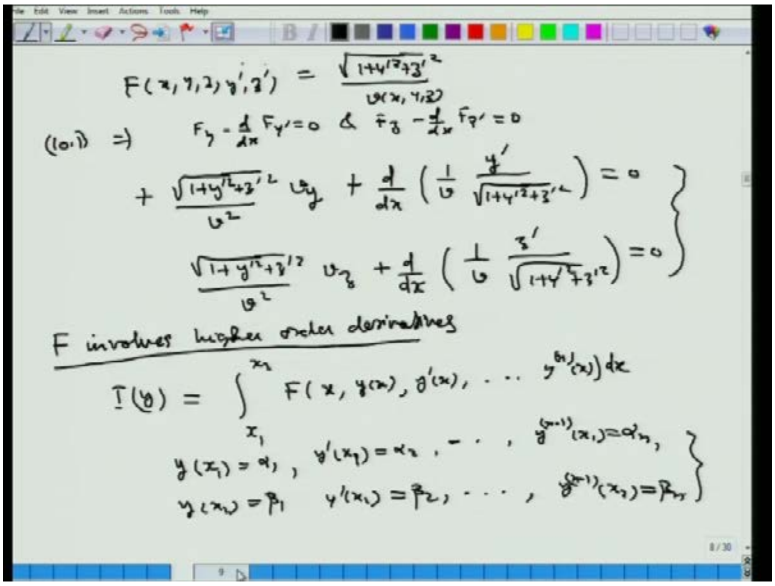

So, it is of the type what we have been considering and. So, here F is x, y, z, y prime ,z prime which is square root 1 plus 1 plus y prime square plus z prime square divided by v of x ,y, z and so, 10.1 implies that which is F y minus d divided by d x into F y prime equal to 0 and F z minus d divided by d x into F z prime equal to 0 gives you here F y gives you square root 1 plus y prime square plus z prime square divided by minus v square and then v into x and minus d divided by d x into 1 divided by v and 1 divided by should be y prime than square root 1 plus y prime square plus z prime square equal to 0.

So, we can multiply by minus and we make it plus here similarly the second one will give they should be y derivative v y.

So, square root of 1 plus 1 plus y prime square plus z prime square divided by v square into v z plus d divided by d x of 1 divided by v z prime divided by square root 1 plus y prime square plus z prime square. So, this is the system which should be satisfied by the velocity v and y prime and z prime.

So, in order to get y and z we solved the system to get the externals. Now next we will consider when higher derivatives are appearing F involves higher order derivative. So, here it will be like I of y equal to integral from x 1 to x 2 F of x, y of x, y prime of x and so on up to nth derivative of x still it is not general in the next one will be where these y 1,y 2,y m are appearing and there higher order derivatives are also appearing.

So, first we tackle this case. So, here this y and its derivatives up to n minus 1 must satisfy these conditions y of x 1 equal to alpha 1, y prime of x 1 equal to alpha 2 up to y n minus 1 of x 1 is equal to alpha n, y of x 2 equal to beta 1, y prime of x 2 is equal to beta 2 and so on to y n minus 1 of x 2 is equal to beta n.

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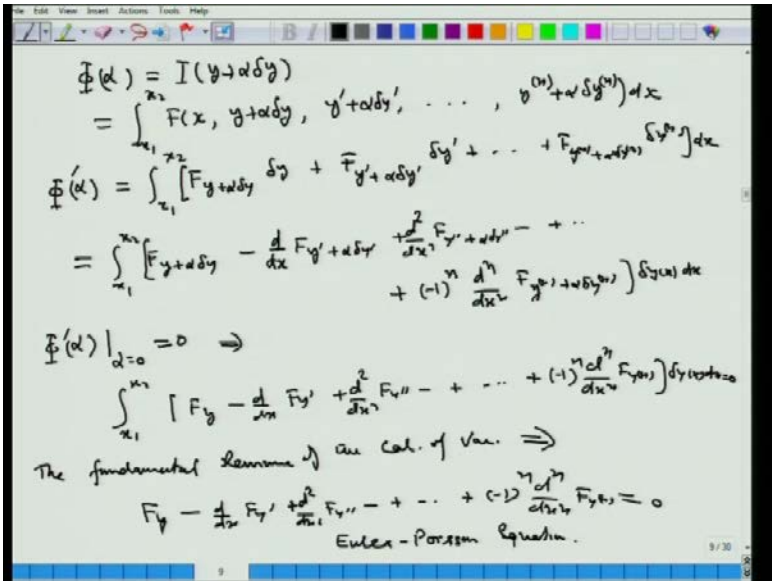

So, you'd like to see what are the conditions being satisfied with by this y in order this I of 2 have optimal value. So, here we proceed in the same manner. We consider this phi of alpha like this which is equal to I of y plus alpha into small delta into y and. So, this is equal to integral from x 1 to x 2 F of (x, y plus alpha into small delta y,y prime plus alpha into small delta into y prime and so on y power of n plus alpha into small delta into y power of n) into d x.

And So, differentiating it at with respect to alpha we get integral from x 1 to x 2 F y plus alpha into y plus small delta into y plus F of y prime plus alpha into small delta into y prime. And. so, this with respect to alpha will leave delta y prime and. So, on plus F of y n plus alpha into small delta into y n into x into small delta into y n into d x.

And proceeding the same manner shifting these derivatives onto F will give us this integral from x 1 to x 2 F of y plus alpha into small delta into y minus and so, delta y will be I mean raised from here that will take out and. So, we will have d divided by d x into F y prime alpha into small delta into y prime plus d power of 2 divided by d x power of 2 into F of y double prime plus alpha into small delta into y double prime minus plus 1 plus the last term will be minus 1 to the power n into d power of n divided by d x power of n into F y n plus alpha into small delta into y n into small delta into y of x into d x.

And so, now the condition is phi prime of alpha alpha equal to 0 must be equal to 0 thus the necessary conditions and so, it implies that integral from x 1 to x 2 F y minus d divided by d x into F y prime plus d power of 2 divided by d x power of 2 into F y double prime and so, minus 1 to the power n into d power of n divided by d x power of n into F y of n into small delta y of x into d x equal to 0.

So, the fundamental theorem fundamental lemma of the calculus of variations implies that F y minus d divided by d x into F y prime plus d power of 2 divided by d x power of 2 into F y double prime minus plus. So, on plus minus 1 to the power n into d power of n divided by d x power of n into F y of n the derivative is equal to 0.

So, this is what is called the Euler poison equation. So, Euler poison equation is the necessary condition for y of 2 has the optimal value of the function. So, here let us consider some examples.

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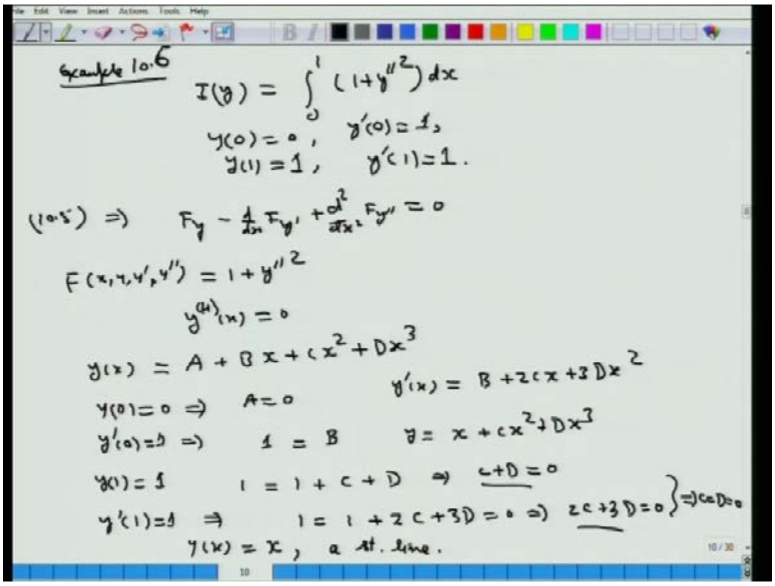

So, this example is now let us see what we had earlier 10.4 or rather we will put it this one as a 10.5 and this as10.6.

Here we consider I of y is equal to integral from 0 to 1 plus y double prime square into d x. So, the conditions are y of 0 is equal to 0, y prime of 0 is 1 and y of 1 is equal to 1 and y prime of 1 is also equal to 1 .

So, this 10.5 that is Euler poison equation.10.5 implies in this case F y minus d divided by d x of F y prime plus d power of 2 divided by d x power of 2 into F y double prime because here n equal to 2

Here since F of (x,y,y prime,y double prime) is equal to 1 plus y double prime square here and. So, we get this y is not appearing explicitly. So, we this term is 0 this term is also 0 only we get y Fourth derivative equal to 0 and. So, we get y of x as. So, it should be a polynomial of third degree. So, it should be of the form A plus b into x plus C into x square plus D into x cube.

Now, this and A,B,C,D will have to be determined by the given conditions. So, y of 0 is equal to 0 implies that A equal to 0 and of y prime of 0 to 1 implies y prime of x is A is 0. So, we get B plus 2 into C into x plus 3 into d x square.

So, this will give us 1 equal to B. So, now, y equal to x plus C x square plus d x cube now and. So, the other conditions that y of 1 is also equal to 1. So, we get 1 equal to 1 plus C plus D. So, this implies that C plus D equal to 0.y prime of 1 equal to 1 implies that 1 equal to 1 plus 2 into C plus 3 into D equal to 0. So, we get 2 C plus 3 into D equal to 0 and this and this would imply that C equal to D equal 0..

We get finally, y of x equal to y equal to only B equal to one and other things are 0. So, we get y equal to x is the extremely a straight line in this case as expected that whenever this is function involving only highest order derivatives we expect that this solution will be a straight line.

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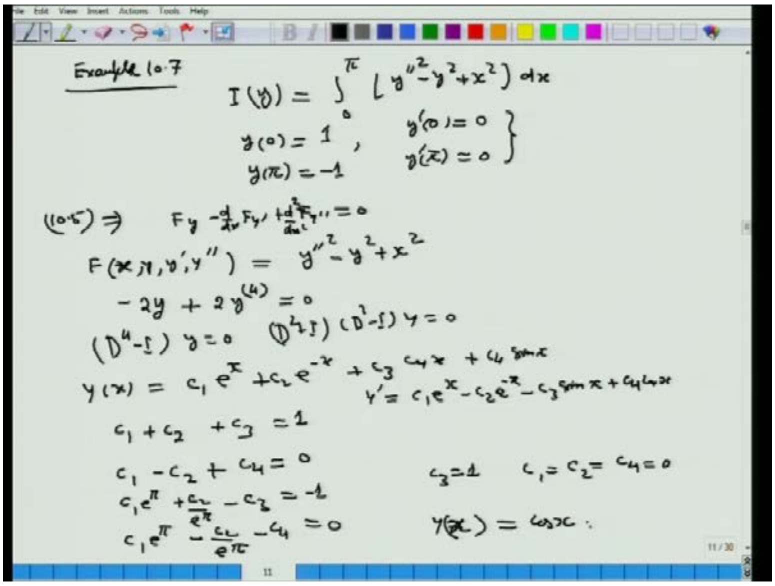

The next one would be this example 10.7. Here this I of y is equal to integral from 0 to pi y double prime square minus y square plus x square into d x. The conditions are y of 0 equal to 1, y prime of 0 equal to 0 and y of pi equal to minus 1 and y prime of pi equal to 0.

So, again this 10.5 implies this case that F y minus d divided by d x into F y prime plus d power of 2 divided by d x power of 2 into F y double prime equal to 0 and. So, here since F is which is (x,y prime, y double prime) is equal to y double prime square minus y square plus x square.So, F y will give us minus 2 into y. So, minus 2 into y here and plus this is no y prime here only y double prime and. So, you get 2 into y Fourth derivatives equal to 0.

So, we get D power of 4 minus I of y equal to 0 or D square plus I into D square minus I into y equal to 0 and. So, as before we get y of x equal to c 1 into e to the power x plus c 2 into e to the power minus x plus c 3 into cos x plus c 4 into sin x

Now, the conditions imply we have this y of 0 equal to 1. So, this is c 1 plus c 2 plus c 3 equal to 1. As the first condition then y prime equal to 0. So, y prime will give us here. So, y prime of 0 means you will have c 1 minus this minus sign will come here minus c 2 this will become minus cos minus sin x and. So, it will give us minus c here this minus sin x will become cos x and. So, the since c 3 will term will not be there plus c 4.

So, let us write y prime here y prime is c 1 into e to the power x minus c 2 into e to the power minus x minus c 3 into sin x plus c 4 into cos x.

So, we will get when we put 0 c 1 minus c 2 this will become 0 plus c 4 and that is equal to y prime 0 equal to 0 thus the second condition and now at pi here minus 1. So, we get c 1 into e to the power pi plus c 2 divided by e to the power pi minus c 3 equal to minus 1 and c 1 into e to the power pi minus c 2 into e to the power pi minus c 4 is equal to 0.

Now, we can see that here we get this system will have unique solution and we can see that here c 3 equal to 1 and c 1 equal to c 2 equal to c 4 is equal to 0 which satisfies this equation become c 3 equal to 1 here and this all 0 here this minus 1 and all the 0.

So, this since these system is having unique solution and. So, we can see that here c 3 equal to 1 and other things are 0 gives you the solution and we get the extremely y as a function of x as cos x.

So, here we considered more general functional now in the next we will consider where more independent variables will be appearing and we will see that we will have domains in the higher dimensions spaces that we will consider in the next nature thank you very much for viewing this lecture.