Calculus of Variations and Integral Equation

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Lecture No. # 01

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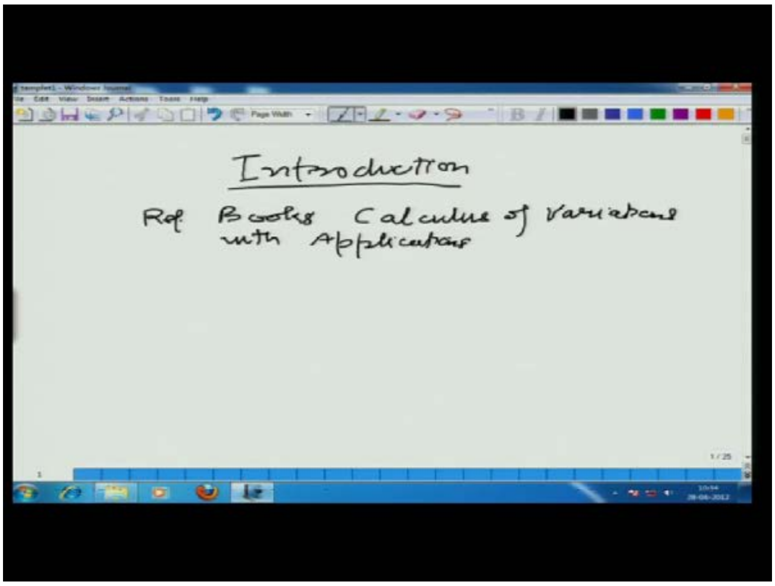

Hello, this is the first lecture of the set of 20 lectures on calculus of variations. This lecture, I will start with the introduction of calculus of variation. Here in this, I will be following the reference book or the books. This is calculus of variations with applications. On calculus of variations, we will be covering many aspects of calculus of variations, how the study of calculus of variation started, and what are the applications of the theory developed for this, that is what we will be discussing in this.

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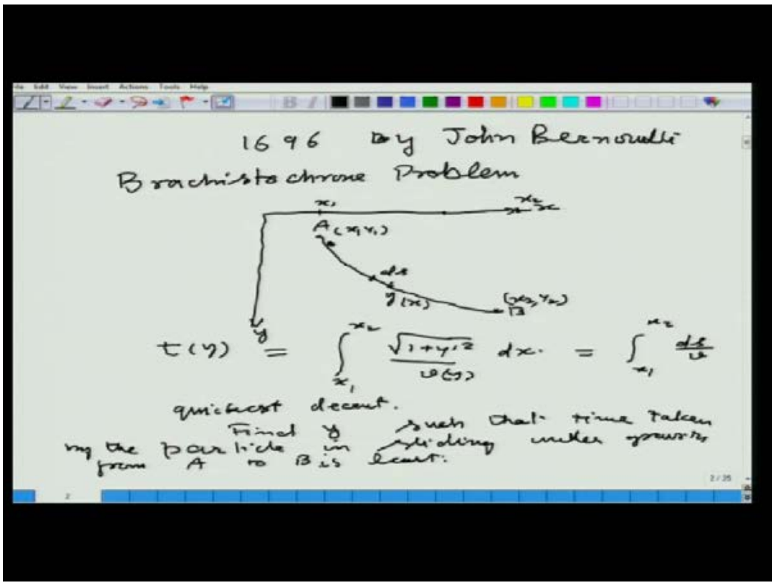

So, in this first lecture, we will give some introductions of problems arising in calculus of variations. The study of calculus of variation started in 1696 by John Bernoulli. He proposed a problem, which is known as Brachisto chrone problem. In this problem, what he proposed to find path along which the particle slides under gravity then it takes the least time. So, mathematically this problem can be stated like this, that in a vertical plane, you have 2 points A and B, not at the same level at different levels. And then if you join this by a smooth curve like this and if particle slides from A to B under the influence of gravity then suddenly the time taken by this particle from coming from A to B is function of this y, where y is this curve as a function of x. So, this is the x axis, and this is y axis and this y is a function of x smooth function of x.

So that at each point, a tangent is defined and so here this time taken by the particle under the influence of gravity sliding from A to the point B is denoted by t y and this will be given by here, let us say this is the point x 1 and this is the point x 2. Here, so this point is (x 1, y 1) and this point is (x 2, y 2). Then here this is actually can be seen. This is like integral from x 1 to x 2 d s upon the velocity, that is what will d s is the element arc length along this curve, the integral is from x 1 to x 2 square root of 1 plus y prime square upon this the d s distance and upon the velocity which is a function of y. And this functional is what called the time taken by the particle sliding from A to B under the influence of gravity.

Here, this time, here will be actually given by this integral x 1 to x 2 square root of 1 plus y prime means d y by d x upon the velocity. Here this actually can be seen. This is like a integral from x 1 to x 2 d s upon the velocity that is what will d s is the element arc length along this curve. So that d s, the time taken from this point to this point, so in this element area d s upon the velocity that will be the time taken by this particle coming from this point to this point. Now here, so if this total time taken by this particle from A to B will be given by the integral x 1 to x 2 integral of d s upon v. So, d s is actually a square root of 1 plus y prime square and the velocity v will be taken as a function of y. So, that is the time taken by the particle from side of a sliding from point A to B.

Now, this is what is called the problem of Brachisto chrone and we are asked to find a curve along which the states the least time. So, that is what is called the problem of quickest decent quickest decent. So, we have to find y such that time taken in sliding time taken by the particle the particle in sliding under gravity from A to B is least. So, this is the problem of Brachisto chrone. Now, this problem was proposed by the famous mathematician John Bernoulli and it was in the same year 1696 was solved by several mathematicians like Newton, Lebanese, Weierstrass, and John Bernoulli himself and some other mathematicians of that time. So, that the problem of calculus of variation is started with that. Now, there are many other problems, which can be posed as the problem of calculus of variations

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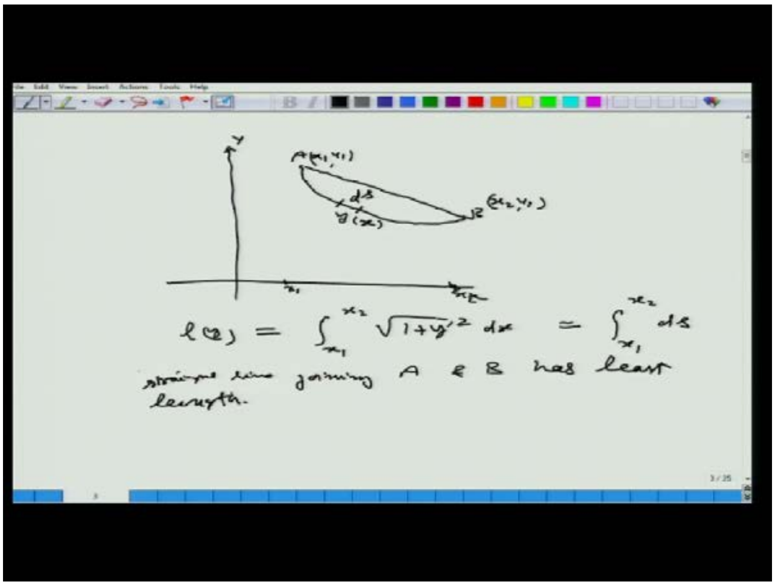

like you are in the plane. Suppose you have a 2 points A and B and then the length of the curve is the function y of x. We know that the length of this curve is given by the integral x 1 to x 2 square root of 1 plus y prime square d x which is again actually given like x 1 to x 2 d s. So, again here d s is the element area element length in terms of arc length s. So, integration of this d s element length from x 1 to x 2, here again this is point x 1 and this point is x 2. So, this is (x 1, y 1) and (x 2, y 2). So, this is what this integral x 1 to x 2 square root of 1 plus y prime square d x which is nothing but the integral of d s element length x 1 to x 2. So, this gives the length of the curve y. So, this is a function of y. If we change y then its length will change. Now the problem of calculus of variation in relation to this is to find the curve such that the length is least. So, we know the answer is that the state line joining these 2 points will be having. So, straight line joining A and B has least length, but how we prove it mathematically? Again it is a problem of calculus of variations.

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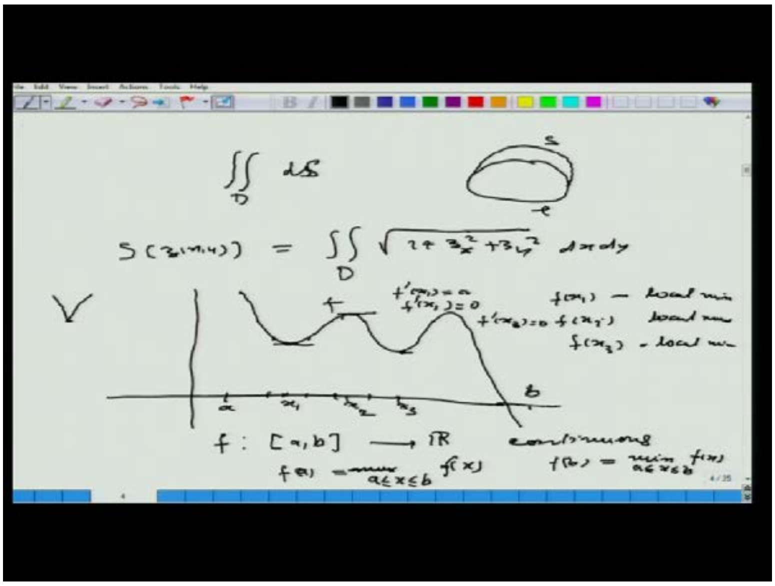

And next problem can be thought as suppose you have this surface area double integral over D then you have surface area d s, now which is bounded by so this surface area over D. So, this is what is the smallest surface area what I mean the curve is given to you and then here let us say this is s surface bounded by this curve and now this length is length of the curve is fixed. Now we want to find what is that surface which will have the least surface area? We know that it will be that flat area, if it is a plane curve. So, this is also a problem of calculus of variation.

Now so with this, here the other problem of surface of area for a given surface z x y, it can be written as double integral D square root of 1 plus z x square plus z y square d x d y. So, here we have to choose that surface S surface z x y such that this has least surface area, so this also a problem of calculus of variations. Now, here this is certainly a generalization of a minimization or maximization of a curve, where you want minimization, maximization of points at which the function has minimum or maximum; for example, here if you consider this curve.

So, let us say this is a point a, and this is the point b here. So, this f is defined from a to b into x into a to b into R and this function has (( )) at least continuous and we may like to have more smoothness property on this function like differentiability at all points in the interval [a, b]. Here we can see that this, at this point f a is maximum of f x between a to b. So, f a is the maximum here and f b, similarly f b is a minimum of f x between. So, this is a global maximum f a is global maximum and f b is a global minimum then there are other points like this, let us say this is point x 1, and this is point x 2, this is point x 3. So, here in this, f of x 1; similarly f of x 2 and f of x 3, now these are the points of local minimum or maximum.

For example, f, this is local minimum, because in the neighborhood of this, here these are the least value. Similarly, in the neighborhood of x 2, here this has the maximum value. So, this is local maximum. Similarly, f 3 is also local minimum, but we have this, f a as global maximum and f b is global minimum. So, at these points, here we can see that the if the function is a smooth, if it does not have corners like this like this, if it is a smooth function like this, where the derivative exists then we can see that here tangent becomes horizontal. Similarly, the tangent becomes here horizontal; that means, derivative is zero, a prime at this point x 1 is zero, a prime at x 2 is 0. So, a prime at x 1 equal to zero, similarly a prime at x 2 is zero and a prime at x 3 is also 0. So, these are actually if the function has continuous derivative at all points in the interval a to b then we can see that at these local minimum or local maximum, we have the derivative of these functions becoming 0. So, here given a functional like this.

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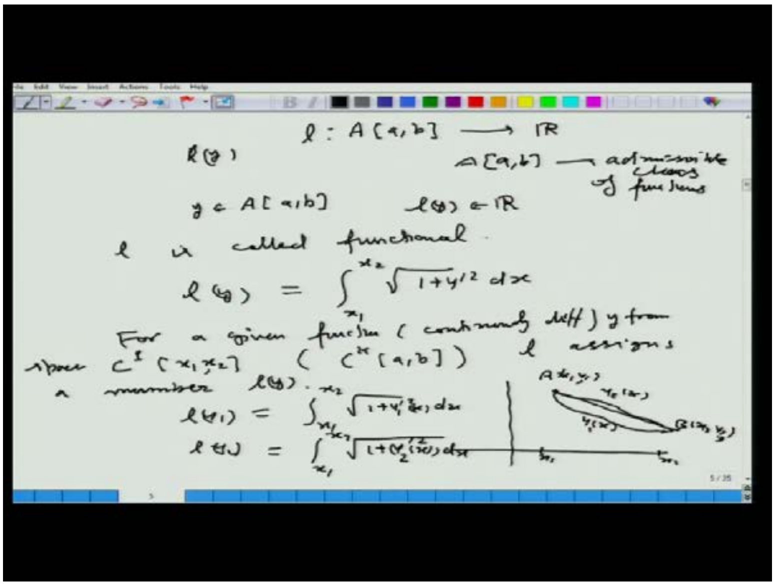

So, we have a functional; for example, here l of y where l is a function from this is function space. So, we call it the admissible function class from here interval a to b. So, this into R, where A is the admissible class a is called the admissible class of functions. So, l y is actually called functional. So, l y, l is for each y here, so each y in a b A a to b, we have associated l y which is a function in, which is the number R. So, given a function in this admissible class, this l which is called functional is a function of functioning loosely speaking. So, here given any so here the argument of l is a function actually whereas, in the previous case, here this f was from number to number. Here in this case, l is from a given function to a number. So, it associates a function to a number. So, such a thing is called functional. So, l is called functional.

For example, this length of y given by this x 1 to x 2 square root 1 plus y prime square d x. this gives you the length of y. So, l for a given function continuously differentiable from the set or the from the function space we call it C 1 x 1 to x 2 from the space C 1 to C 2. C 1 means continuously differentiable the set of the space of all continuously differentiable function from the interval x 1 to x 2 into R. So, this is the space of all continuously differentiable function, let us say usually we will consider C k, k being integer nonnegative integer C k a to b which will denote the space of all continuous functions whose k eth derivative will be continuous. So, for k equal to 1, we will call it C 1 x 1 to x 2 and the values are real numbers. So, we are considering only the real valued functions here, and so an element here will be called an admissible function here. So, this is the admissible class like a comma b. So, here this a is x 1, b is x 2 and this A here is actually C 1. So, that is the class of that is the admissible class of functions for this functional to l to be defined here. So, here in this case, for any given function y from this is space l assigns a number l y. So, that is what is the actually functional which assigns to each function y in this is space admissible space of functions here for this functional to be defined.

If we have odd derivatives appearing here then we will take R order function spaces. So, this for given y, the length of y will be given by this integral here. So, if we change for example, here you have this x 1, x 2 here. So, this the point A x 1 y 1, and this the point B x 2 y 2, and this the curve y x here, which joins these 2 points a and b. So, this functional will give the length of this y, if you change this let say this is y 1, if we consider the another function y 2 here. So, then we will l of y 1 will be given by integral x 1 to x 2 square root 1 plus y 1 prime square x d x.

Similarly, l of y 2 will be integral x 1 to x 2 square root 1 plus y 2 prime x square d x like that. So, we change this y, this l will be changed. Now we want to find what is that y for which we get the least number, which will be the least length? The length of the curve for joining these 2 points a to b such that its length is the least one. We know the answer that it has to be a straight line joining these 2 points, but how do we prove that. So, that will be a problem of calculus of variations. So, now before going into the calculus of variations, we need to develop certain tools for that. Similarly, we will be having this higher order.

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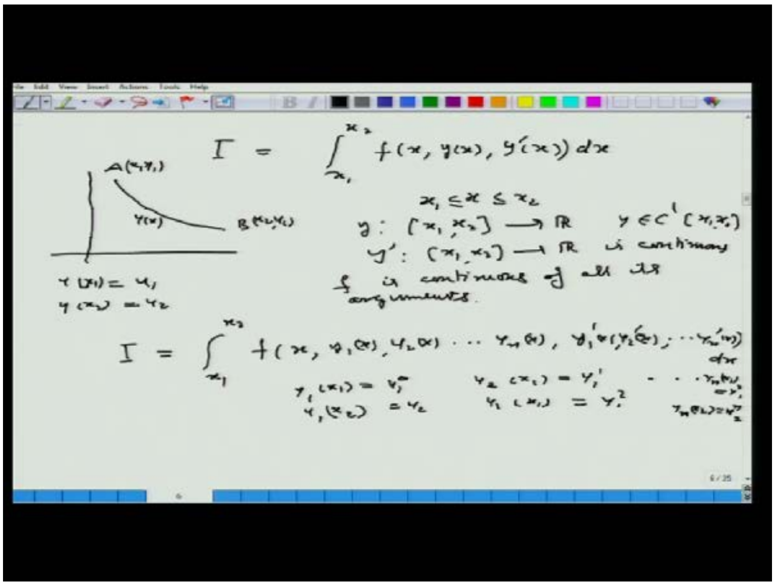

So, here the functional will be of this type like the integral I going from x 1 to x 2 certain function x, certain function f, which is function of these variables x is an independent variable y as a function of x and then d y by d x that is y prime x d x. Now, again here this f is function of 3 variables x, y, and y prime. So, here and these functions like these arguments of course, x is an independent variable which is varying, so x is lying between x 1 to x 2, and y is a function from x 1 to x 2 into R such that y prime and y being C 1 x 1 to x 2 like this. So, that y prime will also be a function from x 1 to x 2 and it will be continuous. So, that y prime is continuous here and this f is continuous of all its arguments. So, there are 3 arguments for this function x, y and y prime and here this f is a continuous function of this variable, this variable and this and y in turn is continuous function between x 1 to x 2 and y prime is also a continuous function. So, over all this integral becomes a continuous function from x 1 to x 2, and then by this remain hypothesis, we know that this integral I exist as a remain integral.

So, here the problem of calculus of variation is then to find y function y such that it is derivative is also continuous and this value is minimum or maximum. So, we call such a thing as optimum value optimal value and why we call as optimal curve for or extremely is called extremal and this integral I will be optimal. So, these are the terms we use that what is that extremal for which I is actually having the value extremal value. So, such a functions y and which are from the admissible class. Here admissible class is C 1 class of functions from x 1 to x 2 into R.

So, here the generalization of this could be like I x 1 to x 2 and f has more variables now like x, y 1 x, y 2 x and similarly this y n x and then there derivatives y 1 prime x, y 2 prime x and so on up to y n prime x d x. So this, the generalization of this, so this can be treated as a particular case of the more general problem where again you have these y 1, y 2, y , they are from the admissible class like, you, we are considering here derivatives. So, they can, derivative should be continuous here in order this integral to exist as a remain integral. So, here f will be actually a continuous function of all these arguments and these depend this x is an independent variable here and y 1, y 2, y n they are dependent variables and these derivatives are also appearing here.

So, again the problem of calculus of variation is to find these functions y 1, y 2, y n and such that this integral has either minimum value or maximum value depending upon the problem posed here and in addition to this like, we have in the this case, we have the curve joining these 2 points a and b. So, here the condition is that it should pass through (x 1, y 1) and (x 2, y 2). So that means, here y 1 at x 1 should be this y 1 here or y at, so y at, so if y joins this. So y, so this y at x 1 should be y 1. Similarly, y at x 2 should be y 2.

So, that this curve passes through those given points so and these points are fixed here. Similarly here, this y 1 x 1 must be some given value like, so again if you have several curves y 1, y 2, y n and they should pass through the these given points A and B. So, this is y 1 x 1. There will be conditions on this like, you have y 1 y 1. Similarly y 1 x 2 will be y 2 and so this should be for all those y 2 x 1 will be rather y 1 1, y 1 0. You can put here y 1 1, y 2 at x 2 is y 2 1 and so on. You have y n x 1 as y n 1 and y n x 2 is y n 2. So, these are the additional conditions to be satisfied, so that the curve pass through those given points here.

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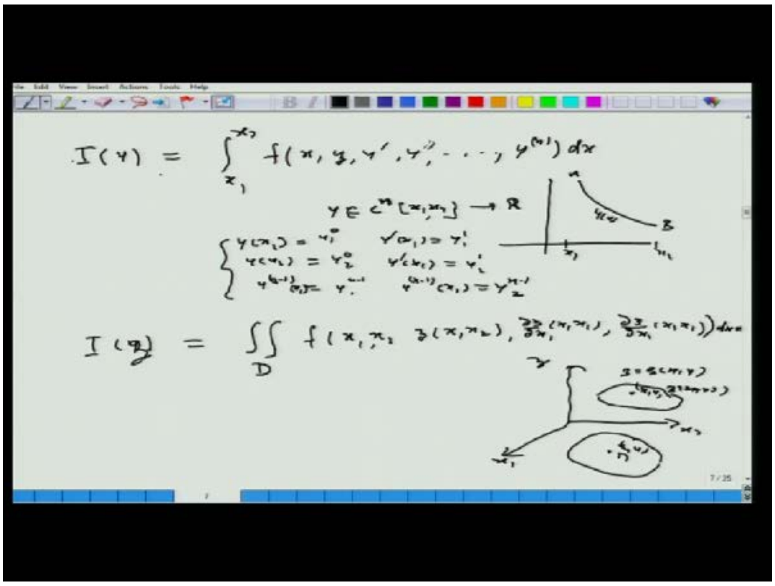

For that generalization could be, here you consider higher order derivatives. So, I again it will be. So, in the previous case, this I is a function of I is a function of y 1 y 2, and y n, and here in this case, let us say I is a or a special case like you have I as y only. But then you have higher order derivatives appearing x 1 to x 2 f of x y y prime y double prime, and up to y n of the derivative is appearing there. So, like this. So, again here the curve y is expected to it is passing through those given points A and B here. So, this is y x and it has a more smoothness now. So, admissible class has to be c n x 1 to x 2. So, y has to be actually in this space of functions c n. So, up to nth order derivative should exist and it should be a continuous function from x 1 to x 2. So, again here you will have certain additional conditions like y at x 1 is y 1, and y at x 2 equal to y 2 like 0 0. And then y prime at x 1 is something y 1 1, y prime at x 2 is y 1 2 and so on, there will be up to y n minus 1 at x 1 y n minus 1 1 ,and y n minus 1 at x 2 y n minus 1 2.

So, these are additional conditions given. So, that this y has to satisfy all these conditions and it has to optimize this integral depending upon whether to minimize or to maximize. Then further generalization could be like I y. Here you have more independent variables, for example, you have let say 2. So, then you have double integral and f is function of let say x 1 x 2, and then some function of x 1 x 2, and then dou z by dou x 1 x 1 x 2, and dou z by dou x 2 x 1 x 2 and dx. Here you have two-dimensional situation here, this is x 1, x 2 and there is some domain d here and on this, z is the surface given like this. So, this is the surface z equal to z x y. So far a given any point, here this is the value z x y x y. Here this is z x y and z x y, this point is x y and z x y. Let us say this is z here like this.

So, here this function f is a smooth function here of all these arguments and this z x y is denoting this surface here. So, here again the problem of sorry this should be z here. So, this integral I is a function of z, if you change this surface this value is going to change and we need to find that z such that it actually the boundaries fixed here. So, long this, z will be satisfying certain boundary condition that z has the fixed boundary here and that z is actually optimizing this functional.

So, that is the condition of that is the problem of optimization, which we will be considering here. So, we will be considering these many such cases where we will have one of these situations here. Now, we will consider certain preliminaries of the concepts, which will be required subsequently.

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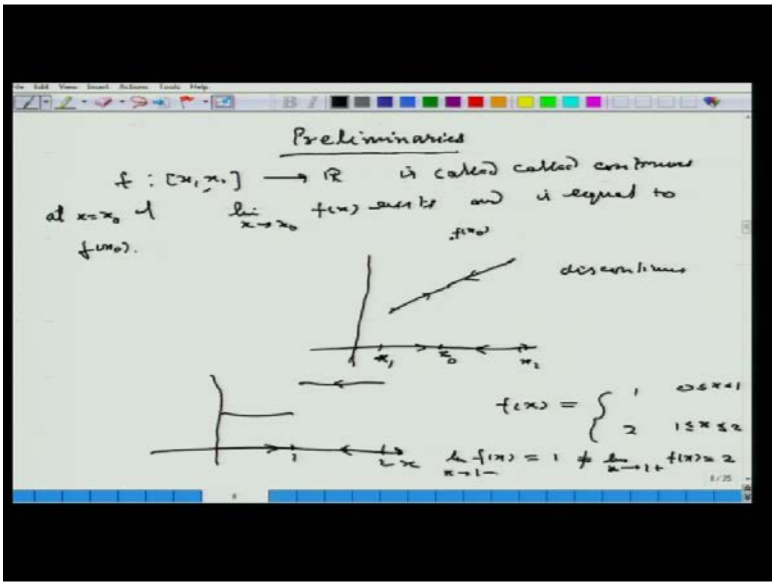

So, we start with preliminaries. So, here we had been considering continuity of a function. So, what do you mean by that? A function f from this x 1 to x 2 to R is called continuous, if continuous at x equal to x 0, if this limit extending to x 0 of f x exist and is equal to f of x 0. For example, here you consider this function. So, here this is x 1, this x 2 and then there is some point here x 0 here.

So, here you can see that let us say, if we consider this is a straight line except at that this point x 0, function is not defined here. So, it cannot be continuous, because it has to be defined at that point. So, let us say at this, the function is here. So, at x, at all these points, function is like this and here all these points on the right, the function is like this up to x 2. But at this point x 0, this value is here. So, this is, although the limit extending to x 0 exist and here that will not be equal to the value of the function which is actually this. So, this is a discontinuous function.

Again the other example is, suppose the function is 1 here and this point onwards 2 here. So, here up to 0 to 1 the value is so f x equal to 1 0 less than x less than 1 and it is 2 for 1 less than equal to x and let us say up to 2 1 to 2. So, this function is discontinuous at this point. Here the left limit exist, right limit exist, but the that is not equal to the value of the function here. So, the left limit here is 1 at this point. So, f x limit extending to x 0 minus that is what. So, this limit extending to 1 minus means if you approach this point 1 from the left side. So, this will be actually equal to 1 and if you take limit extending to 1 plus; that means, you approach from this side. So, of f x that we know that we are approaching from this approaching to this point to from this side like this, it will go to 2. So, these two are not equal and therefore, this limit does not exist and therefore, the function is discontinuous which can be seen intuitively also from this figure that there is a discontinuity here.

Although here in this case, it is not so apparent here, but then we can see that if we approach to this point from the left, then it is going to this value here and similarly if you approach to this point like this from here. So, values are approaching along this and it is going to this. So, these left limit and right limit they are same and so limit exist, but it is not equal to the value of the function, which is also a condition for continuity here, because f of x 0 is here f of x 0 is here, which is not here. So, therefore there is a discontinuous. So, these kinds of things should not happen for a continuous function.

So, these are the concepts of left limit and right limit and the value of the function they should all be equal. So, then we say that the function is continuous. When discontinuity can occur in many ways? But we will be considering only discontinuous function of the first kind. So that, here what we have, only jumped discontinuities like this. So, in this case, we have left limit and right limit exists and they can have the difference here. So, that difference is called the jump here. At this point, the jump is this, the right you are approaching to this point, and from the left we are approaching to this point here. Therefore, there is a jump discontinuity here.

So, that is what is called the discontinuity of the first kind. The discontinuity of second kind, where limit does not exist at all, one of the limits left or right does not exist. Such a thing will be called discontinuity of the second kind. We will not be considering such functions and we will consider only the functions having only discontinuities of the first kind. So, that they have jumps at a finitely many points. So, such functions are called piecewise continuous function.

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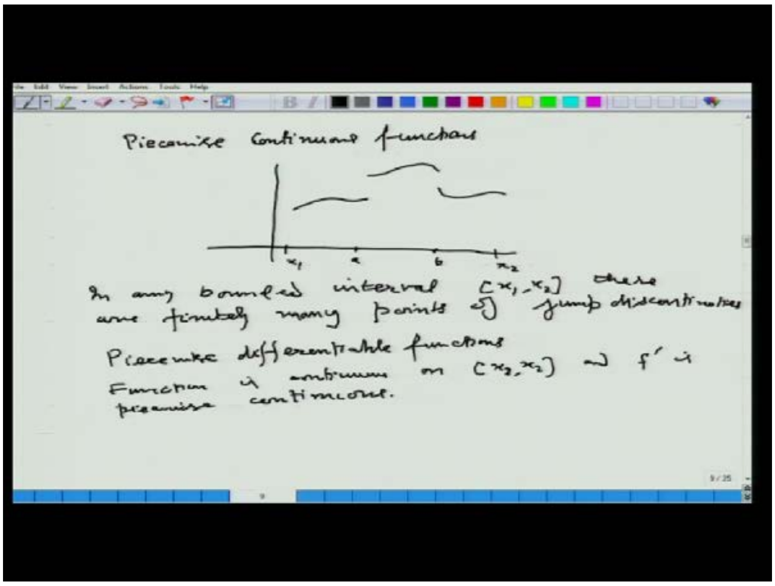

So, we have piecewise continuous function. So, these are the functions like, you have left and right limits existing in the interval like this x 1 x 2. So, here this is the point of discontinuity, this is the point of discontinuity here. So, let us say at a and b in the interval x 1 to x 2. Here at this point, there is a jump discontinuity and the here also there is a jump discontinuity of this. So, we can have a finitely many in any in any bounded interval like x 1, x 2; there are finitely many points of jump discontinuities. So, there are there are only, like in this interval, there are only 2 points here at this point and this is at the point a and at the point b, there are jump discontinuities here.

So, this is a piecewise, this is an example of piecewise smooth function. So, in any interval of finite length, there should be only finitely many points where the function can have jump discontinuities. So, such a function is called piecewise continuous function.

Similarly, we will have the piecewise differentiable function. So, piecewise differentiable function means here you have the function will be continuous function is continuous on x 1 to x 2 and a prime is piecewise continuous. So, a prime has only jump discontinuities at finitely many points in the interval x 1 to x 2. So, such a function is called piecewise differentiable, if it is, if the function is continuous on x 1 to x 2 and a prime has left derivative and right derivative existing at all points and there are only finitely many points where they differ. And it will eliminate the (( )) case of the points, where the left derivative and the right derivative are not equal only if they are equal then it will be actually equal to the value of the function.

So, if there cannot be a case, in this that the left derivative and right derivative are equal, but they are not equal to the value of the derivative the value of the derivative at that point. So, such a case will be avoided in this case, because function is continuous and if the left derivative, right derivatives are equal, then the derivative at that point will also exist and it will be equal to the value of the function.

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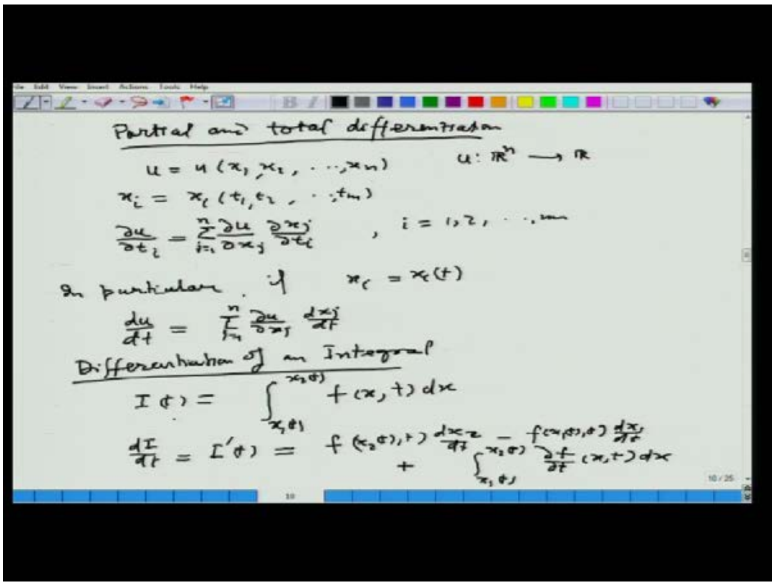

Next, we will be considering a partial differentiation and total differentiation. Here we consider a function u as a function of several variables u, let us say x 1, x 2, x n. So u, here this u is from R n to R or in certain domain, which is a subset of R n and the values are real numbers here. So, and these x 1, x 2, x n are in turn and this x 1 or x i is function of let us say t 1, t 2, t m. So, then we will be considering this dou u over dou t I, it will be actually dou u over dou x j and then dou x j over dou t i summation over j equal to 1 to n. So, j is running from 1 to n and this i is running from 1 to m. So, this i equal to 1, 2, to m.

So, these are the partial derivatives. Here u is a function of several variables x 1, x 2, x n and these x i's in turn are functions of some n variables t 1, t 2, t m. Then the partial derivative of u with respect to these variables t i's will be given by this kind of chain rule, summation j equal to 1 to n dou u over dou x j and dou x j over dou t I and summation over this j going from 1 to n.

Now, if the in particularly if this x i's are, so in particular, if x i's are just function of one variable, x I of t only. Then we can have, then this derivative will be total derivative like d u by d t and you have this summation dou u by dou x j and then d x j by d t then j equal to 1 to n. So, this will, here then this u itself becomes a function of single variable t and therefore, this partial derivative reduces to ordinary derivative that is known as total derivative here. Although u is function of several variables x 1, x 2, x n, but these x i's are now in this case function of a single variable t and therefore, u is a function of single variable t, we get d u by d t as a total derivative in this case.

Now the next one would be differentiation of integral at several places, we will have the integrals coming into our picture as functional and we will be required to differentiate it with respect to its arguments appearing in the integrant. So, here like I of certain variable t integral x 1 function of t x 2 of t and f let us say x and t d x. So, this is x is a variable of integration here and this as limits x 1, x 2, but these f as well as this x i's are functions of t. They are continuously or piecewise a differentiable functions of t then this d I by d t is given by the Lebanese rule is called that is also denoted short I prime t, which is f at the upper limit. You have x 2 t t and then d x 2 by d t minus f at x 1, which is function of t t d x 1 by d t and then plus integral x 1 t x 2 t and the dou f by dou t of x t d x. So, this is what is called the Lebanese rule, which can be seen by the first principle, I prime t will be actually.

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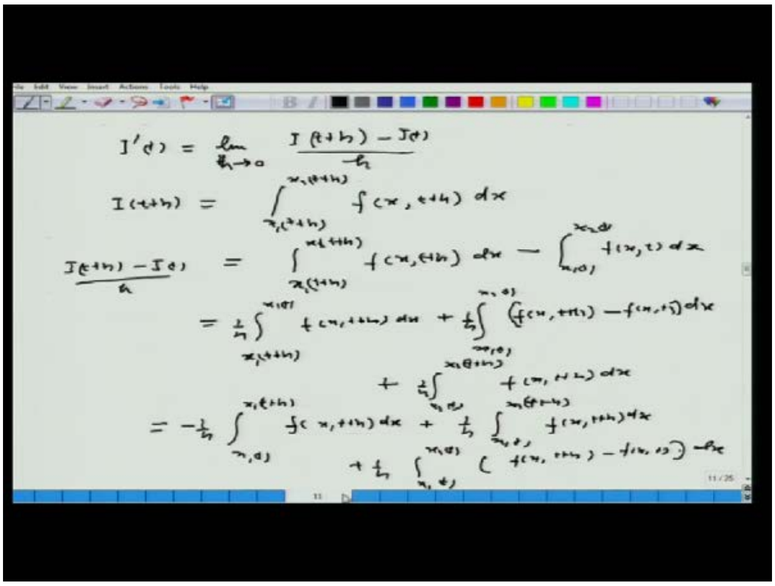

See here it can be calculated by this, what is I prime t? I prime t is by definition, it is limit t tending to or h tending to 0 I t plus h minus I t upon h. So, we can apply this here. So, we will have this I t plus h where t will be replaced by t plus h; here x 1 t plus h, x 2 t plus h. So, that is what we will have I t plus h will be given by x 1 t plus h x 2 t plus h f of x t plus h dx, and then subtracting here. So I of t plus h minus I of t will be x 1 t plus h x 2 t plus h f of x t plus h dx minus x 1 t x 2 t f of x t dx .Then here we will break it into 2 parts, x 1 t plus h to x 1 t, and then...

So, this first integral can be written as x 1 t plus h to x 1 t and f of x t plus h d x plus x 1 t to x 2 t f of x t plus h. So, this you can take with this minus f of x t d x and plus then you have x 2 t to x 2 t plus h f of x t plus h d x and then dividing by h here. So, we get 1 by h here, 1 by h here, 1 by h, then letting h tends to 0 , we get the first term by this, that the second term with the, because this we will write as minus of this thing. The first term, we will write it as minus 1 by h x 1 t to x 1 t plus h f of x t plus h d x and this the last one is anyway, here as it is x 2 t to x 2 t plus h f of x t plus h dx.

And the middle one will be that 1 by h of x 1 t 2 x 2 t. This 1 by h we will take it inside and then f of x t plus h minus f x t d. So, 1 by h taking inside and using the continuity or differentiability property of f, we can see that the last term. Here this one will be given by which will give you the partial derivative with respect to t here. Like in this one and this term, with the that minus sign, here is the limit of this, and the first term will be the limit of this. So, we get the, this, what is called the Lebanese rule of differentiation of the integral. The next tool will be the integration by parts. Thank you very much for viewing this lecture.