# Introduction to Queueing Theory

**Prof. N. Selvaraju**
**Department of Mathematics**
**Indian Institute of Technology Guwahati, India**

## Lecture - 07
## Classification Properties of Markov Chains

Hi and hello, everyone. In the previous lecture, we have seen the definition of the Markov chain and some examples and certain few properties, not properties, that you know with respect to something like a multi-step how you can compute and everything. Now, what we will do in this lecture is we will look at a little bit more properties of Markov chains, and the first thing we will talk about is what we call here accessibility.

- Accessibility: State $j$ is said to be accessible from state $i$ if there exists $n \geq 0$ such that $p_{ij}^{(n)} > 0$, where $p_{ij}^{(0)} = \delta_{ij}$.

  ► If $j$ is not accessible from $i$, then $P\left(\text{Ever be in } j | \text{starting from } i\right) = 0$.

  So, more of these properties that we are going to see, you will see that you can prove analytically, but it is intuitively in a probabilistic way if you think. Then things will be clear; you do not need to worry too much get into the proofs or anything. Because it should be true in an intuitive way, that is how now you will see; that is what we call probabilistic reasoning would be sufficient enough to understand the properties and why and how that should be true. So, it is more of this. So, accessibility means that you are starting at some state $i$. Say, for example, you are starting at some state $i$, and you will reach state $j$ if there is a way to reach maybe, going through different states ultimately. It cannot be possible in one step; maybe it is possible in two steps, maybe it is possible in three steps, and so on; there is some step at which it is possible. Then you would see that it is $j$ is accessible from $i$, or we will say that $j$ can be reached from $i$ if this is true.

  Now, communication; communication means it is also the reverse way; also, if it is accessible simply saying, state $j$ is accessible from $i$ and state $i$ is accessible from $j$, then we say both $i$ and $j$ communicate with each other.

- Communication: Two states $i$ and $j$ are said to communicate if $i$ and $j$ are accessible from each other, *i.e.*, there exist $m \geq 0$ and $n \geq 0$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$.

  Notation: $i \to j$: $j$ is accessible from $i$.
  $i \leftrightarrow j$: $i$ and $j$ communicate.

  ► Communication is an equivalence relation (i.e., it satisfies (Reflexivity) $i \leftrightarrow i$, (Symmetry) $i \leftrightarrow j \iff j \leftrightarrow i$, and (Transitivity) $i \leftrightarrow k$ and $k \leftrightarrow j \implies i \leftrightarrow j$).

► This relation partitions the state space into equivalence classes (known as communicating classes).

So, this the whole state-space $S$ now will be broken into classes where each of these is a communicating state would be there in any particular class like, so that is what it. So, this basically partitions means it divides the state space into equivalence classes.

- Irreducible/Reducible: An MC is said to be **irreducible** if all states communicate with each other, *i.e.*, there is a single communicating class. A chain is **reducible** otherwise.

Most of the time, we will be dealing with this irreducible Markov chain, but of course, the theory is also required for how to handle if the chain is reducible. So, this is what are the properties of accessibility communication and irreducible or reducible.
Then we have the notion of what we call a closed set.

- Closed: A subset $A$ of the state space $S$ is said to be **closed** if no one-step transition is possible from any state in $A$ to any state in $A^c$.

If that is the case, then this is called a closed set which means that once you reach into that set, then you do not have a way of going outside of it in a way. That is what it is you call closed sets.

Absorbing state, we have seen it in the earlier case we just mentioned, but this is what precisely it means.

- Absorbing State: If a closed set $A$ contains only a single state, then the state is called as **absorbing** state.
► A state $i$ is absorbing if and only if $p_{ii} = 1$.

We have seen in that example that one of the examples that this situation was true that $p_{ii}$ for some $i$ if it is equal to 1. That means it is there currently in $i$, and it will be there in $i$ in the next step as well, and forever it will be. So the process got absorbed into the state; that is the terminology used in this particular case.

- If $S$ is closed and does not contain any proper subset which is closed, then we have an *irreducible* MC. If $S$ contains proper subsets that are closed, then the chain is *reducible*.

So, you can also talk irreducible and reducible in this manner as well. You can see that if there is a single communicating class, and that is a closed class, it does not contain any proper subset which is closed. Because all of them form a single class, then we have an irreducible Markov chain. Suppose if it is not the case, then you will have subsets that are closed more than one, then the chain becomes a reducible chain.

■ If a closed subset of a reducible MC contains no closed subsets of itself, then it is referred to as an *irreducible sub-MC*.

Because then that particular sub-Markov chain will have each transition probabilities, it will behave as if that is itself a Markov chain because that p with itself be a stochastic matrix. So, one can handle that chain separately as an independent of all other states, that particular class alone; you can treat it as if it is an irreducible Markov chain, and you can handle that. That is the reason why you look at it that way.

**Example.**

$$P_1 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we will define after these properties of state; now we will define "hitting time" what do we mean by that?

- Hitting Time: For any $A \subseteq S$, the hitting time $T_A$ is defined by

$$T_A = \inf\{n \geq 1 : X_n \in A\},$$

with the convention that $\inf \phi = \infty$.

▶ $T_A$ is the first time after 0, when the chain enters $A$.
▶ $T_A$ is also called **first passage/return time** to $A$.
▶ $T_{\{i\}}$ will be denoted by $T_i$, $i \in S$.
First passage time or first return time plays a crucial role in the analysis of different Markov models. So, what does that mean is that it is basically the first time something is reached a particular suppose you are looking at building wealth and then you are starting at some 100, and you are looking at when you will reach 1000. Then you will say that for the first time, 1000 is reached; when what is the timeline? What when it is needed? So, that is what would represent in this particular case.

Then based upon this quantity $T_A$ the hitting time, we can also classify the states differently, and these are all the basic definitions of that.

- Classification of States:
  - A state $i$ is called **recurrent** (or persistent) if $P\{T_i < \infty | X_0 = i\} = 1$.
    ▶ State $i$ is recurrent if and only if $f_{ii} = P\left\{X_n = i \text{ for some } n \geq 1 | X_0 = i\right\} = 1$.
  - A state $i$ is called **transient** if $P(T_i < \infty | X_0 = i) < 1$.
  - A recurrent state $i$ is called **null recurrent** if $E(T_i | X_0 = i) = \infty$ and **positive recurrent** (or non-null recurrent) if $E(T_i | X_0 = i) < \infty$.

Remember, it could be a probability, it could be a distribution, but the expected value need not be finite for every distribution. So, that is what it would amount to here. So, now, let us be more precise on this.

- Let $f_{ii}^{(n)}$ be the probability that a chain starting in state $i$ returns for the first time to $i$ in $n$ transitions. Then $f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)}$. For a recurrent state $i$, since $f_{ii} = 1$, $\{f_{ii}^{(n)}\}$ defines the first-return time or recurrence time distribution and the mean recurrence time is $M_{ii} = E(T_i|X_0 = i) = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$.

▶ Recurrent state $i$ is positive recurrent if $M_{ii} < \infty$ and null recurrent if $M_{ii} = \infty$.

Similar to this, one can also talk about "first passage time." So, here $f_{ii}^{n}$ starting from $i$, you are looking at the quantities which you are coming back to $i$. Now, this starting from $i$ if you are looking at some state $j$ which is not equal to $i$, then the exact same thing we will be talking about that $f_{ii}^{n}$ would be the probability that chain starting in $i$ reach a state $j$ in $n$ steps for the first time. Then its properties and its distribution whenever that is equal to 1 and so on if it can be done if it can be reached then always this will be 1, and its distribution will be first passage time distribution it will be similar to this. That is what you might see.

Another concept that is also relevant is mainly in discrete-time. Of course, in continuous-time, you do not have this notion of this periodicity, but in discrete-time, you have this.

- Periodicity: The period of a state $i$ is defined by the greatest common divisor of all integers $n \geq 1$ for which $p_{ii}^{(n)} > 0$, *i.e.*,

$$d(i) = \begin{cases} \text{gcd}\left\{n \geq 1 : p_{ii}^{(n)} > 0\right\} & \text{if } \left\{n \geq 1 : p_{ii}^{(n)} > 0\right\} \neq \phi \\ 0 & \text{if } \left\{n \geq 1 : p_{ii}^{(n)} > 0\right\} = \phi. \end{cases}$$

  If $d(i) = 1$, then the state $i$ is said to be aperiodic.
  If $d(i) = \gamma > 1$, then the state $i$ is said to be periodic with period $\gamma$.

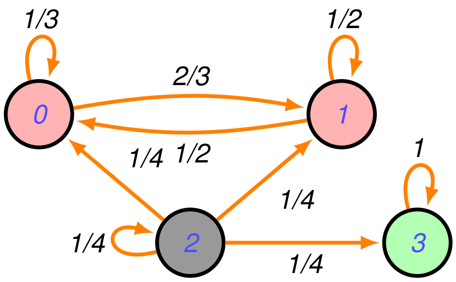
  **Example.**
  *Consider an MC with $S = \{0, \pm 1, \pm 2, \ldots\}$ and with $p_{i,\,i+1} = a$, $p_{i,\,i-1} = b$, $p_{ii} = c$, where $a + b + c = 1$, $a > 0$, $b > 0$, $c \geq 0$.*

  *Determine the period of states (different cases).*

Now, you see here that the previous example that we said we would look at it later, and you will see here what happens.

**Example.**

*Consider an MC with state space $S = \{0, 1, 2, 3\}$ and with TPM* $P = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.




*MC has three classes $\{0, 1\}$, $\{2\}$ and $\{3\}$ and hence reducible.*

So, in any Markov chain, whenever it is reducible, it will so happen that there is one, there will be n number of open communicating classes, and you will have that recurrence transition. So, these are all recurrent or positive recurrent states and so on; you can analyze this. So, with some more results in hand, one can do that thing. So, these are some more results that we have with respect to this. These are all simply we are stating it, but intuitively also, it should make sense you can think a bit more probabilistically to understand that this is actually the case that will happen here.

1. (Number of Visits) For any state $i$, let $N_i$ be the number of visits to state $i$. Then,

   a $i$ recurrent implies $P\{N_i = \infty | X_0 = i\} = 1$.

   b $i$ transient implies $P\{N_i = n | X_0 = i\} = f_{ii}^n (1 - f_{ii})$ for $n = 0, 1, 2, \ldots$, where $f_{ii} = P\{T_i < \infty | X_0 = i\}$ is the probability of returning to $i$ starting from $i$. Thus $N_i | X_0 = i \sim Geo(1 - f_{ii})$.

   *Corollary:* A state $i$ is recurrent iff $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ and transient iff $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$.

   ▶ $P\{X_n = i \text{ for infinitely many } n | X_0 = i\} = 1$ or $0$ iff recurrent or transient.

2. If the state space $S$ is finite, then at least one state must be recurrent.

3. Positive recurrence, null recurrence and transience are all class properties. Also, all the states in a class have the same period.

4. All states of a finite irreducible MC are positive recurrent.

5. Let $i$ be recurrent and $i \to j$. Then $f_{ji} = P\left\{T_i < \infty | X_0 = j\right\} = 1$ and $j$ is recurrent. [Note: Not true if $i$ is transient.]

6. Suppose that $\{X_n\}$ is irreducible and recurrent. Then for all $i \in S$, $P_\mu \left\{T_i < \infty\right\} = 1$ for any initial distribution $\mu$.

If it is irreducible, all states communicate with each other, and it is recurrent, it will come back to 1. So, no matter what is the initial distribution $(\mu)$, $P_\mu\{T_i < \infty\} = 1$ will happen; that is what it means; here, the thing is that the independent of this, this will happen; that is what it is.

So, these are some of the properties of the Markov chain that we are seeing in this lecture. We will see a few more properties, which will complete our discussion of the Markov chain which will be used. So, to develop for to use to come to that kind of conclusion, we need all these notions. Accessibility, communication, irreducibility, closedness, or periodicity, recurrence, positive recurrence, null recurrent, transient because this is all is what. So, how can one determine what they actually mean in a Markov chain scenario? So, all these things are what we have seen. So, we will continue in the next lecture with some more properties.

Thank you, bye.