Introduction to Queueing Theory

Prof. N. Selvaraju
Department of Mathematics
Indian Institute of Technology Guwahati, India

Lecture - 05
An Overview of Stochastic Processes

Hi and hello, everyone; in this lecture, we will look at "An Overview of Stochastic Processes or random processes," as you might know already. Queueing theory is a part of the theory of applied stochastic processes because this is the theory that is developed in stochastic processes to analyze the system or create queueing models so that real systems can be analyzed. So, it is basically a theory of part of the applied stochastic processes. So what is this stochastic process, then? So, when we said that if you have the knowledge of stochastic process, in particular, Markov chain discrete and continuous, you will find the next few lectures will be the things that you already know; if not, you do not need to worry too much on that because we are going to review in some sense. This is not going to be a complete detailed study on the required material, but we will just review what is relevant, what are the basic ideas and, and how things come up, and we will simply use it in our queueing analysis. So, we are not concerned about the theorem-proof approach of this particular theory of stochastic process; instead, we will be using the results, and what is the background to the results, what the basis for these results is, and how the proper applicability of this result can be ensured and that would be our objective. Because one needs to understand on what basis of assumptions this particular result is derived, just knowing the result is not sufficient because you will make mistakes when trying to apply it to the systems. So, that is the thing that we want to keep it. So, we will just brush up and review this whole series of stochastic process lectures for the next two weeks, but to start with, you know, we will give a broad overview of stochastic processes. So, you will know like where exactly or what is the typical type of stochastic process you use and the other processes that might come in handy if you are going a little bit higher in the analysis of such even our queuing systems. So, we will start with what is a stochastic process. So, as you know already, if not you know you will you can see that a stochastic process or a random process is a family or a collection of random variables $\{X(t), t \in T\}$, where T is some index set, defined on a common probability space $(\Omega, \mathcal{F}, P)$, you call simply as a stochastic process. So, any collection, a family if you take a sequence of random variables, could also be considered a stochastic process.

When the index set is some interval on the real line, we will all be worried about the real line and T being subset of whether this is a finite set or it is a countable set, or it is, some intervals and the whole of the nonnegative interval, nonnegative real numbers is what we might have for this particular T, and this is the index. So, we are just denoting in the family we are

denoting the random variable by a certain index, and that is what is the index set, which is what we will also call a parameter. And this T is what we call the parameter space. So, that is the terminology that we will use; the parameter space means that the index where it takes values and all these random variables take so far particular values. So, for example, X_1, X_2, X_3 suppose if you have in a discrete manner or any $X_{0.5}, X_{2.7}$ and so on any random variable if you take, they all take value in some particular set so, if you put together all like in the possible values that a particular random variable in this family, not just one in this family it takes, then that is what is called common support for the all the random variables in the stochastic process that is what we call it as the state space and notation that will be using is S. T is the parameter space, and S is the state space. And X is a random variable, meaning $X : \Omega \to \mathbb{R}$ is what each of these random variables is, and on top of it, we are putting one more index, t. So, basically, a SP is a (measurable) mapping from $T \times \Omega$ to $\mathbb{R}$ and hence X is a function of both ω and t, i.e., $X(t, \omega)$. So, it is a mapping measurable mapping that is what you know already.
Now in this particular case, suppose if you fix $\omega \in \Omega$, then you can look at $X(t, \omega)$ as a function of t defined on T. It is a function as a function of t alone the index alone; you can think because you are fixing ω. And such a function is what is called sample-path or realization of a stochastic process. So, you can view the stochastic process, which is nothing but $\{X(t), t \in T\}$ family of random variables, as if it is a family of these kinds of functions. One for each of these omega you have, and then you put this collection together, which is what you call the stochastic process. So, basically, a sample path here is a collection of time-ordered data describing what has happened to a dynamic process in one particular instance that one particular instance is what is a fixed ω would mean. So, this stochastic process is then a probability model describing the collection of time order random variables that represent the possible sample-path. So, if you put all possible sample paths together, one for each omega, then that is what you will get; this whole collection is what would be represented by a stochastic process. On the other hand, if you fix an index, if you fix the index in this index set T which is t, then as we said, this is just a single random variable. This is just a single random variable. So, that is a normal view that is also that is what you know we have said in the beginning that it is a collection of random variables one for each t.

So, that is a random variable. So, in many stochastic processes, this T often represents time; in our particular case of queueing theory models, this will be more of time, but this is a generic word time; it need not be even time; it could be in any sense even it could be an area that could be represented by. Suppose you have a cloth and you are looking for the defects. So, the area the number of defects in a particular area could also represent the stochastic process, in that case, t represents the area, but in our case, it is just simply the time. And we refer $X(t)$ as the state of the process at time t. So, remember this word state of the process when we mean state of the process at time t means we are representing this $X(t)$. Many a time, for again notational convenience, instead of writing this as $X(t)$, we may write it as X_t, but you should understand that this is a function of t; that is how you have to understand.

Now, let us look at a few simple examples of such a process that you could get here. As usual, in any probabilistic experiment, you always talk about coin tosses. So, this is one such situation.

So, you have a coin, and you are repeatedly tossing the coin, and you are trying to count the number of heads. So, when you start, the number of heads is 0, so the process suppose if I call it as $X(t)$, $X(0) = 0$. Then when the first toss comes, 1 toss is made; either there could be one head or 0 head accordingly; suppose if this is a fair coin that you had assumed the probability of these two would be half each. So, that is, the number of heads could be either 0 or 1. So, the randomness starts creeping in. Now, when you go to two heads, you could have the possibility that the number of heads could be 0, 1, or 2 with corresponding probabilities and so on. So, when we go to n number of tosses, then what we are getting is the number of heads would be anything anywhere between 0 and n. So, this is a random process in this case if you are looking at the number of heads in the first n tosses in a sequence of coin tosses. The number of people affected by COVID. So, here is the first example the index set is basically 0, 1, 2, 3, and so on; the state space is also 0, 1, 2, 3, and so on. So, this is in both cases; it is nonnegative integers. So, that is the parameter space and the state space for this particular example; like the other examples, you can also make it up. For example, the number of people affected by COVID in India as a function of time. Now, this could be like if you assume the time to be continuous at any after every instant, and you are counting the number of people affected by COVID in India. So, then that is essentially you know the number would be a discrete quantity, but the time is continuous the parameter space is in some interval in this particular case or daily closing prices of a particular stock in National Stock Exchange. The temperature at Guwahati as a function of the time are all some simple examples that you can easily think about. Now, recall we know completely about a random variable if you know the probability distribution of the random variable. Because any question that you have with respect to the random variable can be answered if you know its distribution, whether it is certain probabilities or some expectations or anything of that nature, it can be answered.

In a similar way, but at least the distributional properties not of the sample path properties because that there is something which you will not be able to do completely, but at least there as far as the distributional properties of a stochastic process is concerned, they all can be answered, or you can have the complete properties of them if you know what is called a finite-dimensional distribution. Just like for a random variable, you want to know the distribution; in this particular case, for a stochastic process, you want to know what these FDDs are. So, what are these FDDs? These are defined in this manner. For each $n \in \mathbb{N}$, for each $t_1, t_2, \ldots, t_n \in T$, the FDD is specified through the joint DF

$$\begin{aligned} F_{\mathbf{X}}(\mathbf{x}; \mathbf{t}) &= F_{X(t_1),\ldots,X(t_n)}(x_1, \ldots, x_n; t_1, \ldots, t_n) \\ &= P\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\}, \quad \forall x_1, \ldots, x_n \in \mathbb{R}. \end{aligned}$$

So, if you can compute these probabilities for all such, if you can compute $P\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\}$ completely, that means then you know completely about the stochastic process, at least their distributional properties. So, now, the question is, is it possible to get this for all kinds of stochastic processes? The answer would be no because there will be very complex situations where you will not know the joint distributions completely. You may know a few joint distributions, but you may not know completely all joint distributions. But for many interesting stochastic processes, it is possible to provide the above specification in simple terms. Now, let us look at how we can group. This stochastic process; we define in a generic way; now, how we can classify them or group them into different so that the particular group which is having a particular property can be studied in detail. That is the normal approach to take when you have this kind of generic quantity that you are looking at it. So, this can be classified or grouped or distinguished based on, say, parameter space T, state-space S the dependence relationship among the family of random variables because you know that when you have more than one random variable, you always have a possibility of whether they are related in some sense or they, are independent; dependency and independent. So, depending upon the nature of the relationship among the family, one can also classify them. These are the generic way of classifying as far as the parameter space is concerned, which is the index set. If this set T is a countable state countable set representing specific time points, then the stochastic process is said to be a discrete-time stochastic process.

Say, for example, in this particular case, $T = \{0, 1, 2, \dots\}$ could be of this nature, and in such cases, this $X(t)$ we very often we will write it as $\{X_n, n \geq 0\}$ when it is indexed by nonnegative integers. So, in the case of discrete, it is common that instead of using this t within the bracket, it is a suffix n like in the coin toss experiment that we have noticed X_n, X_2, X_3, X_4, and so on. So, it is often denoted in this manner. So, if this is the case, then it is always assumed that you have the index, which is discrete when you are using the index n. Now, if T is an interval of the real line, then it is called a continuous-time stochastic process. For example, $T = [0, \infty)$, r it could be even the whole of $\mathbb{R}$ in some cases and so on.

So, in this case, like, this could be written $\{X(t), t \geq 0\}$ or many a time for simplicity because just we want to avoid this, I mean, looking little complex, but nothing really complex here. So, it is basically like $\{X_t, t \geq 0\}$; also, you can make it here. This is based on the parameter space you are dividing into discrete-time and continuous-time, and depending upon the state space again, if the S is a countable set, then we call this a discrete state stochastic process, and whenever $S = \{0, 1, 2, \dots\}$ of this nature we call $\{X(t), t \in T\}$ a chain rather than a process; we normally tend to refer to this kind of thing when the state space is discrete; we refer to the stochastic process as a stochastic chain; I mean, the chain is the word that is used in such scenarios. If S is the interval of the real line, then it is called a continuous state stochastic process, but, in this case, S could be even $(-\infty, \infty)$ or nonnegative real numbers or whatever is the number. Now, if it is S is a Euclidian-n space, then it is an n-vector stochastic process.

So, depending upon these two, one can have a discrete-time, discrete state, stochastic process, and the combinations the various four combinations also you could have that is one classification. But for example, if you look at here the number of heads in the first n toss, this is the same example that we have seen earlier in the first case; both are discrete; in the second case, the state space is discrete, but the parameter space is continuous when you are looking at it as a time. Daily closing prices of the stock on NSE. So, the price could be anything like its continuous any nonnegative quantity, but daily; that means the time you are discretizing is a day. Temperature as a function of time, both are continuous here in this particular case.

Some more examples like, in a brand switching model for consumer behavior, you are making a survey of the number of people whom you observe on a monthly basis who buy a certain brand of an item, say coffee or something. Like this in the consumer behavior when you want to do the market research as you would do this. So, then the number of people is what on a monthly basis you can see now what is the situation. The number of people waiting at a bus stop at any time of the day. So, time is continuous; you are looking at it. Suppose the same thing if you are looking at every hour, then the time becomes discrete the number of people also becomes discrete. That is why the specification is important at any time of the day or at every hour; as you know, if you specify accordingly, then accordingly, your state space and index space would take the difference. Size of the population at a given time. Waiting time of the 10th person of a day who arrives at a bus stop. In a time-sharing computer system or in a production system, the number of jobs waiting at any time and the time a job has to spend in the system, the number of jobs would be discrete; time is continuous, the time a particular job has to spend in the system is continuous. So, you can look at that. So, the classical and the more useful classification of the stochastic process is by the nature of the dependency relationship among the random variables. That is what makes the study of the stochastic process distinct and more useful than classifying based on the nature of parameter space and index space. So, once you make this kind of grouping or relationship, then that particular group can be studied in detail, and it helps in stochastic modeling. Because then you have already developed so many tools for the application, and then the moment you see such a scenario existing in your real-life system, then all these tools can be applied in an immediate manner, that is what is the idea. So, now, we look at based upon the nature of dependency of the stochastic process what are the different classes of stochastic process that we can, of course, there are plenty like, but we will see the major ones the main important ones, the independent process. So, this is the most trivial stochastic process where there is actually no dependency among the random variables. So, if you are looking for the FDDs, the joint distributions, then they are just the product of their marginal distributions. So, if you know at each time point what is the marginal distribution, then you know completely about the process. So, the example could be an IID noise or a simple white noise process some people call it in say areas like signal processing and other places where it has applications are basically the independent one. This IID sequence essentially or IID family in this particular case. IID means you just want independence only, but if its distribution is also the same, then it becomes IID in a way. And another group is what we call stationary processes. So, again the word stationary, we are referring with respect to time. So, a stochastic process is said to be stationary or sometimes strictly stationary if its joint distribution functions, which we are written down as these $F_X(x, t+\delta)$ FDDs, are invariant to shifts in time. Meaning for any given constant δ the following holds: $F_{\mathbf{X}}(\mathbf{x};\mathbf{t}+\delta) = F_{\mathbf{X}}(\mathbf{x};\mathbf{t})$. So, if you know one n-dimensional joint distribution, then you practically pretty much have known all n-dimensional marginal distributions of the process. It reduces to a great extent the requirement of testing whether you know the process is stationary or knowing the FDDs, the Finite-Dimensional Distributions.

So, basically, the process is, in some sense, probabilistic equilibrium, and the particular times of the process being examined are of no relevance right like any other time in some sense. Because whatever the distribution that I am going to have at the time t now 5 units down the line or 4 units prior to that, I would have had the same distributions; that is what $F_{\mathbf{X}}(\mathbf{x};\mathbf{t}+\delta) = F_{\mathbf{X}}(\mathbf{x};\mathbf{t})$. would mean. This is what we call stationarity in the strict sense because we require that to be in that in terms of distributions. A weaker sense is what is called a wide sense stationary or covariance stationary or weakly stationary if it possesses finite second moments. And $E(X_t) = E(X_{t+\delta})$, which means the mean is equal across if even if the time is different, but the covariance which is defined to be $Cov(X_t, X_{t+\delta}) = E(X_t X_{t+\delta}) - E(X_t)E(X_{t+\delta})$ depends only on the δ for all $t \in T$; δ means that the length of the interval in which you are looking at the two random variables from the same stochastic process X_t and $X_{t+\delta}$. So, it depends only on the δ. If it is a function of delta alone, then you call this wide sense stationary or covariance stationary. You require not in the distributional form, but only the first two moments joint moments as well; obviously, a strictly stationary process would be wide sense, but need not be the other way, and it finds application in time-series signal processing so on in different places. Then comes a process that is also relevant to our studies. I would highlight like which are the process that is more relevant to our studies while making these classifications. So, a stochastic process is said to be a process with independent increments if the increments of the process $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent for any $t_0 < t_1 < \cdots < t_n$ and for all values n. What are the increments mean? So, you are looking at this $\{X_t\}$ you are picking up the time points t_0, t_1, t_2, t_3, and so on up to t_n you are looking at these time points, and for all arbitrary n, some $n+1$ time points of this nature.

Now, you look at the process how you know it has moved from t_0 to t_1 and t_1 to t_2 and t_3 to t_4 and so on. So, that difference $X_{t_1} - X_{t_0}$ in the state of the process, this is what is that representing that random variable, $X_{t_1} - X_{t_0}$ is what is called as an increment of the process in a time in an interval (t_0, t_1). So, this is what you call the increment. I mean, it need not be an increment; it does not mean it always goes up, but it is just the change that you are making. So, we call it as increments of the process. So, $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are the increments. So, $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ if they are all independent so; that means, what has happened in the previous interval and what is going to happen the next interval, starting in two distinct intervals they are independent that is what it would mean. So, this is what we call a process with independent increments. Now, if the increments depend only on the length of the interval but not on where this interval is placed, that means the increment $X_{t_1+h} - X_{t_1}$ depends only on h and not on the time t_1, . Then you say this is the process with stationary increments. Now you could have a process with both independent increments as well as stationary increments; for example, the one that we are going to use very frequently and extensively in our queueing analysis is the Poisson process, which is a process with both independent increments and stationary increments. And another process which is also will be used in the advanced level of queuing systems in the analysis is Brownian motion which is also of this nature.

So, this is a particular class then you have another class which is called martingales. A SP $\{X_t\}$ is called a **martingale** if $E(|X_t|) < \infty$ for all t and $E(X_{t_{n+1}} | X_{t_0} = a_0, X_{t_1} = a_1, \ldots, X_{t_n} = a_n) = a_n$ for any $t_0 < t_1 < \cdots < t_{n+1}$ and for all values $a_0, a_1, \ldots, a_n$.

► This states that the expected value of $X_{t_{n+1}}$ given the past and present values of $X_{t_0}, X_{t_1}, \ldots, X_{t_n}$

equals the present value of X_{t_n}.

Say, for example, in the game situations, this could represent a gain, and if what is going to be, the gain in the next time point after playing one more game and the current. So, the expected value you could actually end up with is either higher or lower, but the expected value of that if it is the same as what I have currently. So, in some sense, you would call this as a fair game. Its a fairness. So, this is, the equality represents, in some sense, the fairness of that. So, this is what you know a martingale means. Suppose if it is greater than and less than then, you have a different nomenclature submartingale, supermartingales, and so on, which anyway we will not. So, this again finds applications in queueing theory as well at a very advanced level along with Brownian motion and martingales, but even I will not be using that here. And also, this is a very important concept in financial mathematics, actuaries, and insurance like whatever you do is martingale theory plays a critical role. Because you want to you know, for example, in financial maths or in insurance, for example, you want to arrive at a premium which is fair in some sense how do you define the fairness or how do you quantify or how do you model that fairness is through this martingale properties. So, that is how it comes there, financial maths again pricing of derivatives, for example, like how you will price what the price that you are going to give is. So, it is the fairness idea that martingale plays a role.

But the most important stochastic process that we are going to use extensively, almost throughout the course in a way, if you have to say, is what is called a Markov process in general. So, what is your Markov process? It is a stochastic process that satisfies the Markov property. So, what is "Markov property"? Markov property is

$$P\{X_{t_{n+1}} \leq x | X_{t_0} = x_0, X_{t_1} = x_1, \ldots, X_{t_n} = x_n\} = P\{X_{t_{n+1}} \leq x | X_{t_n} = x_n\}$$

for any $t_0 < t_1 < \cdots < t_{n+1}$, for any n and for all values $x_0, x_1, \ldots, x_n, x$.

So, this is what is the Markov property. So, basically, what you are trying to do is that you are trying to connect the random variables in this stochastic process through a one-point dependency or one-time point dependency, which means $X_{t_{n+1}}$ is in some sense connected to X_{t_n} only, not an $X_{t_{n-1}}$. What was the process value or what was the state of the process at time t_{n-1}, t_{n-2}, and where it started has no bearing on these computations.

Of course, this definition would be true, and you can do only if $P\{X_{t_{n+1}} \leq x | X_{t_0} = x_0, X_{t_1} = x_1, \ldots, X_{t_n} = x_n\}$ are well defined; we are assuming that they are well defined. So, if it is defined on the left-hand side, the right-hand side is automatically defined. So, we are assuming, and if that is the case, then you have if this is true for all $t_0 < t_1 < \cdots < t_{n+1}$ and for all n and for all $x_0, x_1, \ldots, x_n, x$, then only the Markov property is true. It is not that for a particular value any, of course, that this is true with respect to any general definition that one gives like in the same thing it is here. So, this is a very simple but highly useful form of dependency which was given by Markov, and hence this Markov process. It is 1907. So, it was 115 years or more than that before these things were defined and used. So, it is a highly useful form of dependency because you would find like, that you are extending the dependency only to the last known state of the process. Now, basically like in an informal way, if you have to say the future depends on the present and not on the past, but in queueing, for example, if you look at the number in the system, what was 10 hours before what was the queue size does not matter; whether the queue is going to build up I mean in 1 hour from now depends only on the current state. So, that is where naturally the applications of this come.

Again, as we said, the process with the discrete state space would be referred to simply as the Markov chain. So, that is a word that you will use the Markov process; we will not use the Markov chain; whenever we say Markov chain, we always mean that the underlying state space is discrete.

So, this is a discrete-time Markov chain, which means a discrete-time discrete-state Markov process, which we will simply call a discrete-time Markov chain; in short, DTMC is the easiest to conceptualize and understand, and in fact, we will start with this DTMC only little later.

So, the duration, in this case, one can understand the duration of the time the process stays in a state is distributed as geometric distribution in case of discrete-time and exponential in case of continuous-time. That is the main property, the major property of our holding time or sojourn time in a particular state of the process, which is exponential or geometric depending upon the situation. And you say like these are the only distribution with this memoryless property. So, that is where this thing comes. So, the Markov process is central to the study of queueing system. So, again it has wide applications in a wide spectrum of areas, including queueing theory statistics, economics, finance, population, dynamics, physics information theory, artificial intelligence, etcetera etcetera.

Now, the birth-death process is a very important special class of Markov chains or Markov processes where the transition takes place only to the nearest neighbors; that is, you have a state-space in an ordered manner. And the transitions, the process moves from one time to the next time whenever it makes a move as it moves only to the nearest neighbor states. So, that is a restriction that we are imposing on that movement, and then you get what we call the birth-death process. Our first queuing models that we are going to start will be based upon this birth-death process; the whole may be about two weeks or something that we are going to see in the model different models they will be based upon this birth-death process. Again as I said, you are defining a different class, say Markov process, again a subclass of Markov process, the birth-death process. The reason is that the finer it becomes that you can get, the more information and more theory that you can develop with respect to that particular class as well, at least that is the idea. Now, if you extend this in a way, what you would get is what you call semi-Markov processes. So, this is a generalization of the Markov process wherein we permit an arbitrary distribution for the duration of the time the process stays in a particular state; rather, in Markov, you see it as either geometric or exponential depending upon the time is discrete or continuous. But if you allow that the time a process stays in a particular state need not be exponential, it could be any other nonnegative distribution; suppose if a continuous if you assume then what you would actually get is a semi-Markov process. But at the time of transitions, it behaves like an ordinary Markov chain; in those instances, we say that we have an embedded Markov chain. And this is the idea that you will be using when you are looking at non-Markovian or semi-Markovian queueing models; they will be based upon these semi-Markov processes. We will see more when we reach there.

Random walk is again is another important stochastic process which is defined as:- A sequence of random variables $\{S_n, n \geq 0\}$ is referred to as a (discrete-time) **random walk** (starting at the origin) if $S_n = X_1 + X_2 + \cdots + X_n, n = 1, 2, 3, \ldots,$ where $S_0 = 0$ and X_i's are IID random variables. [If the index is from a continuum, then we have a continuous-time random walk; eg. Brownian motion]. So, here the emphasis is on the position of the process after n transitions; that is what a random walk would be. So, a popular random walk is a simple random walk wherein the process makes a transition only to the neighboring state, and symmetric random walk, if the probability of making it to the neighbouring states if they all have equal probabilities, that is what you call symmetric random walk and so on. So, one can have many different properties derived for such specific cases. As opposed to random walk case in a renewal process, $\{N_t, t \geq 0\}$ is a SP for which $N_t = \max\{n : S_n \leq t\}$, where $S_0 = 0, S_n = X_1 + X_2 + \cdots + X_n, n \geq 1$ for IID non-negative random variables X_i, for $i \geq 1$. But here, in your renewal process, the interest is counting the transitions that take place as a function of time. So, the number of events that have happened, not the position of the process at that point of time, we are looking at the number of events that have happened by time t. So, that is what you define $\{N_t, t \geq 0\}$ to be the $max\{n : S_n \leq t\}$, and this process $\{N_t, t \geq 0\}$ is what then you call a renewal process. So, that is the difference between the interpretation of a random walk and the renewal process that you see here. Then again, a Gaussian process which means that the finite-dimensional distribution, if they are all multivariate normal distributions, then you call the process a Gaussian process, for example, this is a very useful in statistical modeling in various ideas and then machine learning and so on.

Then you have the Brownian motion, which is also called a Wiener process if it satisfies

- (a) $W_0 = 0$
- (b) $\{W_t\}$ has independent increments (for every $t > 0$, the future increments $W_{t+u} - W_t, u \geq 0$, are independent of the past values $W_s, s \leq t$)
- (c) $\{W_t\}$ has Gaussian increments ($\{W_{t+u} - W_t\}$ is normally distributed with mean 0 and variance u)
- (d) $\{W_t\}$ has continuous paths: W_t is continuous in t.

Again, this has a major application in different places; one of the main areas is finance, and it originated from physics because of Einstein and others, statistics etcetera. So, like this, there are many different classes of the stochastic process one can define and depending upon, for example, levy process and so on; anyone can define it depending upon certain properties that if it satisfies certain properties, you call that particular group by some name and then start developing the properties. So that it can be used readily, so, that is the purpose of you know developing the theory part of any field, and in the stochastic process as well that holds true. And once you have the tools and everything available. So, you can simply use it when you are trying to apply that particular process to model certain real phenomena. So, this is the overview; the most important processes are the Markov process, the birth-death process has a special case at its property discrete-time, and continuous-time is what we will see for the next two weeks. And once we understand this, we will apply or make use of the results or apply these results to the modelling, and once you model and then the analysis of queueing systems, you will use these tools. Later on, when we require like, we will introduce the renewal process and semi-Markov process to analyze more generic queueing models semi-Markovian queueing models rather we will see. So, that we will do it, we will come later, but the next two weeks will be based upon Markov chains discrete-time and continuous-time. So, that is what we will see next; you will see that. Thank you, bye.