Introduction to Queueing Theory

Prof. N. Selvaraju
Department of Mathematics
Indian Institute of Technology Guwahati, India

Lecture - 04

Laplace and Laplace-Stieltjes Transforms, Probability Generating Functions

Hello, everyone; in this lecture, what we are going to see is about the transforms in general and mainly "Laplace" and "Laplace-Stieltjes Transform" and the "Generating Functions," which is what is going to be used extensively in our analysis that we are going to see. In queueing theory, when we analyze a particular queueing system, it is often the case except for the simplest of models, it is often the case that you are not able to obtain explicitly or even not able to do some sort of analysis to take it a little bit forward to get some insights; in an explicit form or in the usual domain. So, what is done in such a situation is that then you transform the domain from one to the other and look at it in the transformed domain, probably like your analysis could be done in an easier way. And that is what is being done generally, and the tools that help us to do all these transforms in general. I mean, people familiar with differential equations, like how much these Laplace transforms are useful, and probably the other ones are Fourier transforms, and their uses in different areas are also known. But in applied probability and stochastic models also, these transforms play a critical role in the analysis. So, in this lecture, what we will try to do is we will just try to recap and try to understand what this is and its certain properties of it. So, that it can be readily referred to later whenever we are doing the analysis with respect to a particular queueing system, so, as we said that, it is basically the time-domain normally when we work on, is what our analysis is. So, any quantity, for example, whether it is number in the system at time t or in the waiting time of a customer how long is wait, these are all in the time domain which queueing has to be transferred to some other domain. So, a useful transform, transform here means; that it is mapping from one space to the other like typically, these are all from mapped from some real space to complex space. And this queueing analysis has a lot of importance in this case.

Now, let us see what a Laplace transform is. It is defined as:- Let $f : [0, \infty) \to \mathbb{R}$ be a suitable function, by a suitable function; I mean the function is piecewise continuous, and of exponential order means it is at least piecewise continuous in any finite domain in $[0, \infty)$. And it does not grow beyond an exponential function, but it does not grow faster than a function of an exponential function; that is what it would mean in general. But anyway technical part, you can always look it up. So, for such a function, the Laplace transform of f is defined. So, this is the notation that typically might be given $\mathcal{L}\{f(t)\}$ or $\bar{f}(s)$ the bar is what we denote by the Laplace transform of f is defined to be

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)dt, \quad (t \geq 0)$$

where s is a complex variable. Under the broad conditions, it can be shown that $\bar{f}(s)$ is analytic in the halfplane where $Re(s) > \alpha$, for some constant α, that alpha is related to this exponential order. And there is a one-to-one correspondence between the $f(t)$ and $\bar{f}(s)$; that is why you would see volumes of books, I mean a book which contains exclusively functions and their Laplace transform given as in a table form for ready reference because for different function you obtain and working with $f(t)$ would be in some sense equivalent to working with $\bar{f}(s)$ that is what we will also be doing it. And there is a one-to-one correspondence between the function and its Laplace transform, which means that if you know the function, you know the Laplace transform, or if you know the Laplace transform, then you know what the function is. So, some common pairs of functions and its Laplace transform, which again for ready reference for anything it comes you can just put it into this and then obtain this integral. But suppose if the function is the constant function 1, then you put this instead of the $f(t)$ you put 1 then what you will end up with $1/s$. And if the function is t, so then it is te^{-st}, so if you evaluate that, you will end up with $1/s^2$ as the Laplace transform. In general, for t^n, $\frac{n!}{s^{n+1}}$ for $n = 0, 1, 2, \ldots,$. So, $1 \longleftrightarrow \frac{1}{s}$, $\quad t \longleftrightarrow \frac{1}{s^2}$, $\quad t^n \longleftrightarrow \frac{n!}{s^{n+1}}$ for $n = 0, 1, 2, \ldots$, is what the Laplace transforms. Now if I have e^{-at}

$$e^{-at} \longleftrightarrow \frac{1}{s+a}, \qquad e^{-at}t^n \longleftrightarrow \frac{n!}{(s+a)^{n+1}} \text{ for } n = 0, 1, 2, \ldots,$$

We can substitute and see. Now

$$\cos bt \longleftrightarrow \frac{s}{s^2+b^2}, \qquad \sin bt \longleftrightarrow \frac{b}{s^2+b^2}, \qquad e^{-at}\sin bt \longleftrightarrow \frac{b}{(s+a)^2+b^2}$$

So, these are some common pairs or any other thing that you normally you encounter; you can always substitute here and try to understand. So, most of the things like we will be concerned with something of this kind of forms we will be arriving at, you will see like in the analysis when you go further.

So, for example, if I take $f(t) = \lambda e^{-\lambda t}$, what is this particular function means; it is the probability density function of an exponential random variable, and if I can compute $\bar{f}(s)$:

$$\bar{f}(s) = \int_0^\infty e^{-st}\lambda e^{-\lambda t} dt = \frac{\lambda}{s+\lambda}$$

.

Now, certain properties that you need to know so that you can use them.

- $\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 \bar{f}_1(s) + a_2 \bar{f}_2(s)$

- $\mathcal{L}\{e^{-at} f(t)\} = \bar{f}(s+a)$

- $\mathcal{L}\{f(at)\} = \frac{1}{a}\bar{f}(\frac{s}{a})$

- $\mathcal{L}\{f^{(n)}(t)\} = s^n\bar{f}(s) - \sum_{i=1}^{n} s^{i-1}f^{(i-1)}(0)$, for derivative $f^{(n)}(t)$ of order $n \geq 1$

- $\mathcal{L}\{\int_0^t f(x)dx\} = \frac{\bar{f}(s)}{s}$

- $\mathcal{L}\{\int_0^t g(t-y)f(y)dy\} = \bar{f}(s)\bar{g}(s)$

- (Limit property) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s)$ and $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\bar{f}(s)$

So, even if you are in the time domain, even if if you are in the Laplacian domain if you want, you do not have the function explicit form, and you want to look at this kind of behavior, and this kind of results are useful, like this a looking at this is same as looking at this. So, these are certain properties of the Laplace transform that we might be using it. Differentials, integrals, convolutions, linearity and shifts, and so on.

A related transform is what is called a Laplace Stieltjes transform, which is defined for a function $F : [0, \infty) \to \mathbb{R}$ as

$$\mathcal{L}^*\{F(t)\} = F^*(s) = \int_0^\infty e^{-st}dF(t),$$

where the integral is the Lebesgue-Stieltjes integral.

So, this integral, as you see, is the Laplace sorry Stieltjes integral, Riemann Stieltjes integral, whereas the first one is Riemann integral. So, $\int_0^\infty e^{-st}dF(t)$ is what; we said in the probability review; also, like we pointed out, this kind of integrals will come. So, no matter, you do not need to worry too much about this; what is this even if you do not know, but it is you will know soon anyway; the alternative way of looking at that where the integral is the Lebesgue Stieltjes integral. So, $\int_0^\infty e^{-st}dF(t)$ is what is defined as Laplace Stieltjes transform as, you might have known from the properties of this substituting both discrete and continuous nature of this function F. Now, in our case, what is our F, of course, it also requires some condition on this particular function, F; of course, e^{-st} is what we have given it. So, now, you need to impose some conditions on this F. So, this integral is well defined. For us, there is not much problem on that count. Because we consider this F to be the distribution function or cumulative distribution function of a nonnegative random variable X.
So, $F^*(s) = E[e^{-sX}]$. You see, whenever you encounter $F^*(s)$, whether it is discrete or continuous or mixed or any random variable for us in our case like the Laplace Stieltjes transform $F^*(s)$, you can always relate to $E[e^{-sX}]$. Anyway, the expectation you know how to compute. So, this is nothing but this expectation, $E[e^{-sX}]$ is what the Laplace Stieltjes transform. And again, we consider our f in Laplace transform whenever we are defining our f would be there the PDF of

it. Now suppose for an exponential random variable with mean $1/\lambda$, you can obtain the Laplace Stieltjes transform $F^*(s) = \frac{\lambda}{s+\lambda}$, which you notice is the same as the one that you obtain as a Laplace transform of its density function. Now, for a discrete random variable X, where X takes value $P(X = 3) = \frac{3}{10}, P(X = 4) = \frac{1}{5}$ and $P(X = 9) = \frac{1}{2}$. You can see that this Laplace Stieltje transform is nothing but $F^*(s) = \dfrac{3}{10}e^{-3s} + \dfrac{1}{5}e^{-4s} + \dfrac{1}{2}e^{-9s}$, again; recall that this is nothing but this $F^*(s) = E[e^{-sX}]$. Even if you do not know the theory of Stieltjes integral business, so, you can always use $E[e^{-sX}]$ to arrive at that, so that is what it is $F^*(s) = \dfrac{3}{10}e^{-3s} + \dfrac{1}{5}e^{-4s} + \dfrac{1}{2}e^{-9s}$, and that is what is the Laplace Stieltjes transform, which we will call LST for, later on, we will simply use the abbreviation LST for this purpose. So, when a nonnegative random variable X has a PDF, then, as we just noticed in the case of an exponential distribution, the Laplace Stieltjes transform of the distribution function, which is $F^*(s) = \frac{\lambda}{s+\lambda}$, equals the Laplace transform of its PDF; the probability density function, which means when you have a nonnegative random variable which has a PDF it is a continuous type random variable when it has a PDF. Then whether you are talking about the Laplace transform of its PDF or the Laplace Stieltjes transform of its CDF, they mean one and the same. More generally, one can say that if I want like this is you see $\bar{f}(s)$ it is not $\bar{F}(s)$; we are not inferring that part. So, more generally, one can show that this is what is the case in that scenario. $F^*(s) = s\bar{F}(s)$, you see here, which is what for any such thing that you can find. But $\bar{F}(s)$ nothing but equal to the Laplace transform of its PDF, that is what. So, this is also like you can keep in mind sometimes this will be useful. But for a discrete, you do not have this Laplace transfer function. So, that is the reason why you always deal with Laplace Stieltjes transform. So, you are incorporating both the quantities in one transform or in one tool that you want to use it.

So, basically, whether it is a continuous like exponential or a discrete like this three-point distribution, you can you will have this Laplace Stieltjes transform. Now, for example, if I pick $F(t) = 1 - \rho e^{-\mu(1-\rho)t}, t \geq 0$, where $\rho = \lambda/\mu$, if this is what is given to be the function, look at this little carefully, this is not of a continuous type, not of a discrete type, but it is of mixed type. Meaning it has a mass of $1 - \rho$ at the point 0, and for $t > 0$, this function is continuous and differentiable. So, for $t > 0$, it has a PDF, and at $t = 0$, there is a mass, so the mass is $1 - \rho$. So, the remaining mass of ρ is what is distributed between $t > 0$ and ∞. So, it is a mixed type random variable now if I use the definition of its Laplace transform of F. So, what you will end up is

$$F^*(s) = \int_0^\infty e^{-st} dF(t) = 1 - \rho + \rho \int_0^\infty e^{-st} \mu(1-\rho) e^{-\mu(1-\rho)t} dt = \frac{(s+\mu)(1-\rho)}{s + \mu(1-\rho)}.$$

Now we can also obtain the above using $F^*(s) = s\bar{F}(s)$; like you have this $\bar{F}(s)$ you have, you can try to obtain s $\bar{F}(s)$ and so on and then you can see this is an exercise that you can try. Now, the properties of LST are similar to the properties of Laplace transform again, like $F(t)$ and $F^*(s)$ both have a one-to-one correspondence with each other which means $F(t)$ can be uniquely determined from $F^*(s)$. So, if I know this, it is equivalent to knowing this. Now, certain properties again, this is one property that has one-to-one correspondence, but the most the other useful property in our analysis turns out to be this part. The LST of the convolution of independent random variables is the product of the LSTs of individual random variables. That is suppose if $F^*(s)$ is the Laplace Stieltjes transform of X and $G^*(s)$ is a Laplace Stieltjes transform of Y. Then the Laplace Stieltjes transform of $X+Y$ if X and Y are independent is given by the $F^*(s)G^*(s)$.

And you know that this is what the convolution component is what the property that is what you know we are stating here. The other used property is this continuity, that is if $X_n, n = 1, 2, \dots$ be a sequence of RVs with DFs $F_n(t)$ and LSTs $F_n^*(s)$, and if as $n \to \infty$ if $F_n \to F$ that has an LST of $F^*(s)$. Then $F_n^*(s) \to F^*(s)$ for $s > 0$, and conversely. So, this is sometimes useful in finding out the limiting form of certain distributions that you might encounter. Now, we can relate by looking at this Laplace Stieltjes transform that you just notice that it is the $E(e^{-sX})$. And recall if you have a nonnegative random variable as we have it in our queueing analysis that the moment generating function $M_X(t)$ is nothing but $E(e^tX)$ which is given by $\int_0^\infty e^{tx}dF(x)$ (if exists) we need to worry about that part. This is similar to LST of F, with $-s$ replacing t, i.e., $M_X(t) = F^*(-t)$. What are properties that I have for you know Laplace Stieltjes transform I can transform now to this moment generating function whenever this exists. So, this is the connection.

And for example, for an exponential random variable, we know that $M_X(t) = \frac{\lambda}{\lambda - t}$, for $t < \lambda$, and $F^*(s) = \frac{\lambda}{\lambda + s}$, which is the same as the Laplace transform of its PDF, or for the random variable, we can refer to that. Now, recall for a nonnegative random variable, we also know this function which is nothing but the characteristic function, which is defined to be $\phi_X(t) = E[e^{itX}] = \int_0^\infty e^{itx}dF(x)$. Again in this particular case, we can relate this to LST with $-s$ replacing this it. So, this $\phi_X(is) = F^*(s)$, that you know you might find. So, for an exponential, for an example, $\phi_X(t) = \frac{\lambda}{\lambda - it}$, what is your characteristic function, but $F^*(s) = \frac{\lambda}{\lambda + s}$ is the Laplace Stieltjes transform that you can see. And from here, like I can get the moments of this right, that is what is important.

- Moments:
  - $E(X^n) = i^{-n}\phi_X^{(n)}(0)$
  - $E(X^n) = M_X^{(n)}(0)$
  - $E(X^n) = (-1)^n F^{*(n)}(0)$

So, why this is relevant because, if you know, we have said that the transform and its function have a one-to-one relationship, so, normally, instead of working in the time domain, we transform to the transform domain and work on that I am not going to simplify it. Finally, you find the inverse Laplace transform or inverse Laplace Stieltjes transform to get back to the time domain. Now, many a time, that can not be possible like it will be very complex in nature. So, in that case, what you do is at least try to obtain moments of the random variable originally. So, you can use $E(X^n) = (-1)^n F^{*(n)}(0)$ directly to get the moments; you do not even need to invert back; if that is what is complex in nature. So, that is a beauty that is an advantage that you deal with such quantities in a different domain. But we will be contained with MGFs, and Laplace transforms, or Laplace Stieltjes transform of mainly continuous random variables meaning that of course, whenever LST means dealing with discrete of course, this will be Laplace Stieltjes transform.

Continuous means one can refer to any one of these in a way. But what I mean to say is that we will deal with MGFs mainly rather than the characteristic function, so we will be content with this.

So, because of this relationship, MGF too has similar properties, so there is a one-to-one correspondence between MGFs and their probability distributions. The MGF of the sum of independent random variables is the product of MGFs of the individual random variables that also you know. So, again then, these are all one and the same, so it has the similar properties you have here. The other quantity that is of interest to us in our queueing analysis is what we generally call generating function. We just said that moment-generating function because it is the moments that are used to construct such a function. So, it generates moments like we just saw this here. So, $E(X^n)$ can be obtained from its $M_X^{(n)}(0)$ the same thing.

So the other one is what we call probability generating function, which is an example of a generating function of a sequence and equivalent to the z transform of the probability mass function. And these generating functions are, in general, useful in solving difference equations. Difference equations not just arise in our queueing theory, but elsewhere in many different fields of applied mathematics, you will encounter difference equations. And while to solve that, the z transform is what is quite helpful in using that; we will also come to now soon. So this is a probability generating function in our language in probability theory; this z transform equivalent in a certain form is what we call a probability generating function, which is defined for a random variable X, a discrete random variable with $P(X = n) = p_n, n = 0, 1, 2, \dots$ and $\sum_{n=0}^{\infty} p_n = 1$. Then this one $P(z)$ sometimes some books call it $P(s)$, it does not matter, is

$$P(z) = E[z^X] = \sum_{n=0}^{\infty} p_n z^n$$

is the probability generating function (PGF). So, what you do here is that you take each of this p_n's, and you put a tag that is z^n and put together and put it in a single basket rather than representing all p_0, p_1, p_2 and so on. You represent it in $\sum_{n=0}^{\infty} p_n z^n$ form, and as a coefficient of z^n, you can always extract p_n. I mean, these are all used in that manner. You represent $P(z) = E[z^X] = \sum_{n=0}^{\infty} p_n z^n$, so this is whether I represent in $P(X = n) = p_n$ form by giving all p_0, p_1, p_2, p_3, and so on, or I just describe this function $P(z)$ it is one and the same.

And so this is what is called the probability generating function because the coefficients generate the probabilities of the discrete random variable, which has support on nonnegative integers. And this is similar to MGF with now again you see, z if you replace by e to the power t then what you are going to get is the, for such random variables if you have the MGF, then you can find the PGF using $M_X(t) = P(e^t)$, or if you have PGF, the MGF is given by in this case. Say, for example, for a Poisson random variable, we know that $M_X(t) = e^{\lambda(e^t - 1)}$ is the MGF. Now, this e^t if I replace by z, $e^{\lambda(z-1)}$ is going to be the probability generating function. So, you can try to expand $e^{\lambda(z-1)}$ here $e^{-\lambda}$ will be outside, and then there is $e^{\lambda z}$; you make it as a power series expansion, and you extract the coefficient of z^n you get the probabilities. So, whether I describe the probability distribution through the description of this p_n's or just by giving $P(z)$, you should know what the transform is. So, we have given, I mean in the tables of distribution on its MGFs. So, MGF, if you have like if you replace e^t in the MGF by z, what you are going to get is the corresponding transforms for appropriate random variables like Poisson and so on.

Geometric, Poisson are the typical random variables that we will encounter, so it is just sufficient if you know a few of these transforms. Now, again this $P(1)$, as, like I mean in simple terms suppose if I think like, suppose if I put in $z = 1$ in $\sum_{n=0}^{\infty} p_n z^n$ What will happen? This is simply the sum of p_n, which must be equal to 1. So, that is what essentially we are saying in the limit for this function $P(1) = \lim_{z\to 1} P(z) = 1$ because we need to worry about the range for this; if it is a little bit more than that, then the properties of power series come into play when you are differentiating and passing the limit inside. So, the series converges absolutely for at least for all complex numbers in $|z| \leq 1$, but if it is slightly more than 1, then we will not have a problem directly we can substitute; otherwise, you have to take the limit is from below, it is a technical part we do not need to worry. But for our cases, we are in good shape; you do not need to worry about that, so you can simply substitute one and then see that $P(1) = 1$ that is a property that you need to remember. Then again, if I differentiate this, for example, simply you can think if I differentiate $p_n z^n$ with respect to z, then I will have nz^{n-1}. So, I will have a form of some $np_n z^{n-1}$. Now, if I substitute $z = 1$, then what am I going to end up with? This quantity $E(X) = P'(1)$. Similarly, $E(X^2) = P''(1) + P'(1)$. So the double derivative or the triple derivative or, in general, the n^{th} derivative evaluated at 1 will give you $E[X(X-1)(X-2)\cdots(X-n+1)] = P^{(n)}(1)$ which is what is called the factorial moment. So, that is what you will get from this; remember, MGF, you differentiate, and put 0, you will directly get the n^{th} raw moment.

But here, you differentiate with and put the value 1 or evaluate the derivative at value 1; then, you will get the factorial moments from the probability generating functions. Now, that is a simple exercise example that you can see. So, you have a scenario where $X_n, n = 1, 2, \ldots$ be independent and identically distributed discrete random variables with $p_k = P(X_n = k)$ and with PGF $P(z) = \sum_k p_k z^k$. Generically we are taking it. Further, suppose that this capital N is also a discrete random variable with the $P(N = n) = g_n$ and with its PGF being $G(z) = \sum_n g_n z^n$. Now assume that this random variable X_n and N, this X_n's already are independent among themselves. Now they are also independent of this N; that is independent of X_n. Now you define $S_N = X_1 + X_2 + \cdots + X_N$ where N is already a random quantity, and let PGF of S_N be $H(z)$.

► Exercise: Show that $H(z) = G(P(z))$.

Easy you have to use the conditional argument that practice should come, so this is one start. So, basically, $H(z)$, you know what $H(z)$ is. It is basically $SN's$ distribution, and if you want to call that as some h_n then using that, you can do that. Say, for example; you can define h_n as the probability of S_n for fixed n equal to some, this n and that n are different, so $P\{S_n = k\}$; is what then is the distribution of S_N. Now, this particular n is also random in our case. Then you condition on that, and then you will use this expression and then you will arrive at finally, this one you can easily do it is not a big problem. Now, what is important for us is that from $H(z) = G(P(z))$, using the properties that we have just stated, you can deduce $E(S_N) = E(X_n)E(N)$. If this is a fixed one, if S is a fixed one, they are all IID. So, S_1 suppose S_10 suppose if you are looking at it. Then there are 10 such random variables. So, $E(S_10)$ is $10E(S_1)$; that means that the fixed value instead of $E(N)$ times $E(X_N)$ is what then you will know. What this says when this is random, then this particular quantity is to be replaced by this expected value of this particular random variable which you can deduce from here. Similarly, the $Var(S_N) = E(N)Var(X_n) + Var(N)[E(X_n)]^2$, you can deduce. So, these are all some simple exercise that helps to get you started. So, you can try these examples. Now, for practice again, you can consider the above means this whole thing; you keep this as generic p_k, but you assume N to be that not just any generic random variables, but this is a Poisson random variable with parameter λ. And see like what would be this. What is that? That means the form of G you know, which is the Poisson random variable you know what its probability generating function is. So, $G(P(z))$ will be then the G form, you know, but the P form you still retain is $P(z)$. So, that is what is the form that you will use. So, you know what you will obtain in this case; you will obtain that as a compound Poisson distribution. Again you can extend the above by considering these X_n's as the continuous random variables than discrete here with a PDF f and Laplace transform $\bar{f}$.

And you can deduce that the Laplace transform of S_N is nothing but $G(\bar{f}(s))$ is what you would obtain here. Again, you know you can extend a similar exercise where it is now continuous. Now, where do these kinds of things occur, and why do we worry about these kinds of things. So, let us look at this typical simple example that we are giving here. Suppose there is a clinic, and the patients visit the doctor in the clinic. The number of patients visiting a doctor is a Poisson random variable with mean λ, and the time taken by the doctor on a patient is IID uniform in $[o, h]$, say in minutes. So, he will take the time to treat one particular, or the time the doctor spends on a particular patient is uniformly distributed between $[0, h]$. So, suppose h suppose if you assume to be some 5 minutes or 10 minutes. So, it is anywhere between 0 and 5 is what he is going to spend on each of his patients; patients' arrival is a Poisson random variable with the Poisson process. So, the number of patients visiting the doctor is a Poisson random variable; this is what then we are assuming here. Now, if you want to find the mean and the variance of the time taken by the doctor to complete the consultation of all the patients, so what you have here is the number of patients you can relate to N. X_n's are the time taken to serve a particular patient. So, this is given to be uniform on $[0, h]$. N is given to be in a Poisson random variable. So, you can simply use $G(\bar{f}(s))$ to get the transform of S_N, and from there, you can obtain mean and variance. So, you are interested in, say, what is the mean time taken by the doctor to complete all the patients to attend to all patients. And what is the variance of that suppose if you see since you do not know how many patients would have arrived during a particular time interval in which you are considering. Say, for example, in a day, it is random according to some distribution, and the time taken is also not deterministic it is also a random according to some distributions. You are assuming that they are independent because there is no relationship between the time taken for a previous patient and the time taken for the next patient. So, that is, you are taking in to be IID for ease and then some distribution in. Then you know you want to see what is the overall mean-variance; suppose this is a simple clinic with a single doctor, it is fine, but if it is a big clinic where multiple numbers of doctors, then to understand how many doctors are needed or how to increase the capacity or whether overcapacity, overstaffing is there everything to analyze you need to understand, but things are random here. To understand all those things because your interest if it is only mean and variance. So, now, remember that if I want to compute the distribution of $S_N = X_1 + X_2 + \cdots + X_N$, it will be very complex, but the mean one I can obtain here. If I know the mean and variance of the N and X_n, I can get the mean and variance of S_N. So, how do we arrive at this? I have to do this analysis.

So, this is the expression that you would have. Again in queueing situation, you would find you know many a type this kind of scenarios, given a time interval how many people would arrive, how much time server would have spent in you know in a similar situation. So, this is you know something similar to that is what then. So, such kinds of situations would arise it is not just in queueing and elsewhere in other stochastic models as well. For example, biology or DC, I mean all these kinds of things like you know you would find these are so common it is not just in queueing, but it is also in the more general stochastic process this kind of thing would come. So, to understand that you know you will need this, but the tools that you use you see here the PGF, I have I can answer, or I can handle this kind of scenarios with this kind of knowledge. So, these are all the techniques that we would need from the transform thing Laplace transform Laplace Stieltjes transform and the probability generating function. We will use mainly LST because it is the same as LT in that it can incorporate discrete to random variables as well and PGF whenever it is much easier than the other one to handle in such scenarios. Because we are dealing directly with the probabilities, we are working with probabilities, and then inversion will give you probabilities as well, so that is what it is. These are all the tools you might need; you can get familiarized with this, and we may get going in the next lecture.
Thank you.