Introduction to Queueing Theory

Prof. N. Selvaraju
Department of Mathematics
Indian Institute of Technology Guwahati, India

Lecture - 39

M/G/1 Queues, The Pollaczek-Khinchin Transform Formula

Hi and hello, everyone; what we saw in the previous lecture was this M/G/1 Queue, and what we have done is that we have derived the PK mean value formula by considering the system at arrival points. By employing the mean value arguments and using results from renewal theory, we have obtained the PK mean value formula. Basically, what we have obtained was this one $W_q = \frac{1+C_B^2}{2}\frac{\rho}{\mu-\lambda}$, as a product of three quantities one is $\frac{1+C_B^2}{2}$, the other is $\frac{\rho}{1-\rho}$, and the third quantity is $1/\mu$, which is basically the $E[S]$ is what, so that is what we have obtained. And from this, we have obtained all other quantities, and we have filled this table; and this table

L_q	$\frac{1+C_B^2}{2}\cdot\frac{\rho^2}{1-\rho}$	$\frac{\lambda^2 E\left[S^2\right]}{2(1-\rho)}$	$\frac{\rho^2+\lambda^2\sigma_B^2}{2(1-\rho)}$
W_q	$\frac{1+C_B^2}{2}\cdot\frac{\rho}{\mu-\lambda}$	$\frac{\lambda E\left[S^2\right]}{2(1-\rho)}$	$\frac{\rho^2/\lambda+\lambda\sigma_B^2}{2(1-\rho)}$
W	$\frac{1+C_B^2}{2}\cdot\frac{\rho}{\mu-\lambda}+\frac{1}{\mu}$	$\frac{\lambda E\left[S^2\right]}{2(1-\rho)}+\frac{1}{\mu}$	$\frac{\rho^2/\lambda+\lambda\sigma_B^2}{2(1-\rho)}+\frac{1}{\mu}$
L	$\frac{1+C_B^2}{2}\cdot\frac{\rho^2}{1-\rho}+\rho$	$\frac{\lambda^2 E\left[S^2\right]}{2(1-\rho)}+\rho$	$\frac{\rho^2+\lambda^2\sigma_B^2}{2(1-\rho)}+\rho$

is what we call the PK mean value formula.

And we have seen that in the case of $M/E_k/1$ or $M/D/1$, we can obtain the directly using this as well the mean value quantities is what we have seen. Now, let us take another example where we will use this PK mean formula to give certain inferences to the system under consideration.

Example.

- Assume that a system is currently working as an $M/M/1$ system with $\lambda = 10$ and $\mu = 12$, per hour.
- The server undergoes a training session at the end of which it is expected that while the mean service time would increase slightly, the variance would see an improvement.
  - ▶ The mean service time now is 5.5 minutes and the standard deviation is 4 minutes.
  - ▶ The system is now an $M/G/1$ system.

- Management is interested to know the impact of the training and whether they should have the server undergo further training.

- *Let us compare L and W:*
  For $M/M/1$: $L = 5$ and $W = 30$ minutes.
  For $M/G/1$: $L = 8.625$ and $W = 51.75$ minutes.

  You see how you thought that with a variability reduction in the service time, system parameters should improve, but you can see deterioration here.

- Hence, it is not profitable to have the server better trained.
  ▶ Here, with training, while the mean increased by 10%, the standard deviation decreased by 20% (from 5 to 4).
  ▶ The performance is more sensitive to mean than to standard deviation.

  That is why even though the standard deviation reduction was 20%, the mean increased by 10%; this was crucial. So, that is what you ultimately see here.

Example 1. • Now, if you ask the question that what is the reduction of variance required to make up for the increase of 0.5 in the mean. So, instead of 4, say, for example, if I have to make for this 5.5, what would be the reduction that would require whether it is 3, whether it is 2 or 4.5, what is the required variance so that you will see the same L at least.

- *We can do this by solving for σ_B^2 in the PK formula for L:*

$$L = 5 = \rho + \frac{\rho^2 + \lambda^2\sigma_B^2}{2(1-\rho)},$$

  where $\rho = 11/12$. This yields $\sigma_B^2 < 0$, which is not possible.
  ▶ *This means that $L > 5$ always (even with $\sigma_B^2 = 0$).*
  ▶ *The minimum value of L, achieved in the $M/D/1$ system, turns out to be $L = 6$.*

  So, you can not reduce it any further if the mean is increased to 0.5; but now you can ask for a slightly different one along the same line.

- *Exercise: Determine the value of σ_B^2 required to yield the same L if the mean service time were increased to only 5.2 minutes after training.*

Because that is what is the idea that, how much variability I can reduce from there you can compute this along using the same line, so this is sort of, a reverse engineering kind of thing. If you want to retain the same L, what would be then the variability how much reduction you would expect after the training to happen so that you retain the same L? So, you can ask this question, of course, and try that. So, this is another example there where we are using this result; basically, we are comparing the systems as well between this and so on. So, you can directly use this PK mean value formula to answer questions of this nature, where you are not concerned about the system size probabilities but only on the mean values.

Now that we said; that was the first derivation using the arrival points, the same formula can be derived by using a similar argument but by using now departure time points.

- We can derive the PK mean formulas given in table earlier considering the queue at times when customers depart from the queue.
- Considering the number of customers remaining in the system immediately after a customer has departed from the system, we can first derive a formula for the expected system size L at departure points.
- This is then seen to be equal to the expected steady-state system size at an arbitrary point in time.

- Instead of doing this, we will treat the steady state system size probabilities at departure points, from which too the PK mean formulas can be obtained.

Even if you do not want to consider the departure point system size probabilities, one can even consider departure points, and again by the mean value formula directly; you can obtain that is the idea behind this. Because it is just a repeat, only thing is the ideas are slightly different, but we will directly go to departure point system size probabilities which for the $M/G/1$ queue, which is what we will it will result in this PK formula or more specifically, now what we are going to get is this PK transform formula.

- Let π_n denote the steady state probability of n in the system at a departure point.

  Excluding the customer who is departed, which means just after the time point; immediately after the time point, what is the number in the system, just after the departure, what is the number in the system that is what we denote it by π_n.

  ► In general, it need not be the case that $\pi_n = p_n$, but it is true here for the $M/G/1$ model.

  p_n is what is the arbitrary time point probabilities a_n's earlier we have considered these arrival point probabilities. Now, π_n in this particular case is what the departure point system size probabilities. In general, this need not be the case; the departure point probabilities need not equal the arbitrary time point probabilities. But here, in the case of $M/G/1$ queue, this is true, so that we will see later. So, if we obtain this π_n, if this is true and true here, then this will also be equal to p_n. And that is the reason why we said that here even though you are looking at the departure times and then obtaining the moments, it will also be equal to arbitrary time point moments. But now, we are looking at the probabilities themselves.

- The $M/G/1$ queue, viewed only at departure times, leads to an embedded discrete-time Markov chain.

We said that we are going to use this embedded Markov chain technique for the analysis of semi Markovian queuing system. So, here for the $M/G/1$ queue, if you view the system at departure points, then you can extract an embedded Markov chain.

- The number in the system process $\{N(t), t \geq 0\}$ is not a Markov process here, because the state of the system after a transition depends not only on the state of $N(t)$, but also on the amount of elapsed service time of the person receiving service, if any. [Together, they form a Markov process]

Now whereas, this was not the case in Markovian queuing models because the elapsed time is again exponential other residual lifetime is again exponential. So, one need not worry about that, but here it does matter because the service time distribution does not have the memoryless property. But this $\{N(t), t \geq 0\}$ is not a Markov process, but suppose if I call the amount of elapsed service time as some other process, say some $Q(t)$, then this $\{N(t), Q(t)\}$ as a two-dimensional process will be a Markov process.

That is what we said together they form a Markov process; $\{N(t), t \geq 0\}$ alone is not a Markov process. $\{N(t)\}$, Markov process means what? Whether it is in one dimensional or two dimensional, or in any number of dimensions, the future evolution of the process depends only on the current state of the process; if that is the case, then the Markovian property is holds true. It does not depend on anything in the past; that is what you see here. So, here $\{N(t)\}$ alone would not determine as obvious, because how long the current customer who is currently undergoing service was in service that will determine how much more time he is going to get service. So, that is required to be known for the future evolution of the process, even if you look at the number in the system so, so that is the reason why this $\{N(t)\}$ is not a Markov process. But together they form because together now you get the complete information about it. But if we look at such a process at some specific time points, then you can extract a Markov chain, or you can see at those time points the system will behave like a Markov chain. And here, in the $M/G/1$ queue, one such point or set of points is the departure point. Because as you see at any point of time t if you look at it, you need to remember what is the number in the system at that point of time and how long has been the service completion happened for the current customer who is currently undergoing service. But if I look at departure points, that elapsed service time is 0. So, if that is the case, then the evolution of this $\{N(t)\}$ from this departure point to next departure point, if I can look at it, I can forget the elapsed time. Now, I need to know that apart from what is there at this point, it is basically a regeneration point or restarting point. Suppose I call this as a departure point, now from and this is the next departure point. From this to this the departure point, what I need to know is what was the number here and how many people who arrived during this period would determine the number in the system at the next departure point. So, if I can get along with this, the number in the system at this departure point say nth departure point; now, if I want to look at the number in the system at $(n+1)$st departure point. Now between this nth and $(n+1)$st departure, what were the number who arrived if I know that, and what was the number at nth departure point. Then I can determine what is the $(n+1)$st departure point, what was the number in the system.

► If we consider the system only at those points when a customer completes his service, there will be no elapsed service time.

► So, in that way, like this evolution of this $N(t)$ which is what our ultimate aim is that the number in the system in the queueing system can be captured very nicely if I look at departure points in this particular case.

So, say, for example, to put it in a more concrete form:

- Let $t_1, t_2, \ldots$ be the sequence of departure times from the system.
- Let $X_n = N(t_n+)$ be the number of customers left in the system immediately after the departure at time t_n.

  That means that this is the number in the system immediately after the departure of a customer at time t_n.

- If $Y(t)$ denotes the number of customers left-behind in the system by the most recent departure. That is, $Y(t) = X_n, \quad t_n \leq t < t_{n+1}$.

  Then $Y(t)$ will give me the number of customers left behind in the system by the most recent departure.

  ► $\{Y(t)\}$ is a semi-Markov process having $\{X_n, n = 0, 1, ...\}$ as its embedded Markov chain.
  ► $\{(X_n, t_n), n = 0, 1, 2, \ldots\}$ is a Markov renewal process.
  ► The sequence of intervals $\{t_{n+1} - t_n, n = 0, 1, 2, \ldots\}$ being the inter-departure times of successive units (or equivalently $\{t_n, n = 0, 1, 2, \ldots\}$) defines a renewal process.

  Go back to your definition of renewal process; we have taken some IID random variables X_ns and used it to define S_n and used it to define the process, say $\{Y(t)\}$, which is a semi Markov process. So, here that $t_{n+1} - t_n$ is what are those X_ns and this t_ns is what correspond to those S_ns and $Y(t) = X_n, \ t_n < t < t_{n+1}$ we are defining it in this manner with semi Markov process. So, this is what you have here, so $\{Y(t)\}$ is what is the semi-Markov process that you are considering under this. So, $\{Y(t)\}$ is the semi Markov process; this $\{(X_n, t_n), n = 0, 1, 2, \ldots\}$ is the Markov renewal process, or, equivalently, $\{t_{n+1} - t_n, \ n = 0, 1, 2, \ldots\}$ is the inter-departure times of successive units. This is what will constitute the renewal process; that is why here we have Markov Poisson process arrival and a renewal service process; renewal service process means, this is what the inter departure times are renewal process is what you have. Then basically, what you have is this semi Markov process, and where X_n is given by $N(t_n+)$, and you think that you are defining this process $\{Y(t)\}$. So, this $\{Y(t)\}$ process is what is the semi Markov process connected with this $M/G/1$ queue is this one $Y(t)$, which is the number of customers left behind by the most recent departure that is semi Markov process that we have here. So, this is what is the Markov chain that we are talking about. So, if I look at here X_n, then we know this $\{X_n, \ n = 0, 1, 2, \ldots\}$ is the embedded Markov chain that is what we were looking for it. Now what we will do? We will treat this $\{X_n\}$ to study like what we can do with this and how one can analyze the $M/G/1$ queue based upon this $\{X_n\}$.

- Let A_n be the number of customers who arrive during the service time of the nth customer. Then, for all $n \geq 1$.

$$X_{n+1} = \begin{cases} X_n - 1 + A_{n+1}, & X_n \geq 1, \\ A_{n+1}, & X_n = 0. \end{cases}$$

Suppose if there is no customer was there in the system when the nth customer left. Then what would have happened? First, one customer could have arrived at some point of time, and during then, the $(n+1)$th departure would be the customer who arrived when an empty system was there, and his departure is what is $(n+1)$th departure. So, what would be the number that he will be leaving behind in the system is exactly the number of customers who arrived during his service time that is simply A_{n+1}. So, that is the relationship here, when $X_n = 0$. Suppose if leaves at least one customer in the system when the nth departure happens, then when $(n+1)$th departure happens. So, the number who were originally there at the point of time of nth departure was X_n, and when $(n+1)$th departure means out of this X_n, 1 will depart. So, out of those X_n, $X_n - 1$ would be now the new number, but during this one customer who was getting served during his service time, A_{n+1} many number of customers would have arrived. So, the total number that will be left behind by the $(n+1)$th customer is basically $X_n - 1 + A_{n+1}$. This is the recursive relationship between the X_ns; basically, the X_ns are the departure point system sizes. Now we want to see that this is a Markov chain. In general, it is easy to see if you know the Markovian theory that if X_{n+1} is a function of X_n and some other random variable that is independent of the past process of this X_ns.

Then that will define a Markov chain; it is obvious because the evolution of this X_{n+1} as long as this A_{n+1} is independent of this $X_n - 1$, $X_n - 2$, and so on. Then $X_n - 1 + A_{n+1}$ quantity, this whole evolution would depend only on the what is the system state at the time n or what is the value of X_n and A_{n+1} and A_{n+1} has no relationship with this past history of X_n. So, if that is the case, this will be a Markov chain that will be clear.

- We see that $\{X_n, n \geq 1\}$ is a Markov chain.
  ♦ Need to show that future states of the chain depend only on the present state – more specifically, we must show that given the present state X_n, the future state X_{n+1} is independent of previous states $X_{n-1}, X_{n-2}, \ldots$.
- First observe that X_{n+1} depends only on X_n and A_{n+1}. If A_{n+1} is independent of $X_{n-1}, X_{n-2}, \ldots$, then $\{X_n\}$ is a Markov chain.
- A_{n+1} is the number of customers arriving during the service time of the $(n+1)$th customer.

Remember, what in the arrival process? Arrival process is a Poisson process, which has stationary increments and independent increments. So, because of those properties, this A_{n+1} depends only on the length of the service time but does not depend on the events that occurred earlier, which means the queue size at earlier departure points. Because whatever be the duration the past, what has happened, like how many arrivals is happening in an interval of length t it depends only on the length t like; where this interval is positioned, it does not matter. So, what has happened before the nth departure, what was the size and duration, it has no impact on the number of customers who are going to arrive during this period. That is the independent increment property that helps you to get. And where this interval is positioned is also because this station increment property is just the length of the interval; that is all it requires.

► Thus, A_{n+1} independent of $X_{n-1}, X_{n-2}, \ldots$ and hence $\{X_n\}$ is a MC.

Of course, if this is the case, A_{n+1} is independent of $X_n - 1$ and so on, and X_{n+1} is a function of X_n and A_{n+1}.

Then it is a Markov chain one can easily show in general that is what it is happening here. So, this is we have shown that this is a Markov chain.

- We now derive the transition probabilities for this Markov chain

$$p_{ij} = P\{X_{n+1} = j | X_n = i\}.$$

- The transition probabilities depend on the distribution of the number of customer who arrive during a service time.

Now, since the nth or $n+1$th or some kth customer, the index does not play a role in was for the distribution of this service time because they are all IID. So, we drop the subscript A_n, and we consider A and S.

- Let S denote a random service time (with CDF $B(\cdot)$) and A denote the random number of customers who arrive during this time (we drop the subscript as the distribution does not depend on the index of the customer). Define, for $i = 0, 1, 2, \ldots,$

$$k_i = P\{i \text{ arrivals during a service time}\} = P\left\{A = i\right\} = \int_0^\infty P\left\{A = i | S = t\right\} dB(t).$$

So, this is what you have obtained where we have written this as a Stieltjes integral. So, even if it is discrete distribution or any other distribution, one can still write this nicely. But whenever the density exist, you can see that $dB(t)$ is equal to the density times dt, then it becomes an ordinary Riemann integral. As we said in the beginning, itself like, most of the results, you will write in this form to make it uniform or fit in more general situations. But now I have to find $P\{A = i | S = t\}$, but what is this probability, this S is fixed at t; I am looking at A arrivals during an interval of length t.

- Note that $A|S = t$ is a Poisson random variable with mean λt, and hence $P\left\{A = i | S = t\right\} = \dfrac{e^{-\lambda t}(\lambda t)^i}{i!}$ giving us

$$k_i = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^i}{i!} dB(t).$$

k_i gives you now the probability of i arrivals during a service time is given by this expression. So, k_i is an important quantity in this analysis. So, you just remember that this is what you have you have obtained as k_i.

- Then from the relationship between X_n's, we get

$$p_{ij} = P\left\{X_{n+1} = j | X_n = i\right\} = \begin{cases} P\left\{A = j - i + 1\right\}, & i \geq 1 \\ P\left\{A = j\right\}, & i = 0 \end{cases}$$

$P\{A = j - i + 1\} = k_{j-i+1}$, and $P\{A = j\} = k_j$; that is what we have just denoted here; that is what is this particular case $P\{A = i\}$ is what is k_i. So, this is $P\{A = j - i + 1\} = k_{j-i+1}$ and $P\{A = j\} = k_j$. So,

$$p_{ij} = P\left\{X_{n+1} = j | X_n = i\right\} = \begin{cases} k_{j-i+1}, & i \geq 1, j \geq i - 1 \\ k_j, & i = 0, j \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- We the have the following transition probability matrix

$$P = ((p_{ij})) = \begin{pmatrix} k_0 & k_1 & k_2 & k_3 & \dots \\ k_0 & k_1 & k_2 & k_3 & \dots \\ 0 & k_0 & k_1 & k_2 & \dots \\ 0 & 0 & k_0 & k_1 & \dots \\ 0 & 0 & 0 & k_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

  This matrix is called $M/G/1$ type matrix; in general, in queuing theory, this type of matrix is referred to as $M/G/1$ type of matrix. So, this is the transition probability matrix of this embedded Markov chain in the case of n M/G/1 queue. Now, this is what I obtained as pgf.

- Assuming that steady state is achievable (which is basically when we have to put find out the condition under which the system is ergodic, the necessary and sufficient condition for the steady-state to exist; which is the usual condition of $\rho < 1$, which you are not going to prove, but you can assume that that is what is the case.), the steady state probability vector $\boldsymbol{\pi} = \{\pi_n\}$ is found in the usual manner as the solution of the stationary equations:

$$\boldsymbol{\pi} = \boldsymbol{\pi} P, \;\; \boldsymbol{\pi e} = 1.$$

  Writing down explicitly, these equations are

$$\begin{aligned} \pi_i &= \pi_0 k_i + \pi_1 k_i + + \pi_2 k_{i-1} + \cdots + \pi_{i+1} k_0 \\ &= \pi_0 k_i + \sum_{j=1}^{i+1} \pi_j k_{i-j+1}, \qquad i = 0, 1, 2, \dots \end{aligned}$$

- Now define the generating functions

$$\mathit{\Pi}(z) = \sum_{i=0}^{\infty} \pi_i z^i \quad \text{and} \quad K(z) = \sum_{i=0}^{\infty} k_i z^i \quad (|z| \leq 1)$$

- Multiplying the steady state equations by z^i, summing, and solving (Exercise!) for $\mathit{\Pi}(z)$ yields

$$\mathit{\Pi}(z) = \frac{\pi_0 (1-z) K(z)}{K(z) - z}$$

  Now, as we say, as usual, we need to determine what is this π_0, for which we have to use the boundary condition or the normalization condition that $\mathit{\Pi}(1) = 1$.

- Using the fact that $\mathit{\Pi}(1) = 1$, along with L'Hospital rule, and realizing that $K(1) = 1$ and $K'(1) = \lambda(1/\mu)$, we find that

$$\pi_0 = 1 - \rho \quad (\rho = \lambda E[S] < 1 \text{ is the condition for ergodicity})$$

  and therefore we obtain finally

$$\boxed{\mathit{\Pi}(z) = \frac{(1-\rho)(1-z)K(z)}{K(z) - z}.}$$

This is known as Pollaczek-Khinchin (PK) Formula or Pollaczek-Khinchin (PK) Transform Formula.

We will later see that we can write $|Pi(z)$ in one more form again using transforms, but here both of them are z transforms is what you see here. So, this is what is known as PK transform formula. So this is what you can go as far as the analysis or the simplification of the process of obtaining the departure point system size probabilities go. Now, what do you need? Once I know $\Pi(z)$, I can obtain the PK mean value formula; like I can differentiate $\Pi(z)$, I can obtain the departure point, equilibrium, or mean values, and then you will get that that will be equal too since we have already seen not shown. We are already observe said that this will be equal to arbitrary time point. So, that will be the L in the general one as well, and hence other measures also one can obtain. So, how do we then work out in such scenarios?

- Given the service time distribution B, we can obtain k_i's and hence $K(z)$. Substituting this, we obtain $\Pi(z)$, the PGF of the distribution of the departure epoch system size. $\{\pi_n\}$ can then be obtained from its PGF.
  ▶ It is the case here that $\pi_n = p_n$.

You see here in this semi Markovian system what we have done? Though the process $\{N(t)\}$ is semi Markovian, one way to extract a Markov process, consider this $\{N(t)\}$ along with the elapsed service time. Then together, they form a Markov process; then, one can apply the Markov process theory, which is called the supplementary variable technique. But instead of what we did, we looked at this $\{N(t)\}$ process only not at all times, but at specific time points, which are the departure epochs, and we could extract a Markov chain, and from the steady of this Markov chain, we obtain for that corresponding Markov chain at those time points what is the system behaviour, whether it is mean number or probabilities and so on. Now, if one can show that that system behaviour at those specific time points would be probabilistically equal to the system behaviour at an arbitrary time point, then we have done the analysis for the complete arbitrary time point analysis. So, for which what we need to see is that this whether this is true. We repeatedly said that $\pi_n = p_n$, but how this $pi_n = p_n$ is what we will see now quickly.

- To prove that π_n, the steady-state probability of n in the system at a departure point, is equal to p_n, the steady-state probability of n in the system at an arbitrary point in time.

- We begin by considering a specific realization of the actual process over a long interval $(0, T)$.

- Let $N(t)$ be the system size at time t. Let $A_n(t)$ be the number of unit upward jumps or crossings (arrivals) from state n occurring in $(0, t)$. So, you fix the state n. So, from the process $N(t)$, how many times it crosses it is going above n is what you are counting that is the number of the unit upward jumps from state n in $(0, t)$ is what you call it as $A_n(t)$. Let $D_n(t)$ be the number of unit downwards jumps (departures) to state n in $(0, t)$. From state n how many times it goes up is what you are counting this $A_n(t)$ and to state n how many times it reaches is what you are counting is what you call it as $D_n(t)$.

- Since arrivals occur singly and customers are served singly, we must have

$$|A_n(T) - D_n(T)| \leq 1. \tag{1}$$

- Furthermore, the total number of departures, $D(T)$, relates to the total number of arrivals, $A(T)$, by

$$D(T) = A(T) + N(0) - N(T). \tag{2}$$

So, $N(0)$ is the initial number when you are starting the system at time 0. $N(T)$ is the current one. Number of arrivals that has happened A of T plus $N(0)$ was the initial one. So, $A(T) + N(0)$ was the total number who were either in the system or arrived in $(0, t)$, and $N(T)$ is the current number. So, the departure is essentially the difference of these two.

- The departure-point probabilities are

$$\pi_n = \lim_{T\to\infty} \frac{D_n(T)}{D(T)}. \tag{3}$$

- By adding and subtracting $A_n(T)$ from the numerator of (3) and using (2) in its denominator,

$$\frac{D_n(T)}{D(T)} = \frac{A_n(T) + D_n(T) - A_n(T)}{A(T) + N(0) - N(T)} \tag{4}$$

- Since $N(0)$ is finite and $N(T)$ must be too because of the assumption of stationarity, it follows from (1), (4), and the fact that $A(T) \to \infty$

$$\lim_{T\to\infty} \frac{D_n(T)}{D(T)} = \lim_{T\to\infty} \frac{A_n(T)}{A(T)} \tag{5}$$

  with probability one. Since the arrivals occur at the points of a Poisson process operating independently of the state of the process,

- Since the arrivals occur at the points of a Poisson process operating independently of the state of the process, we invoke the PASTA property that Poisson arrivals find time averages.

- Therefore the general-time probability p_n is identical to the arrival-point probability $a_n = \lim_{T\to\infty} \frac{A_n(T)}{A(T)}$, which is in turn, equal to departure-point probability from (5).

- Thus, all three sets of probabilities are equal for the $M/G/1$ problem (i.e, $a_n = p_n = \pi_n$).

And hence what we have obtained so far, whether we look at the mean value formulas arrival point departure point the departure point system size probabilities, will all be the same even if you consider at the arbitrary time point. So, that is what you are seeing it here. So, all three sets of probabilities are equal for $M/G/1$ probability. So, what we have obtained though it is a departure point system size probabilities; they can also be considered as the arbitrary time point system size probabilities, arbitrary time point mean value formulas, and so on. Let us quickly take an example before we close.

Example.
If we set the service time distribution as exponential, then $M/G/1$ should reduce to $M/M/1$.

Take $B(t) = 1 - e^{-\mu t}, t \geq 0$ *(and* 0 *otherwise). Then*

$$\begin{aligned} k_i &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^i}{i!} dB(t) = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^i}{i!} \mu e^{-\mu t} dt \\ &= \frac{\lambda^i \mu}{i!} \int_0^\infty t^i e^{-(\lambda+\mu)t} dt = \frac{\lambda^i \mu}{i!} \frac{\Gamma(i+1)}{(\lambda+\mu)^{i+1}} \\ &= \left(\frac{\mu}{\lambda+\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^i, \quad i = 0, 1, 2, \ldots \end{aligned}$$

So, the number of arrivals during an exponential service time is what is geometric distribution.If you know like a Poisson process, of course, if you have two independent Poisson processes, then also one can get in in this manner easily, so that is what it is actually. *Therefore,* $K(z) = \dfrac{1}{1+\rho-\rho z}$*, where* $\rho = \lambda/\mu$*. Using* $K(z)$*, we can obtain the PGF* $\Pi(z)$ *as*

$$\Pi(z) = \frac{(1-\rho)(1-z)\frac{1}{1+\rho-\rho z}}{\frac{1}{1+\rho-\rho z} - z} = \frac{(1-\rho)(1-z)}{(1-z)(1-\rho z)} = \frac{1-\rho}{1-\rho z},$$

which is the PGF in the $M/M/1$ *model (equal to* $P(z)$*), as required.*

Even if you have some other distribution here, this is the process that you need to follow. k_i either you find directly $K(z)$, or you obtain k_i, and then you find $K(z)$. Now once I obtain $K(z)$ given the service time distribution, I can obtain $K(z)$. $K(z)$ substitute in this PK transform formula to get $\Pi(z)$. Now extract from this to get the system size probabilities. So, that is the way one does the analysis with respect to the $M/G/1$ model.

So, we will stop the discussion of $M/G/1$ model at this point of time which is basically we what we have considered is the PK mean value formula PK transfer formula, we are given to obtain the mean values as well as the system size probabilities. So, given any service time distribution now, you can think about any arbitrary service time distribution; it could be one-point distribution, two-point distribution, or a discrete distribution, continuous distribution, or anything you can think about it. This is the steps that you will follow to arrive at $\Pi(z)$; how complex is that depends on the distribution that you are picking it up. So, in this case, for example, it is very easy; it is exponential; we wanted to show that this formula reduces to the exactly same formula corresponding to $M/M/1$ case. So, that is what we have done here. So, we will stop here; we will continue with our discussion further some more ideas of $M/G/1$ in the the following lectures.
Thank you.