Introduction to Queueing Theory

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Lecture - 19

Transient Solutions: M/M/1/1, Infinite-Server and M/M/1 Queues, Busy Period Analysis

Hi and hello, everyone. We will see in this lecture the Transient Solution or time-dependent solution to the Queueing system or Queueing model. As you know, we have considered so far only the steady-state solution or equilibrium solution, or what is the long-run behaviour. So, it does not take care of what might happen or what might be happening with the system in the beginning when the system is basically starting. If you want to know about that part, then what we require is the transient solution or time-dependent solutions, and this one only like as this time tends to infinity only what we are going to get is the limiting distribution or stationary distribution or steady-state distribution that is what we have been concerned with. So, that is the reason being that it is easy to handle, and for most of the models right that you might have in mind, that might be a sufficient one to get a first-hand understanding of what the system is all about. But you know one needs to remember that the steady-state results, the drawbacks are what then probably like we are listing down here some of them, but may not be all.

★ Steady state results exist only under severe restrictions unlike the transient ones.
  And the conditions that we impose for the stability of the system may not be true, or it may be a little severe on the system.

★ In many potential applications steady state measures of system performance do not make sense for systems that never approach equilibrium.

★ They are inappropriate in situations wherein the time horizon of operations is finite.

★ Time dependent analysis helps us to understand the behaviour of a system when the parameters involved are perturbed.

  Like what could be the immediate impact of the perturbation in the parameters. It might keep changing. So, that is what is happening that you can immediately see rather than waiting again for the system to become an equilibrium state and then looking at its involvements.

★ Two models may have the same steady state results but their transient behaviour may be different.

  We have already seen with respect to the discrete arrivals queue and the M M infinity model that this is what is happening.

★ The transient analysis contributes to the costs and benefits of operating a system.

★ Transient analysis is very useful in obtaining optimal solutions which lead to the control of the system.

[**Whitt, W.** (1983) Untold horrors of the waiting room: What the equilibrium distribution will never tell about the queue-length process. *Mgt. Sci.*, **29**, 395-408.]

I mean, for this, I am giving one reference, but there has been a lot of work after this as well on why the transient analysis is needed to be studied in a queueing model as opposed to only in the steady-state analysis. Of course, the steady-state analysis you will do in the first level. But, if you want a finer understanding, then; obviously, you need to know the transient analysis. Even in the case where the steady-state does not exist, say, for example, in an M M 1 queue, we know that for rho greater than or equal to 1, we do not have an equilibrium distribution. But even in such cases, if you want to know how it is behaving, it does not have an equilibrium. So, it does not reach an equilibrium, but in a finite amount of time, it is not going to explode. So, things are going to. So, how is it behaving? Even within that, how is its behaviour if you want to know the only recourse that you have in your hand is the transient analysis or the time-dependent analysis. Since the the the quantities, the equilibrium the limiting probabilities are 0 in that particular case, so, how quickly it is reaching 0 and what is the rate at which you know it is reaching. If you want to, you can look a little bit more closely at even the instability of the system; if you want to study, you can look at more closely how quickly it is going into that particular state. So, that is what would be helpful nature. So, this does not require any condition of that nature in that case.

- In many practical applications, an analysis that deals with a system's operating behaviour for a fixed, finite amount of time and takes into account the initial conditions is more relevant.

- Analytical solutions of time-dependent nature are very complex (as we will show now!).

We will take three examples to exhibit the complexity in the solutions as opposed to their corresponding equilibrium behaviour; what could be the level of complexity that one needs to handle, and whether that much effort is worth putting in it depends on the situation if it is required then; obviously, you need to put. But then, one needs to prepare for the more complex analysis of the queueing model as opposed to a much easier way of handling the equilibrium scenario case. So, let us get started. So, if you go beyond Poisson exponential assumption, this becomes even more complex, but we will restrict, but even out of the models that we have considered just to we are going to give an overview. We are not going to go into every detail of this solution; we will just give an overview to show you how complex is the solution is; how critical the analysis would become if you want to really get into that, then you need a lot of machinery than what we have at hand right now, but still, you want to give an overview so, that you get a feel of what is this transient analysis or transient behaviour would mean and how one can go about doing it. So, let us get started with the simple model of $M/M/1/1$; very simple model, there is one server, and there is no queueing happen. So, this is the loss system sort of thing. $M/M/1/1$ model is very simple.

- There is no waiting and $\lambda_0 = \lambda,\ \lambda_n = 0,\ n > 0,\ \mu_1 = \mu,\ \mu_n = 0, n > 1$. So, this is what is the simple two-state irreducible Markov chain scenario.

  The forward Kolmogorov equations satisfied by $p_n(t) = P\{N(t) = n\}, n = 0, 1,$ are

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t)$$
$$\frac{dp_1(t)}{dt} = -\mu p_1(t) + \lambda p_0(t)$$

And, $p_0(t) + p_1(t) = 1, \forall t \geq 0$. Assume that the initial probability distribution is given by $p_0(0) = \alpha_0, p_1(0) = \alpha_1$ and $\alpha_0 + \alpha_1 = 1$.

♦ We need to solve the above problem to get the transient solution.

- Substituting $p_0(t) = 1 - p_1(t)$ in the second equation, we get

$$p_1'(t) + (\lambda + \mu)p_1(t) = \lambda \qquad \text{and solving this linear ODE,}$$
$$\Longrightarrow p_1(t) = Ce^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda + \mu}.$$

But $C = \alpha_1 - \frac{\lambda}{\lambda+\mu}$ from the initial conditions.

- We finally obtain the transient solution, for all $t \geq 0$, as

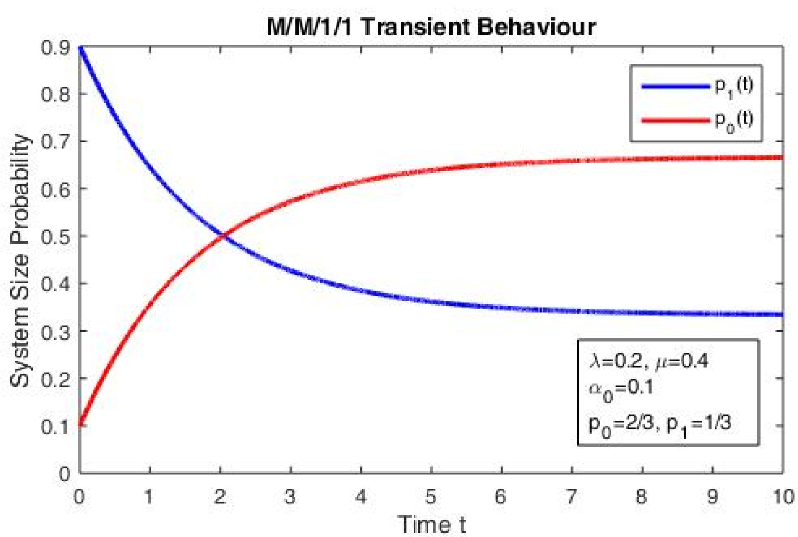
$$p_1(t) = \frac{\lambda}{\lambda + \mu}\left[1 - e^{-(\lambda+\mu)t}\right] + \alpha_1 e^{-(\lambda+\mu)t}$$
$$p_0(t) = \frac{\mu}{\lambda + \mu}\left[1 - e^{-(\lambda+\mu)t}\right] + \alpha_0 e^{-(\lambda+\mu)t}.$$

Now, this is what we call the transient solution or the time-dependent solution.

- The limiting (steady-state, equilibrium) solution can be found as the limit of the transition solution above as $t \to \infty$. Now, once when you do that, as $t \to \infty$, $e^{-(\lambda+\mu)t} \to 0, e^{-(\lambda+\mu)t} \to 0$. And $p_1(t)$ goes to $\frac{\lambda}{\lambda+\mu}$, and $p_0(t)$ is $\frac{mu}{\lambda+\mu}$, which is essentially

$$p_1 = \frac{\rho}{\rho + 1} \quad \text{and} \quad p_0 = \frac{1}{1 + \rho}, \qquad \rho = r = \lambda/\mu.$$

Now let us look at how this behaves for you know certain parameter values from this graph.



So, this is the first model that we have considered. Now, the second model that we considered is the ample server model or $M/M/infinity$. So, in this case, what we have here.

- Assume that the initial system size at time 0 is 0, i.e, $p_0(0) = 1$ and $p_n(0) = 0, n \geq 1$.

  Suppose if I assume that the system starts with some i customer, then this initial will be according to that, or it starts with a probability distribution really with one customer with the probability two customers with this probability. Then, of course, I will have a much more complex situation to make my life easier I will take this one easily.
  Since $\lambda_n = \lambda, \mu_n = n\mu$ here, the forward Kolmogorov equations satisfied by $p_n(t) = P\{N(t) = n\}$ are

$$\begin{aligned} p_0'(t) &= -\lambda p_0(t) + \mu p_1(t) \\ p_n'(t) &= -(\lambda + n\mu)p_n(t) + \lambda p_{n-1}(t) + (n+1)\mu p_{n+1}(t), \qquad n \geq 1 \end{aligned}$$

- Let $P(z,t) = \sum_{n=0}^{\infty} p_n(t)z^n$ be the generating function of the probabilities $\{p_n(t)\}$. For $n \geq 1$, we multiply the n^{th} equation by z^n, sum on n, and then add the equation for $n = 0$ to get

$$\begin{aligned} \sum_{n=0}^{\infty} p_n'(t)z^n &= -\lambda \sum_{n=0}^{\infty} p_n(t)z^n - \mu z \sum_{n=1}^{\infty} np_n(t)z^{n-1} + \lambda z \sum_{n=1}^{\infty} p_{n-1}(t)z^{n-1} + \mu \sum_{n=0}^{\infty}(n+1)p_{n+1}(t)z^n \\ \frac{\partial P(z,t)}{\partial t} &= -\lambda P(z,t) - \mu z \frac{\partial P(z,t)}{\partial z} + \lambda z P(z,t) + \mu \frac{\partial P(z,t)}{\partial z} \\ &= \lambda(z-1)P(z,t) - \mu(z-1)\frac{\partial P(z,t)}{\partial z}. \end{aligned}$$

Now, this is the partial differential equation that is satisfied by $P(z,t)$. Now, you can solve this so that we are not getting into that business. But you can solve this; there is a standard procedure to solve this is a very simple PDE; by the way, it is not very complex.

- With the initial condition that $p_0(0) = 1$ (which is equivalent to $P(z,0) = 1$), we can solve this partial differential equation to obtain

$$P(z,t) = \exp\left[\frac{\lambda}{\mu}(1 - e^{-\mu t})(z-1)\right]$$

  which is the generating function of a Poisson random variable with parameter $(\lambda/\mu)(1 - e^{-\mu t})$. Hence,

$$p_n(t) = e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})} \frac{\left(\frac{\lambda}{\mu}\right)^n [1 - e^{-\mu t}]^n}{n!}, \qquad n = 0, 1, 2, \ldots.$$

  As $t \to \infty$, we have $p_n(t) \to p_n$, where

$$p_n = e^{-\frac{\lambda}{\mu}} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}, \qquad n = 0, 1, 2, \ldots.$$

  Now, you go back to the discouraged arrivals queue, which also has the same probability distribution in equilibrium or in steady-state, but for $M/M/\infty$, $e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})} \frac{\left(\frac{\lambda}{\mu}\right)^n [1-e^{-\mu t}]^n}{n!}$ is the transient solution. Now, for the same model that we have said about the discouraged arrivals queue, which is $\lambda_n = \lambda/(n+1)$ and $\mu_n = mu$. This will also have the same equilibrium distribution. If you look at its transient solution that will be much more complex expressions, you will get that, but that is different from this one; one can compute and for fixed values of n and t, and then you can show that this is different, but both of them tends to the same limit this as $t \to \infty$. So, if you want to look at now like the transient behaviour of this really they are different one can exhibit. So, this is the $M/M/\infty$ model this is the second model that we are giving an overview of the transient behaviour.

- Thus, this model is not too difficult to analyze for its transient behaviour.

Now, let us take up what we thought was the simpler model to start the queueing analysis, which is the $M/M/1$ model. We started with our queueing analysis with this $M/M/1$ model. Now, let us look at the transient behaviour of this $M/M/1$ model. This transient solution of this $M/M/1$ is quite complicated. So, we present only an outline of it, and there are many different approaches that give you different forms of answers to transient solutions for this particular model; of course, all of them would lead to the same equilibrium distribution. But their transient behaviour can be expressed in many different ways depending upon which approach that you are picking; there are plenty of approaches to obtain the transient solution and hence different kinds of expressions, but if you compute, all of them will land at the same value that is the different matter. Obviously, it has to be.

- Assume that the initial system size at time 0 is i for some $i \geq 0$, i.e., $p_i(0) = 1$ and $p_n(0) = 0$ for $n \neq i$. Since $\lambda_n = \lambda$ and $\mu_n = \mu$ here, the differential-difference equations governing the system size are

$$p_0'(t) = -\lambda p_0(t) + \mu p_1(t)$$
$$p_n'(t) = -(\lambda + \mu)p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t), \quad n \geq 1$$

- We outline an approach (due to Bailey) to solve these equations using a combination of PGF's, PDEs and Laplace transforms. Define

$$P(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n, \quad (z \text{ complex}, \quad |z| \leq 1)$$

  The summation is convergent in and on the unit circle with its Laplace transform

$$\bar{P}(z,s) = \int_0^{\infty} e^{-st} P(z,t) dt, \quad (Re(s) > 0)$$

- After obtaining the PDE satisfied by the PGF, then taking Laplace transform of it and solving, we get

$$\bar{P}(z,s) = \frac{z^{i+1} - \mu(1-z)\bar{p}_0(s)}{(\lambda + \mu + s)z - \mu - \lambda z^2}$$

  where $\bar{p}_0(s)$ is the Laplace transform of $p_0(t)$.

  Now, we need to determine $\bar{p}_0(s)$, this is a typical approach that is also applicable later on; you will see immediately when we are looking at the general Markovian model, even in the steady-state analysis of some complex Markovian type model itself, this similar approach would be valid. So, you will have the generating functions or its Laplace transform, which will have one constant. Now, we need to determine $\bar{p}_0(s)$; how do we determine that? To determine this $\bar{p}_0(s)$, we use this fact; what is this fact?

- Since the Laplace transform $\bar{P}(z,s)$ converges in the region $|z| \leq 1, \text{Re}(s) > 0$, wherever the denominator of the RHS has zeros in that region, so must the numerator.
  We use this fact to determine $\bar{p}_0(s)$.

- The denominator has two zeros, since it is quadratic in z and they are (as function of s)

$$z_1 = \frac{\lambda + \mu + s - \sqrt{(\lambda+\mu+s)^2 - 4\lambda\mu}}{2\lambda},$$
$$z_2 = \frac{\lambda + \mu + s + \sqrt{(\lambda+\mu+s)^2 - 4\lambda\mu}}{2\lambda},$$

where the square root is taken so that its real part is positive. Observe that

$$|z_1| < |z_2|, \qquad z_1 + z_2 = (\lambda+\mu+s)/\lambda, \qquad z_1 z_2 = \mu/\lambda.$$

- The following theorem from complex analysis has wide use in queueing analysis. So, we are going to use it when we are talking up next a more complex Markovian models equilibrium analysis again; this theorem would be applicable. So, this theorem is important in that context. So, it is better that you know not just from the transient analysis of this model, but even otherwise, even in equilibrium analysis, we will refer to this theorem from the complex analysis and we will use this. So, you better be aware of this theorem and how we are actually using it. So, what is this theorem says is the following:

Theorem. ***Rouche's Theorem:*** *If $f(z)$ and $g(z)$ are functions analytic inside and on a closed contour C and if $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.*

So, you have a closed contour, and you have two functions, f, and g, which are both analytic inside and on, which means the boundary of this contour. And, what you are observing is $|g(z)| < |f(z)|$ see these are the complex number. So, you have this complex function that you have here. So, $|g(z)| < |f(z)|$ on the boundary; then, inside the contour C, you have f, which is the dominant one, and $f + g$, the dominant one, and plus the dominated one, so the dominant one and the dominated plus the dominant one both have the same number of zeros inside the contour C; that is what you know it says. So, let us see how we are using it here.

- For $|z| = 1$ and Re$(s) > 0$, we observe that

$$|f(z)| = |(s+\lambda+\mu)z| = |(s+\lambda+\mu)| > \lambda + \mu \geq |\mu + \lambda z^2| = |g(z)|$$

Hence, from Rouche's theorem, $f(z) + g(z) = (s+\lambda+\mu)z - \mu - \lambda z^2$ has only one zero in the unit circle. This is z_1 (as $|z_1| < |z_2|$) and this must also be a zero for the numerator. Thus, we have

$$\bar{p}_0(s) = \frac{z_1^{i+1}}{\mu(1-z_1)}.$$

Inserting this into $\bar{P}(z,s)$ and writing it in an infinite series form, we find that (noting $N(0) = i$)

$$\bar{P}(z,s) = \frac{1}{\lambda z_2}\sum_{j=0}^{i} z_1^j z^{i-j} \sum_{k=0}^{\infty}\left(\frac{z}{z_2}\right)^k + \frac{z_1^{i+1}}{\lambda z_2(1-z_1)}\sum_{k=0}^{\infty}\left(\frac{z}{z_2}\right)^k, \quad (|z/z_2| < 1).$$

- Observe that $\bar{p}_n(s)$ is the coefficient of z^n in $\bar{P}(z,s)$. Extracting it and then inverting $\bar{p}_n(s)$, we will get $p_n(t)$, utilizing some properties of the transforms of the Bessel function of the first kind $I_n(y)$. The final result is

$$p_n(t) = e^{-(\lambda+\mu)t}\Big[\rho^{(n-i)/2} I_{n-i}(2t\sqrt{\lambda\mu}) + \rho^{(n-i-1)/2} I_{n+i+1}(2t\sqrt{\lambda\mu}) + (1-\rho)\rho^n \sum_{j=n+i+2}^{\infty} \rho^{-j/2} I_j(2t\sqrt{\lambda\mu})\Big], \qquad n \geq 0,\ \rho = \lambda/\mu.$$

Here, $I_n(y) = \sum_{k=0}^{\infty} \frac{(y/2)^{n+2k}}{k!(n+k)!}, n > -1,$ is (in series form) the modified Bessel function of the first kind.

- Using the properties of Bessel functions, one can now show that, as $t \to \infty$,

$$p_n(t) \to \begin{cases} p_n = (1-\rho)\rho^n, & \rho < 1 \\ 0, & \rho \geq 1, \end{cases}$$

  which agrees with our earlier result known to us.

So, this is one outline; we will not worry about how we are obtaining this, and everything is just to give you an outline of this. So, here what you will take with you forward is this Rouche's theorem and how we have applied this Rouche's theorem in this particular situation which we are going to use even later, even in equilibrium analysis. So, this result is what then you have to remember going forward, but that is the result that we are using to sort of explaining the nature of the transient solution, but the advantage disadvantage both are there; ρ greater than or equal to one there is nothing like equilibrium analysis that you can do, and then you can get something, but here I can understand I can see how quickly it tends to 0 by looking at this $p_n(t)$ for each n and so, on. So that I can get an idea about how things go or how quickly the equilibrium is reached here.

So, what do we have now three models that we have seen with the different complexities and the more complex the solutions will become if the model is going beyond even Poisson exponential assumptions within this framework itself, if the rates are complex, things are becoming very very complicated. And there are many different models for which, even there is no solution transient solution is not yet been found because it is so complex in that case. So, it is extremely difficult to obtain, but then what is the alternative? That you know, remember this is ultimately, after all, is a system of differential equations based upon an initial condition that you are trying to solve. So, any numerical procedures, anything available to do that one can utilize, which can be successfully employed to at least get a numerically solvable one. So, often what happens? The special structures, like in the case of BDP, you have this Q matrix. So, this $Q - matrix$ is a tridiagonal structure. So, these special structures always have some specialized procedures which can be executed in a more efficient way to get the solution in a much lesser time, and there could be approaches that might rely on such structure. Say, for example, a continued fraction approach or even any other combinatorial approach or many things that one can talk about with respect to the numerical procedure itself with respect to even birth-death queues, forget about more complex queues.

But the more complex queues that you have, like, these numerical techniques, will become a bit more complex, but still, it is doable. So, whenever you are not able to do this, you can always appeal to numerical techniques to get the solutions. So, that is about the transient solution of this queue; probably, you would have got an idea about how complex the transient solutions can become even for the simplest of the simple queueing model, which is $M/M/1$. And, if there is a need if you have to do, you have to do it if you are not able to do analytically appeal to numerical techniques, and there are plenty of numerical methods to solve Markov chains. So, that can readily be used here. So, we are not concerned about that part, or one can go for approximate solutions things like that, but that is beyond the level that we are talking about here. Now, once we have done this transient analysis, this can be used to, some kind of analysis which is also required to be done for some kind of models that if you have to look at it, then how one can adapt this analysis approach to do this and that is called busy period analysis. So, this is also relevant for many queueing models you want to do. What is that Busy period?

- A "$busy\ period$" begins when a customer arrives at an idle channel and ends when the channel next becomes idle.

  A busy period is basically the time when a customer arrives at an idle channel when the system is empty and ends when the channel next time for the first time becomes idle again, meaning the system was idle, now a customer arrives. So, the time count starts there; now, this customer is getting served first. So, we are looking at it in the context of $M/M/1$. So, this arriving customer service starts immediately because he is arriving in an empty system. Now, when next time the system becomes idle again, meaning what could have happened is that during the service time of this first customer, there might be customers arriving. Now, if you take into all their services and during their service time, another customer might come and so on.

  So, it goes on, and the server is kept busy serving customers, and at some point of time again, the server will become idle; in a steady-state situation, that will happen. It will become idle then only the steady-state. Because any state is recurrent, it has to be positive recurrent means it will come back in a finite amount of time. So, there is always a time at which it will become empty again, or it will become idle again. So, from this point till that point, that is what is called the busy period, meaning is also very clear that it is a busy period for the server; the server is kept busy during that period.

  ▶ A busy period is a random variable T_{bp}, being the first passage time from state 1 to state 0.

- A *busy cycle* is the sum of busy period and adjacent idle period (denoted by T_{bc}). The idle period could be the previous one or the succeeding one preceding one, or the succeeding one; whichever way it is, both of them are the same.
  ■ Time between two successive departures leaving an empty system, or two successive arrivals to an empty system.

- For $M/M/1$, the idle period $T_{\text{idle}} \sim Exp(\lambda)$ (i.e., the residual interarrival time in a Poisson process).

- If we know the distribution of the busy period, then the distribution of the busy cycle is the distribution of the sum of these two random variables (idle and busy periods).

- The CDF of the busy period can be found by considering the original $M/M/1$ equations with an absorbing state at 0 (by taking $\lambda_0 = 0$ and an an state of 1 (i.e, $p_1(0) = 1$).
  ▶ Then, $p_0(t)$ is the CDF of the busy period and $p_0'(t)$ is its PDF.

- The equations are

$$
\begin{aligned}
p_0'(t) &= \mu p_1(t) \\
p_1'(t) &= -(\lambda + \mu) p_1(t) + \mu p_2(t) \\
p_n'(t) &= -(\lambda + \mu) p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t), \qquad n \geq 2
\end{aligned}
$$

- Carrying out the analysis similar to the transient analysis of $M/M/1$, we get

$$
\bar{P}(z, s) = \frac{z^2 - (\mu - \lambda z)(1 - z)(z_1/s)}{\lambda (z - z_1)(z_2 - z)} \qquad [z_1, z_2 \text{ as earlier}]
$$

- $\bar{p}_0(s)$ is the first coefficient of the power series $\bar{P}(z,s)$ and found as $\bar{P}(0,s)$. Thus

$$\bar{p}_0(s) = \frac{2\mu}{s\left[s+\lambda+\mu+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}\right]}$$

  Using the properties of Bessel functions, the PDF of the busy period can be obtained as

$$p_0'(t) = \frac{\sqrt{\mu/\lambda}e^{-(\lambda+\mu)t}I_1(2\sqrt{\lambda\mu}t)}{t}.$$

- We can obtain $E(T_{\text{bp}}) = -\frac{d}{ds}s\bar{p}_0(s)\Big|_{s=0} = \frac{1}{\mu-\lambda}$.

- An alternative way to find the mean length of the busy period is to use the simple steady-state ratio argument that

$$\frac{1-p_0}{p_0} = \frac{E\left[T_{\text{bp}}\right]}{E\left[T_{\text{idle}}\right]} = \frac{E\left[T_{\text{bp}}\right]}{1/\lambda}$$
$$\implies E\left[T_{\text{bp}}\right] = \frac{1}{\mu-\lambda} \qquad \left\{\text{using } p_0 = 1-\lambda/\mu\right\}$$
$$\implies E\left[T_{\text{bc}}\right] = \frac{1}{\lambda} + \frac{1}{\mu-\lambda}$$

  ► The above result holds true for all $M/G/1$-type queues as long as the arrivals are Poisson process, for any service time distribution; this holds because in this derivation like the service time distribution did not play a role here in this case at least.

  So, this busy period is basically another performance measure with respect to the server; if you want another performance measure, then that is this is the idle period busy period is what you can talk about; we said that there are different angles to the system when you look at the performance measure.

But this busy period analysis if you want in detail about its distribution or the duration of the busy period or you want to know say for example, how many customers are being served during a busy period and so on, then you need a much more detailed analysis which will have similar complexity as the transient analysis case, but with slightly less complexity because you would need only like this kind of one term depending upon this. But this average case analysis one can get very easily using this kind of argument, and anyway, we already said that this result holds true for any $M/G/1$ type queues. So, like this, like then you can have it for multi-server case you can easily extend this idea. So, we can talk about very variations of this busy period, but we will not be concerned more about this, but if you are interested, of course, one can do this kind of analysis, but this will be a little bit complex; that is what you need to keep in mind. So, with this then, we end our discussion on the complete birth-death queueing systems. We have seen, even given an overview of this transient analysis in this lecture, the take-home from here is mainly the Rouche's theorem, and it is how we can apply it here and the complexity of the transient solution if you want to obtain. Now alternative could be that numerical procedures can be applied, and this helps us to get the busy period analysis as well, and this result is true for $M/G/1$ type that is what you know we will take it home. We will see later.
Thank you, bye.