Discrete-Time Markov Chains and Poisson Processes

Professor Subhamay Saha

Department of Mathematics

Indian Institute of Technology, Guwahati

Module: Poisson Processes

Lecture 29

Poisson Thinning 1

Hello everyone, welcome to the 29^{th} lecture of the course discrete-time Markov Chains and Poisson Processes. In the last lecture, we learned several properties of Poisson processes. A Poisson process is a counting process, which starts from 0, it has stationary independent increments, and the number of events that occur in an interval $[0, t]$ is Poisson distributed with parameter λ_t, then you say that it is a Poisson process with parameter or with rate λ. And then we saw that the inter arrival times, which is basically T_n is called the n^{th} inter arrival time, if it is the time between the occurrence of the n^{th} event and $(n-1)^{\text{th}}$ event. So, we saw that this inter arrival times are exponentially distributed with parameter λ. And that gave us a corollary that the time of occurrence of the n^{th} event has Gamma(n, λ) distribution. And we also saw that Poisson process has this Markov property or in other words probabilistically, it restarts itself at each point of time, which means that if you start observing a Poisson process from a point, then you can pretend that the Poisson process actually starts from that point. And you do not need to really know that when it actually started. So, in order to say things about future you can very well pretend that the process started from that point. So, this property is called Markov property. And I also mentioned this that this Poisson processes is one of the simplest example of a much wider class of stochastic processes, which is continuous time Markov chains. Today we will continue with further properties of Poisson processes.

Let $\{N_t\}_{t\geq 0}$ be a Poisson process with rate λ. Suppose, that each time an event occurs, it is classified as either type 1 or type 2 event. So, you are classifying events or events are classified into 2 types, type 1 and type 2. Suppose, further that each event is classified as type 1 event with probability p or a type 2 event with probability $1-p$ independently of all other events. So, once an event happens it is with probability p it is classified as type 1 and with probability $1-p$ it is classified as type 2 and again this each time an event happens you classify it independently of everything else. Now, let $N_{1,t}$ and $N_{2,t}$ denote respectively the number of type 1 and type 2 events occurring in the time interval $[0, t]$. Now, it is easy to see that N_t is basically $N_{1,t} + N_{2,t}$ because the N_t is counting the total number of arrivals, $N_{1,t}$ is counting the arrival of events of type 1 and $N_{2,t}$ is counting the arrival of events of type 2, where each event is classified as type 1 with probability p and as type 2 with probability $1-p$. So, now what is the theorem?. The theorem says that this $N_{1,t}$ and $N_{2,t}$ are again both Poisson processes with respective rates $p \times \lambda$ and $\lambda \times (1-p)$. Furthermore, the 2 processes are independent. You said like 2 processes are independent means like information about one process does not give you any extra information about the other process like for example, if I tell you the number of type 1 events that has arrived in the same time interval $[0, t]$ is, say k, then that does not give you any information about the number of type 2 events. That is roughly what independence means. So, if you classify each arriving event as type 1 and type 2 with probability p and $1-p$ respectively, and then you just count the number of type 1 events and number of type 2 events, then individually, they are again, Poisson processes. But now the rates are $p \times \lambda$ and $\lambda \times (1-p)$. And you can see where this p and $1-p$ is coming from, because you classify an event as type 1 with probability p, and an event as type 2 with probability $1-p$.

So basically, you can think of it is like there is a single arrival stream, and then you break it into 2 parts. That is why this is called this Poisson thinning. So, you are kind of thinning or separating the arrival the total arrivals into 2 groups. Like one real life example that you can think of is like suppose customers are arriving in a departmental store. And then each arriving customer is a male customer with a certain probability p and a female customer with certain probability $1-p$. If the original arrival process was Poisson with rate λ, then now the arrival of male customers will be Poisson with rate $p \times \lambda$, and the arrival of female customers will be Poisson with rate $\lambda \times (1-p)$, and these 2 processes will be independent. That is one real live way of thinking about this Poisson thinning.

Now let us see an example of this concept of Poisson thinning. What does it say, phone calls arrive at a call center according to a Poisson process with rate 10 per hour. Again, I mentioned this in previous lecture as well, the unit is important, like here, the unit is per hour. Now, if each arriving call is a complaint with probability $\frac{3}{5}$, again, here you are doing some kind of classification. A call that is coming to a call center will be a complaint call with probability $\frac{3}{5}$ and something else with the remaining probability, but it will be a complaint call with probability $\frac{3}{5}$. Then the question is what is the probability that no complaints arrive between 10 to 10:30 am?. Calls are arriving according to a Poisson process with rate 10 per hour. Now, each arriving call is a complaint call with probability $\frac{3}{5}$. And this is independent of all other calls. Each arriving call is a complaint call with probability $\frac{3}{5}$ and it can be some other kind of with some other issue or other type of call with the remaining probability, but we are not bothered about that for this particular problem. So, each arriving call is a complaint call with probability $\frac{3}{5}$. Now, you are asked that what is the probability that there will be no complaint calls between 10 to 10:30 am. Now, how do you solve this problem using the concept that you learned in the previous slide, the concept of Poisson thinning. Now, using previous theorem, complaint calls arrive according to a Poisson process. The original rate was 10. So, $\lambda = 10$. And here what is p?. p is $\frac{3}{5}$. So, the rate of arrivals of this complaint calls is $p \times \lambda$, which is $10 \times \frac{3}{5}$, which is equal to 6. So, using the previous theorem, we get that complaint calls arrive according to a Poisson process with rate $10 \times \frac{3}{5}$, which is 6 per hour. This is what we get from the previous theorem, which said that the if we look at this $N_{1,t}$, which is the counting the number of type 1 events, so that is Poisson with rate $p \times \lambda$, so here you can think of time one as a complaint calls. So, then it is a complaint call with probability $\frac{3}{5}$. So, the complaint calls arrive according to a Poisson process with parameter $10 \times \frac{3}{5}$, which is 6. Now, what you are asked?. You are asked what is the probability that no complaints arrive between 10 to 10:30 am. Now, you know that this Poisson process has this property of stationary increments. So, if you want to say something about the number of events in a particular interval, then you do not, you only need to know the length of the interval and not exactly where the interval is situated. So, that is what is stationary increments. We know that Poisson process has this property of stationary increments, which tells you that the number of events in a particular time interval depends only on the length of the interval and not exactly where the interval is situated on the timeline. So, you are basically asked that what is the probability that in an interval now, see, here unit is important it was 10 per hour, and here the duration is 30 minutes. So, this is basically in terms of hour, t is $\frac{1}{2}$, so, here the unit is hours, so, 30 minutes is $\frac{1}{2}$, so you are basically asked that, what is the probability that there will be no arrival in a time interval of length $\frac{1}{2}$. So, what will be the required probability?. Now the new arrival rate is 6. So, $e^{-\frac{6}{2}}$, so this by 2 is coming because now the time unit is $\frac{1}{2}$ because the original time unit was hour, and you are asked about a 30 minute window, so the time unit is $\frac{1}{2}$, so you get $e^{-\frac{6}{2}}$ which is e^{-3}, which is roughly like 0.05. So, it is a very small probability anyway, but the important thing is how we are using this theorem on Poisson thinning to solve this problem. So, each arriving call is a complaint call with probability $\frac{3}{5}$. So, the complaint calls arrive according to a Poisson process with rate $10 \times \frac{3}{5}$ which is 6. You are now asked what is the chance that in 30 minutes duration, there will be no complaint calls, but since the rate is in terms of hour, so, 30 minutes in terms of hour is just $\frac{1}{2}$. So, you are basically asked that what is the chance that there will be no arrival in a time period of length half and that is just $e^{-\frac{6}{2}}$ which is e^{-3}. So, the problem is very simple if you understand this concept of Poisson thinning properly. We will end today's lecture here. Thank you.