Discrete-Time Markov Chains and Poisson Processes

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Lecture 13

Module: Classification of States

Lecture: Passage Time and Excursion

Welcome to the 13$^{\text{th}}$ lecture of the course Discrete-Time Markov Chains and Poisson Processes. Just recall that the in the previous lecture, we have talked about strong Markov property. Strong Markov property basically tells us that if I know that at some stopping time the state is i, then Markov chain actually starts afresh from there, the only thing is that, the initial distribution is δ_i and the transition probability matrix is same as the previous transition probability matrix and not only that, the two part of the Markov chain becomes independent. We have seen this property in the last lecture and then we have seen how this can be used to solve or find out probability of some random variables concerning the Markov chain. Today, we are going to see what is called classification of states that basically means that we will going to define two kinds of states and each states will be either one or the other kind. So, it is kind of an exclusive that in the sense that I mean exhaustive and exclusive in the sense that one state i will be either transient or recurrent these two states we are going to see, one is transient state and other is recurrent state. Let us start with.

The idea behind these transient or recurrent states are as follows that suppose suppose I have a state i and I start from the state i and the question is that whether I will come back to the same state the i^{th} state again in a finite time or not. If I come back in finite time, we call that corresponding state is a recurrent state and if we do not come back to that state in a finite time or rather there is a possibility that we will not come back to the state in a finite time, then that particular state is called transient state. So, if I come back in finite time for sure then it is called a recurrent state if I do not come back for sure, then basically it is called a transient state. The thing is something similar to think about your hometown. You may be because of your college or because of your job, you may be outside of your hometown, but in finite time you actually come back to hometown to see your relatives, your parents, maybe your family, my other family members and all members there. So, it is some kind of that, but that is basically the hometown is something close to our recurrent state. And then suppose that we have a third city maybe we went there for some visit maybe I went to Darjeeling to visit it or maybe I went to Koshani to visit since I like the hill station. So, I go into the spaces, but I may not come back again to that place I may come back I may not go back to that those spaces for sure. So, this kind of places which is not my hometown may work as an example or something close to transient state. This is the intuition, now come to the mathematical definition of this thing. So, to define this thing, we have to start with something which is called the first passage time. What is first passage time, the first pass is time for a state i of a Markov chain X_n is defined by T_i and T_i is a random variable and it is defined by this particular expression, what is this?. This is nothing but T_i is the infimum of the set. I am taking the collection of all possible n such that n is greater than equals to 1 such that X_n is i i.e., $T_i := \inf\{n \geq 1 : X_n = i\}$. So, this T_i is called the first passage system. Let us try to understand what does this mean. See, we have seen something very similar to this, which we call the hitting time and we define the hitting time T^i or in general T^A as infimum of the set n greater than equals to 0 such that X_n belongs to A i.e., $T^A := \inf\{n \geq 0 : X_n \in A\}$ or when A is singleton i, we define T^i which is equals to infimum of $\{n \geq 0\}$ such that $X_n \in A$ and here A is a singleton set so, I can write $X_n = i$. So, the only change between hitting time and that first passage time is that in this k^{th}, I have taken n is greater than equals to 0 and in this k^{th}, I have taken n is greater than equals to 1 and the difference that makes is that if I start from i then $T^i = 0$, but if I start from i that $T_i \neq 0$, the reason being that 0 is not in the set and so, infimum cannot be 0, $n \geq 1$. So, infimum cannot be 0. So, this is the main difference. Now, let us try to understand these in word. So, what happens is that a Markov chain actually starts from 0 and maybe that it just proceed like that, and maybe this is the state i. Now, what happens it starts from somewhere and then maybe this is the time 1, time 2, time 3, so on so forth, maybe this is the time 5. At the time 5, it actually comes to the state i for first time. And so, this set if you see 5 will be there in this set. And then maybe some more than 5 values some n greater than 5 values will be there. But when I am talking about the infimum of the state that turns out to be 5, because 5 is the minimum value in this particular set along with some n is there which are greater than 5 which are basically these points when we visit the state i because $X_n = i$. So, that means that difference between these two basically can be written in this way that T_i is basically the first time the chain visits state i after time 0, 0 excluded. But in case of T^i this is basically the same thing the first time the Markov chain enters state i after 0; here 0 included. So, this is the difference between the hitting time and the first passage time in case of hitting time 0 is included. So, if I start from i in 0 time I will be nice, hitting time will be 0. But in case of the first passage time, it is basically once I see i, then after how many times I will again come back to i that is the important in case of the first passage time. So, the 0 is not included. That is why the changes that both of them is after 0, but in one case that 0 is included in the other case that 0 is excluded. So, this is basically the first passage time and now using the first passage time, we are going to define what is called recurrent and what is called transient?.

The transient state is defined as follows. A state i is called transient, if $P_i(T_i < \infty) = 1$ recall that this T_i just let me let me draw the picture. I start here from i, the state is i and the time is 0, then in the finite time if T_i is basically nothing but from i again I come back to i and in between there is no i. So, that means that I start from i and I come back to i in the at the time T_i and that the state i is recurrent, if I come back in finite time starting from i if I come back for sure in finite time, we say that the state is recurrent. So, $P_i(T_i < \infty) = 1$ that basically mean that starting from i, I come back to i in a finite time for sure. And on the other hand, the transient state means that a state i is called transient if this probability is strictly less than 1 that means that there is a positive probability that means that $P_i(T_i = \infty) > 0$ and that says that there is the positive probability that starting from i, I will not come back to i again. So, there is a chance that I will not come back to i starting for i, if such thing happens for a state i we call that corresponding state as transient state, if for sure we come back we call such kind of states as the recurrent state. Now, let us see this remark, the remark is important in the sense that gives us some intuition about recurrent state such kind of statement, we will see later about the transient state, let us first talk about the recurrent state. It says that the state i is recurrent, if and only if this thing happens, what does this mean?. This basically means that if I start from i, that $X_n = i$ for infinitely many n this probability is 1. Conditioning on the fact that I am starting from i, I will come back to i for infinitely many n that probability of that event to happen is 1. So, for sure, I will come back to i again and again. So, this basically means that these infinitely many n, $X_n = 0$ that the state i is visited again and again. So, this remark basically tells us that if i is a transient state, then starting from i this chain will visit the state i again and again. So, it will be visited for infinitely many times that $X_n = i$ for infinitely many n that is going to happen with probability 1 if the chain started from state i. How we can prove that?. The proof is simple using the strong Markov property. What happens suppose at the time 0 the state is i because I am trying to find out the probability under the condition that $X_0 = i$. Now, after some time it will come back to state i for the first time which is the time T_i according to our definition. Now, let us take T equals to this and this T is basically now, we are actually talk about this is a stopping time, we can show this as a stopping time there is no difficulty in showing then the proof is just the same as showing that hitting time is a stopping time. That is first passage time is a stopping time showing this one is almost similar to that of showing hitting time is a stopping time. So, just check that whether you can able to show that or not. So, this is a stopping time. Now in the strong Markov property, I take this T_i as the time with respect to which I am conditioning because I know at T_i the state is also i. Now condition on two facts, one is $T_i < \infty$ and $X_{T_i} = i$ that is true. That is always true because this is the first passage time. So, that is always true at the time point T_i the state is i. There is no concern about that, but I am conditioning on $T_i < \infty$. So, what happens?. The strong Markov property says that this part actually again starts from a fresh. This part is a fresh Markov chain which is basically starting from i. So, it is a fresh Markov chain, but the it is the initial distribution is δ_i, it is starting from i. So, this part and this part are independent that is the strong Markov property tells us. Now, use this thing. If I use this thing, then the Markov chain is starting from here a fresh and it has no effect of the previous thing. So, if I rescale this time to 0, then after some time again the i^{th} state will be visited. So, that means that I come back to i again. Again starting from here, here the state is i again come back to the state i after some time. So, the state i will come back again and again using the strong Markov property with probability 1. And that actually shows that one part that if i recurrent, then it shows that this condition holds true and the other side is trivial that if this quantity holds true, that means that starting from i the state i is visited again and again that basically means that starting from i at least once I am coming back to i for sure in finite time for sure and that is exactly the definition of the recurrent state. Let us go through this argument once again. Let us remove this part and let us write these things very clearly that so, basically, I start with the fact that suppose i is recurrent.

Now I use the strong Markov property. What the strong Markov property tells?. That well if I start from here basically, I am talking about I am starting from i at the time 0 and there is a time T_i when I again come back to i for the first time, then this one is a stopping time. Now, strong Markov property says that this part Markov chain is a fresh Markov chain and starts a fresh and there is no effect of this side on this part. And now, I use the strong Markov property again and again, i is recurrent. So, I will come back to here for sure that $T_i < \infty$ for sure. Now, starting from i, again I come back in the state i the reason being that is that Markov chain starts from here. Now, if I rescale this time to 0, then it is basically I can just forget about this part, this is basically again a new Markov chain having the same transition probability matrix. That means that starting from here and here the state is i, I again come back to i in finite time. Next, starting from this i again I will come back to i in finite time. So, this will keep on happening. And this is exactly the statement says that starting from i, I will come back to i again and again infinitely many times with probability 1 and it is exactly saying that using the strong using the strong Markov property, I will come back to i again I come back to i again I come back to i in some time and it is just repeating again and again, again and again. Now, what about the reverse that if I am given with now, suppose that this statement is hold true,

let us call this a star, suppose that star is true, if star is true, what does this mean?. This basically means that starting from i, I will the state i for infinitely many times. Now, if starting from i, I again, come back to i for sure if these have a keep on happening. That means starting from i, in a finite time, I will come back to i at least once. Because if I am here, I come back to here that was in finite time I have come back and for finite time I have come back, come back for sure. So, that is why this condition holds true. That is why this implies that $P_i(T_i < \infty) = 1$ and this implies that i is recurrent. And that actually completes this proof. So, this remark is important in the sense that the definition tells us starting from i, I will come back to i in finite time for sure. For recurrent, i is recurrent if starting from i, I come back to i for sure in finite time, with probability 1 that is the recurrent state. And this remark basically tells us that starting from i, the chain will visit the state i, again and again, again and again, again, and again, for sure with probability 1 and infinitely many times if the chain starts with state i. So, starting with state i is important when we are talking about recurrent or transient state. Let us talk about that we will start with i, then whether we will come back to i again and again or whether we will come back to i in finite time for sure or not, if it is sure then we say that it is that it is the recurrent state. If it is not sure, we call that it is a transient state. Keep on my keep in mind that for this transient state, I may come back in infinite time, but I may not be also come back in finite time.

Let us proceed now, let us now talk about what is called the passage time, k^{th} passage time. In the previous slide, we have talked about the first passage time. So, now, we are going to take the talk about k^{th} passage time and then we are going to talk about what is called an excursion and the length of excursion. Let us talk about first the k^{th} passage time, what is k^{th} passage time?. The k^{th} passage time is defined by this expression. This is look quite complicated. Let us try to understand this one step by step. First thing that the Zeroth Passage time is by default defined by 0. So, $T_i^{(0)} = 0$ by definition. Now, for $k \geq 0$, that definition is given by it is $T_i^{(k+1)}$ is defined by $\inf\{n : n > T_i^{(k)}, X_n = i\}$ if $T_i^{(k)} < \infty$ and it is infinity otherwise. Let us see what this means, that when $k = 0$, $T_0 := 0$ there is no problem about that. Now, let us talk about $k = 1$ case. So, $T_i^{(1)}$, that by definition is that it is infimum of the set, let us first talk about the first case, infimum of the set such that $n > T_i^{(0)}$ and $T_i^{(0)} = 0$ in this case and $X_n = i$ i.e., $T_i^{(1)} = \inf\{n : n > 0, X_n = i\}$ because $T_i^{(0)} = 0$ that when I am talking about $T_i^{(1)}$ it will come into this framework. Now basically that means that it is nothing but same as what we have defined as T_i. So, this is the first passage if I take $k = 0$ here, actually I am taking 0 so $k = 0$ here I get $T_1^{(k+1)}$, so 0 plus 1 is same as that n is strictly greater than 0 because $T_i^{(0)} = 0$, $T_i = 0$ and then $X_n = i$ this is the first passage time. So, $T_i^{(0)}$ is by default is that 0, then $T_i^{(1)}$ is the first time when I visit the state i after the time 0, 0 is excluded here. Now look into $k = 1$ so I am now having $T_i^{(2)}$ what is $T_i^{(2)}$?. By definition it is $\inf\{n : n > T_i^{(1)}, X_n = i\}$. If $T_i^{(1)} < \infty$. That means that if this one is less than infinity this one is finite that means that after $T_i^{(1)}$ again when I come back to this one, so, I am taking the all possible values of n after the time $T_i^{(1)}$. So, it starts from this point onwards and then I am taking only those points where $X_n = i$. So, I am taking this point, in this point and then if another I happen see at that point I will take all these points and then I take the infimum that means, this gives me this point. So, that means, $T_i^{(2)}$ is basically nothing but after $T_i^{(1)}$ at the time when again we come back to the state i after the first visit to the state i when we come back again in the state i that is basically $T_i^{(2)}$. So, $T_i^{(2)}$ is nothing but the second time I visit the state i after time 0, 0 excluded, second time I visit the state i that time is denoted by $T_i^{(2)}$. Now, why this infinity otherwise infinity part is given, that is given because if $T_i^{(1)} = \infty$ that means $T_i^{(2)}$ will not be visited. If $T_i^{(1)} = \infty$ that means, I will not visit the state i again so, $T_i^{(2)}$ has to be infinity or a second time visiting is also infinity that time visiting is also infinity and so on so forth. So, that is the why that otherwise if it is not true that means if $T_i^{(k)} = \infty$, then the next visit time has to be infinite there is no question about that. That means $T_i^{(2)}$ is the second time I visit. Similarly, $T_i^{(3)}$ is the third time the state i is visited, $T_i^{(4)}$ is the fourth time the state i is visited and so on so forth. So, the k^{th} passage time is nothing but the time of the k^{th} visit of the state i. Now, let us now proceed and see that remark, that remark tells that if i is recurrent, then for all $k \geq 0$, starting from i, I visit the state i for the k^{th} time in finite time for sure, I will visit the state i for the k^{th} time in the finite time is for sure that is basically the statement. And the proof of the statement again using the strong Markov property the similar argument I mean the same argument we have given for this one that if i is recurrent, I will visit the state again and again. So, if I visit the state again and again that means that the time to visit the i^{th} state for the k^{th} passage time is finite, I visit the state again and again, again and again if I use recurrent and that implies that the k^{th} passage time that is which is basically nothing but the time at which we visit the state i for the k^{th} time that has to be finite because I keep on visiting the state. So, that has to be finite with probability 1 if I start from i.

Let us proceed and now let us see what is called the length of the k^{th} excursion. The length of the k^{th} excursion actually defined by this it is nothing but the time difference between two consecutive visits to the state i. In the figure it can be given in this way that I have visited the state i for the first time here. So, $S_i^{(1)}$, which is basically nothing but $T_i^{(1)} - T_i^{(0)}$, and we know that $T_i^{(0)} = 0$. So, that it is basically $S_i^{(1)}$ is basically $T_i^{(1)}$, which is basically the time difference here go from 0 to the first time. Then $T_i^{(2)}$, which is basically nothing but $T_i^{(2)} - T_i^{(1)}$ the $S_i^{(2)}$ is nothing but $T_i^{(2)} - T_i^{(1)}$ so, which is nothing but the time difference here. Similarly, this one is $T_i^{(3)} - T_i^{(2)}$ and this is the time difference here. Again, this is only defined when I have the this S_i only defined if this quantity is finite, and this one is defined if this quantity is finite in this particular manner, otherwise we see it is 0. So, if this time is finite, then this time this difference we take as 0 and if this one is finite, but this one is infinite, then the length is infinite. So, the length of the k^{th} excursion is nothing but the time difference between two constitutive visit only little bit difference in the definition of $S_i^{(1)}$ which is basically $T_i^{(1)}$, the rest of the thing is the time difference between two consecutive visits. So, we define the k^{th} passage time, we define the length of the k^{th} excursion, the question is that what is the excursion?. Excursion is basically nothing but the path of the Markov chain between two consecutive visit of state i and here basically length of the k^{th} excursion to state i that is basically nothing but the time length between two consecutive visit to the state i and of course, the $S_i^{(1)}$ is different a little bit different way it is $T_i^{(1)}$ for rest of them is basically the time difference between two consecutive visits.

Now, let us talk about the distribution of these. Note that this ${S_i^{(k)}}^{'s}$, are random variable because ${T_i^{(k)}}^{'s}$ are random variable. So, ${S_i^{(k)}}^{'s}$ are also random variables. Now, let us talk about the definition of this thing. Distribution goes like that for $k = 2, 3, \cdots$, conditional on $T_i^{(k-1)} < \infty$, $S_i^{(k)}$ is independent of this thing, what is this thing?. I will explain and the probability that $S_i^{(k)} = n$ given this one this quantity is finite is same as $T_i = n$ starting from i. So, let us first try to understand this theorem using the graph. What is says suppose, let us take $k = 2$ here. If I take $k = 2$ these theorems tells that conditioning on $T_1^{(1)} < \infty$ recall that $T_1^{(1)}$ is same as T_1 or rather I just write the general one i. So, $T_i^{(1)}$ is same as T_i notice conditioning on $T_i^{(1)} < \infty$. So, this point this time is less than infinity $S_i^{(k)}$ is independent of this what is this?. This is I am taking the collection all $\{X_m : m \leq T_i^{(k-1)}\}$. So, when I take a $k = 2$, I am basically talking about X_m such that $m \leq T_i^{(1)}$. So, it says that $S_i^{(2)}$ this one is independent of all Markov chain in this part and and that $S_i^{(2)}$ are independent. $S_i^{(k)}$ is independent of the previous part. Similarly, when I talk about $S_i^{(3)}$, these actually consider the Markov chain in this part when I talk about $S_i^{(3)}$. So, $S_i^{(3)}$ is independent of the random variables in this part. So, the random variables in this part is independent of $S_i^{(3)}$ that is the independent part says. Now, come to the next part, next part tells that again if I talk about $S_i^{(2)}$, then the distribution of probability that $S_i^{(2)} = n$ given $T_i^{(1)} < \infty$ is same as probability starting from i, $T_i^{(1)}$ or $T_i = n$, what does this mean?. This means that well, if I know this one is finite, then the distribution of these one is same as if I am because here the state is i as even now, I am starting from i and the for the first time I am visiting the state i. Note that from here to here, if I just forget about this part, the Markov chain actually start here from refresh using the strong Markov property. Now from here I visit the state i here again. So, from here, the first time visit is here, starting from here the first time visit to the state i is here. And this basically tells that that the distribution of $S_i^{(2)}$ is same as if I am starting from i and then I visit the state i again for the first time, so that these theorems actually quite helpful theorem to find out the distribution of $S_i^{(k)}$, it basically says that $S_i^{(k)}$ is independent of the previous part of the Markov chain. And not only that, the distribution of $S_i^{(k)}$ conditioning on the previous time is finite is same as, as if I am starting from i and I am visiting the state i for the first time for the first time the distribution of that time is same as the distribution of $S_i^{(k)}$.

The proof actually, the intuition I have given using this graph, now, let us write down those things clearly. As I mentioned, the proof can be done with the help of the strong Markov property where the stopping time can be used as the $(k-1)^{\text{th}}$ passage time, when I talk about these passage time, as the stopping time in the strong Markov property. Now, of course, in this case, T_i, X_T is i, because this is the passage time and at this time, the state is i, provided that the $T < \infty$. If $T = \infty$, then X_T does not make any sense provided $T < \infty$ that this quantity makes sense and then at that X_T at the time T, which is same as this the state is i. Now, conditioning on this fact that $T < \infty$ that means this time is less than infinity, which is exactly this condition, this is basically $T < \infty$, because we take that T is this one. So, conditioning on these, this part is a Markov chain with the same transition probability matrix and initial distribution δ_i that the same thing that if I start from 0, I come at $T_i^{(k-1)}$ at this time the state is i. So, Markov chain starts a fresh here using the strong Markov property conditioning on that this quantity is finite. That means that this is a Markov chain in this part of the Markov chain, which is again having the same transition probability matrix, but the initial distribution is δ_i, it is starting from i and this part is independent of this part, these two parts are independent. This is the strong Markov property tells us. Now, they look into the definition of this. The definition of this was that X_k was basically the time difference if this time is $T_i^{(k-1)}$ if this time is $T_i^{(k)}$, then this is the time difference, is basically $S_i^{(k)}$. Now, if I look into the next part, if I start from here on this side, then this time difference can be written as after this point when I again visit state i, so, this is nothing but $n \geq 1$ such that $X_{T+n} = i$, take the collection of all such chains and I take the infimum. After this again I see state so, this is basically this one. Notice that this T is same as this quantity, so, this is basically my T here, so, $T + n$ is basically means that after this time when again I see these state i. That means this is after this what is the first passage time after this what is the first passage time?. So, $S_i^{(k)}$ the first passage time of this Markov chain. If $S_i^{(k)}$ is the first passage time of this Markov chain and the strong Markov property says that this is a Markov chain having the same transition probability matrix starting from i and this Markov chain now is independent of X_0 to X_T which is basically this part of the original Markov chain. So, this shows that this first quantity holds true because $S_i^{(k)}$ is the function of this part of the Markov chain only. And the previous part this part is basically this quantity. So, clearly $S_i^{(k)}$ has to be independent of the previous part. And then, the next part is basically given by the fact that once I come here then everything start afresh. So, from here it is basically again when I first time visit this one such from state i this is basically nothing but when I again visit the state i. So, this is basically the proof. The takeaway from today's lecture is that we have defined what is called that passage time and using the first passage time we have defined what is called the recurrent state and transient state. Recurrent state basically means that starting from the state i, I will come back to the state i in finite time for sure and transient state means that starting from i there is a positive probability that I will not come back to i in a finite time. Now, the intuition of the recurrent state is that starting from i, I will come back to the state i again and again, again and again for sure. And then we talk about the distribution of the k^{th} passage time and we pointed out the fact that the k^{th} passage time is independent of the previous part of the Markov chain and that conditioning on the fact that that $T_i^{(k-1)} < \infty$, the distribution of the k^{th} passage time is same as distribution of the first passage time starting from i. With that, I stopped in the lecture. Thank you for listening.