Mathematical Portfolio Theory

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Module 03: Mean-Variance Portfolio Theory
Lecture 03: Multi-asset portfolio and Efficient frontier

Hello viewers! Welcome to the next lecture on the MOOC course on Mathematical Portfolio Theory. I would recall that so far we have talked certain basic aspects of modern portfolio theory; particularly, the expected return and risk which forms the two pillars of this theory. And then, we talked about in the last class about the expected return and risk of a two and a three asset portfolio and we looked at what is going to be the minimum variance portfolio that is we determined what are going to be the weights which will lead us to obtain the portfolio of minimum variance. And also, we talked about something which is called an opportunity set or feasible set which essentially is the collection of all possible combinations of the weights resulting in different portfolios. So, in todays class we start off with a discussion on the opportunity set in the context of a portfolio comprising of n risky securities.

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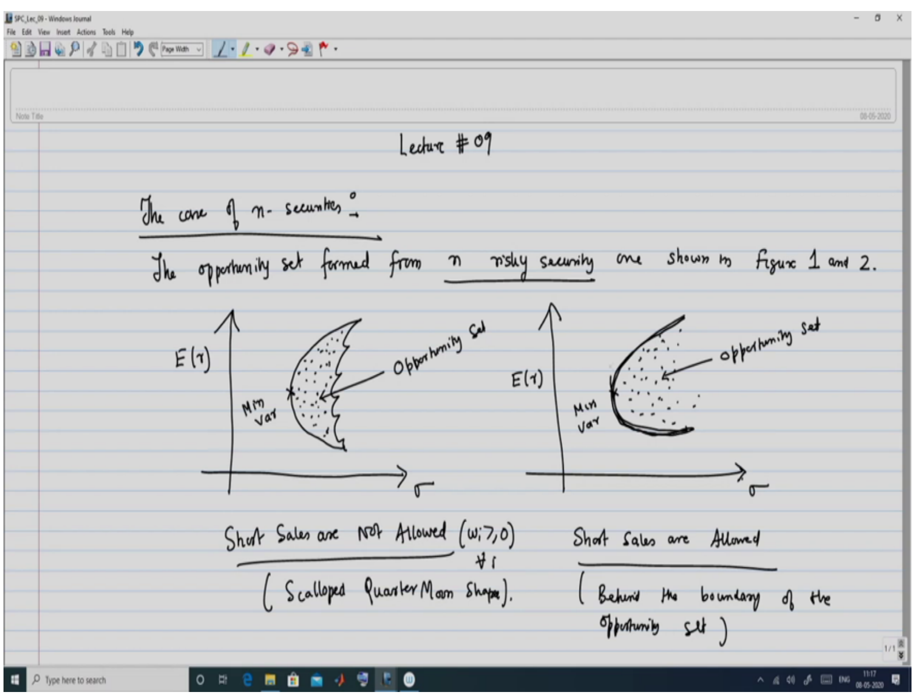

So, we start off with n securities as an extension to the case with two and three securities. So, we begin with recalling the opportunity set and we give a visualization of this formed from n risky securities and these are shown in figure 1 and 2. So, let me graphically give what the opportunity set will look like. So, if we remember correctly the opportunity set will be given not in terms of weight, but rather in terms of the expected return and risk of the portfolio obtained by that risk combination. So, we look at two scenarios.

So, one scenario, I will have a graph like this and in the second scenario, I will have a graph like this, with all these dotted points representing different portfolios and the entire thing together is what constitutes the opportunity set. So, here we will have the minimum variance portfolio and here, we have the minimum variance portfolio and this entire set this is the opportunity set and the opportunity set here and both of them are for n risky securities. So, why do I have two graphs? So, the first graph is basically when short sales are not allowed that means, all your weights have to be greater than or equal to 0. That means, $W_i \geq 0$ for all i and the second case it gives the opportunity set when short sales are allowed. So, in the first case, so basically the opportunity set in the first case this looks like scalloped quarter moon shaped and the second case, the opportunity set is basically behind the boundary of the opportunity set. So that means, that all the feasible portfolio in case no short sales are allowed will lie within this structure which is a scalloped quarter moon shape and in case a short sales are allowed that means, your W_i can be any real number, it is essentially going to be the collection of all portfolios which lies beyond this particular boundary, alright.

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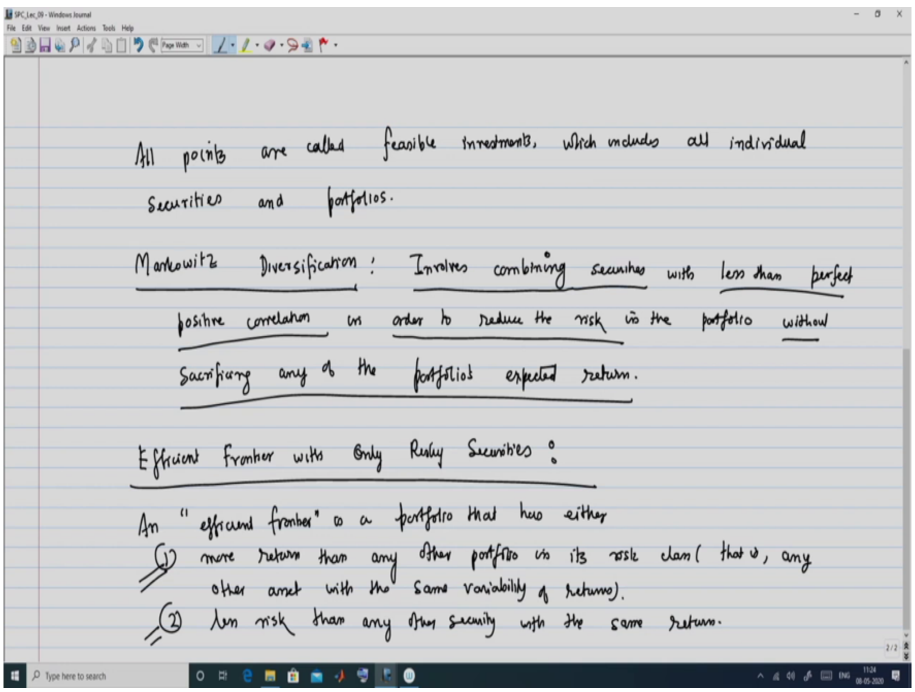

So, once you have done with this graphical representation, so we can make the observation that all these points that I have plotted which are actually infinitely number; all points are called feasible investments which includes all individual securities. That means, you could have a portfolio of just one asset that is also a legitimate portfolio in the opportunity set and all possible combination of portfolios in general ok. So, this brings us a to Markowitz diversification which forms the backbone of modern portfolio theory. So, Markowitz diversification involves combining. So, the way this Markowitz diversification is carried out is that it involves combining securities with less than perfect positive correlation in order to reduce the risk in the portfolio without sacrificing any of the portfolios expected return. So, just to sort of give a greater clarity to this particular statement, what do you look at is that we said that the Markowitz diversification, what is the essence of this it means that you are combining securities; that means, you are creating portfolios out of assets with less than perfect positive correlation. So, you are going to essentially in order to attain the benefits of diversification in the sense of Markowitz diversification. What you need to do is that you need to create a portfolio picking up assets whose returns have a correlation that was less that are less than perfect. That means, correlation coefficient of returns being less than strictly less than 1 and this is done in order to achieve a reduction in the risk of the overall portfolio, while retaining the characteristic of the expected return of the portfolio alright. So, this brings us to the concept of what is known as the efficient frontier. So, efficient frontier is a part of the opportunity set with a special characteristic that make at the it very attractive to just pick up a portfolio that lies on the efficient frontier. So, accordingly, we start off with efficient frontier with only risky securities. So, the reason we specify risky securities is because eventually we will talk about an efficient frontier with risky securities along with risk free security. So however, for the time being we will just focus on the portfolio of n number of risky securities. So, accordingly we define that an "efficient frontier" is a portfolio that has either of the following and I will enumerate them more return than any other portfolio in its risk class. And what do you mean by risk class? This means that any other asset with the same variability of returns and secondly, it has less risk than any other security with the same return. So, the first thing it means that it says that if we have for a given level of risk that is either as sigma or sigma square, the portfolio on the official frontier would be the one which has the highest level of return amongst all those portfolios which have an identical amount of risk. And or equivalently, it could also be that you fix a level of return and you will have numerous portfolios at the given level of return and then, you pick up the one which has the least amount of risk. So, this means that maximum return given a level of risk or minimum risk given a level of return or simply the minimum variance portfolio.

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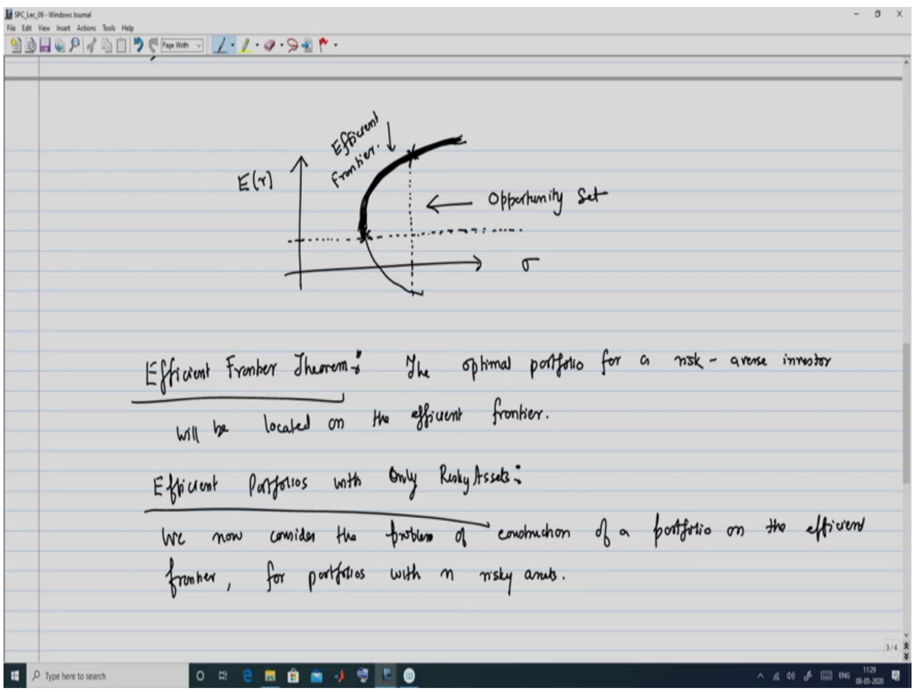

So, let us revisit this graphically. So, graphically again, we look at the sigma E r diagram and then, we look at the opportunity set. So, remember that this entire thing is the opportunity set and I am looking at the case, where short sales are allowed. Then, the efficient frontier is going to be this part of the opportunity set. So, this is what is going to be the efficient frontier. So, the reason this is the efficient frontier is the following that if we as if we have a certain level of risk, say I fix a sigma here then at this level of risk there are so many different infinitely many possible combinations of portfolios at that level of risk. And among them, I choose the one with the highest amount of return which lies on this line or equivalently, if I choose a given level of return then what I get is that at that given level of return, there are so many portfolios with different level of risk. And amongst them, I will essentially choose the portfolio that has the minimum risk. So, it is the collection of such points which give the highest return for a given level of risk or which give the minimum level of risk for a given level of return, the collection of all these points is what is known as the efficient frontier and note that, this will also include the minimum various portfolio which is at the tip of the border. So, it is basically the part of the border of the opportunity set which satisfies these two conditions. So, now, we state the efficient frontier theorem. So, the optimal portfolio for a risk averse investor will be located on the efficient frontier. So, essentially for a risk averse investor, who obviously are driven by the two considerations that are enumerated in the definition of efficient frontier. For them, the optimal portfolio is going to be a portfolio that lies somewhere on the efficient frontier which will be driven by. And, the particular optimal portfolio will be driven by the level of the risk appetite that means, the level of risk that they are willing to take ok. Now, let us look at what are going to be the efficient portfolios. Now, that we have got the efficient frontier. So, this is basically a collection of an extremely large number of portfolios; that means, combinations of words W 1 to W n resulting in those particular values of expected return and risk. And so, the next thing to do is that to figure out what is going to be those weights such that, that those weights constituting the portfolio will end up giving you the expected return and risk lying on the efficient frontier and these are what we are going to call is the efficient portfolios. So, efficient portfolios with only risky assets; so, remember we are still considering in the paradigm of our risky assets. We now consider the problem of construction of a portfolio on the efficient frontier for portfolios with n risky assets.

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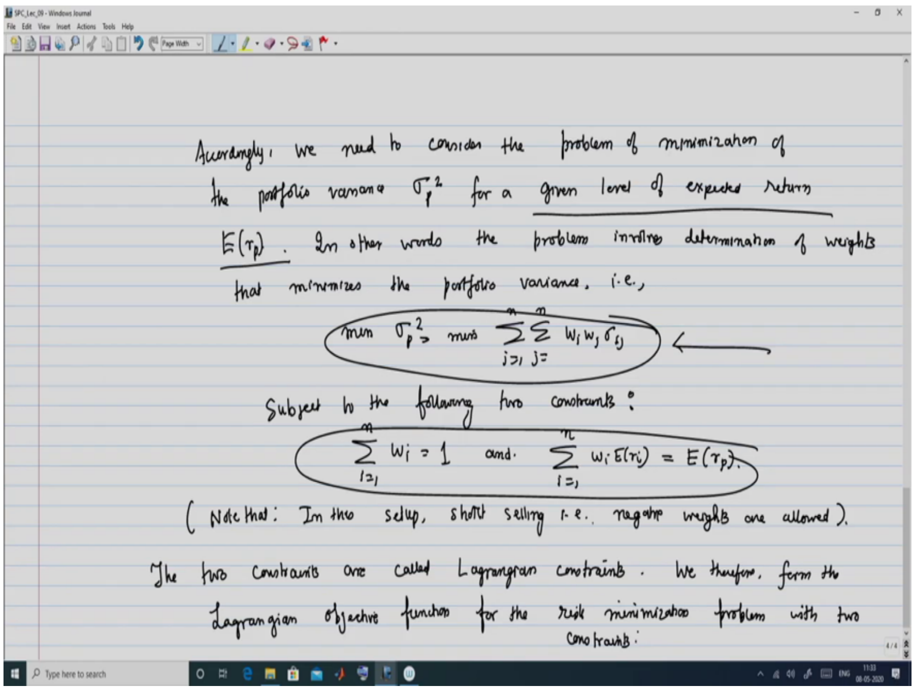

Accordingly, we need to consider the problem of minimization of the portfolio variance, σ_P^2 for a given level of expected return $E(r_P)$. So, this basically means that it is the minimization of the portfolio variance given a level of expected return. So, it is this problem that when I fix the expected return and then, amongst all the portfolios at that expected return, I will try to obtain that this portfolio which gives me the minimum risk for that given level of returns. So, that I get the weights of the portfolios on the efficient frontier. So, for a given level of expected return $E(r_P)$, so, in other words, so we can now formulate the problem in more concise term. So, in other words, the problem involves determination of weights that minimizes the portfolio variance that is minimum of σ_P^2 and this is nothing but minimum of you recall that it is $\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_P ij$. And this of course, has to be subject to the following two constraints; namely. So, the obvious constraint is the sum of the weights equal to 1 which you have seen previously. And, we now have a new constraint that the given level of expected return is fixed to be $E(r_P)$. So, accordingly this will be $\sum_{i=1}^{n} W_i E(r_i)$ that is the expected return of the portfolio. This is going to be equal to $E(r_P)$ that is pre specified, ok. So, note that just to make an observation that in this setup, the way this problem has been formulated in terms of the minimization of this function and this constraint. So, in this setup, short selling that is negative weights are allowed. So now, let us look at these two constraints. So, there is the two constraints that we have mentioned above are referred to as what are known as Lagrangian constraints and we therefore, form the Lagrangian objective function for the risk minimization problem with two constraints as identified above, ok.

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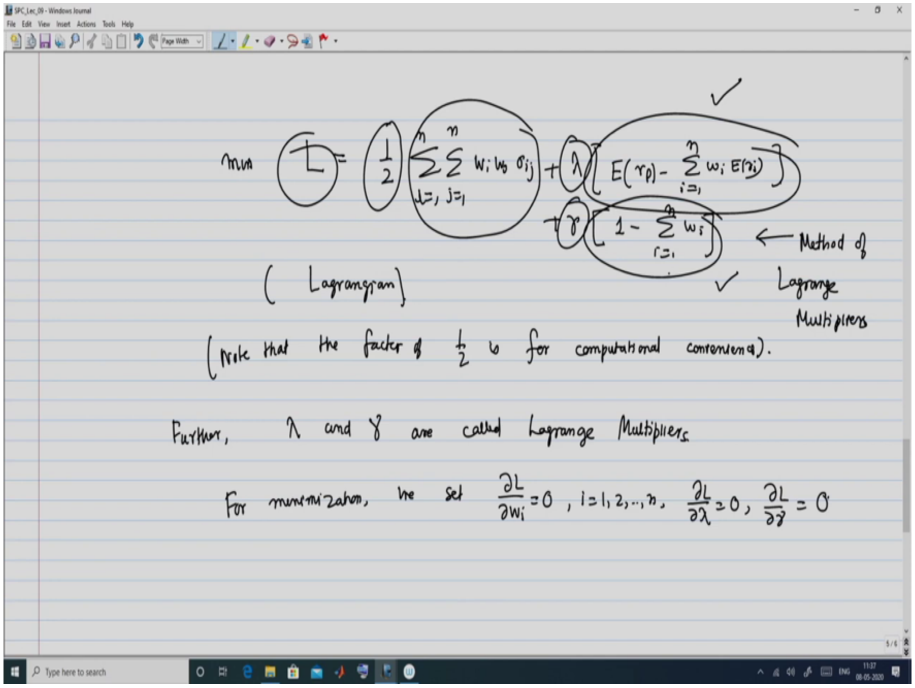

So, then the problem becomes

$$Min(L) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_P ij + \lambda\left[E(r_P) - \sum_{i=1}^{n} w_i E(r_i)\right] + \gamma\left[1 - \sum_{i=1}^{n} w_i\right].$$

So, this is what is known as the Lagrangian. Now, observed that this Lagrangian, what it does is it accounts for the two constraints as well as the function to be minimized; the only thing you will notice that we have added a factor half here. So, that is for convenience notational convenience and also recognizing the fact that minimization of this expression is the same as minimization of half of this expression. So, note that the factor of half is for computational convenience. Further, lambda and gamma, remember that we have introduced two constants here. So, for this discussion I refer you to consult some calculus book on method of Lagrange multipliers. So, here, λ and γ are called Lagrange multipliers ok. Now, we need the minimization of this L. So, for minimization, we set that

$$\frac{\partial L}{\partial w_i} = 0,\ i = 1, \cdots, n,\ \frac{\partial L}{\partial \lambda} = 0,\ \frac{\partial L}{\partial \gamma} = 0.$$

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So, this gives us the following set of equations. So,

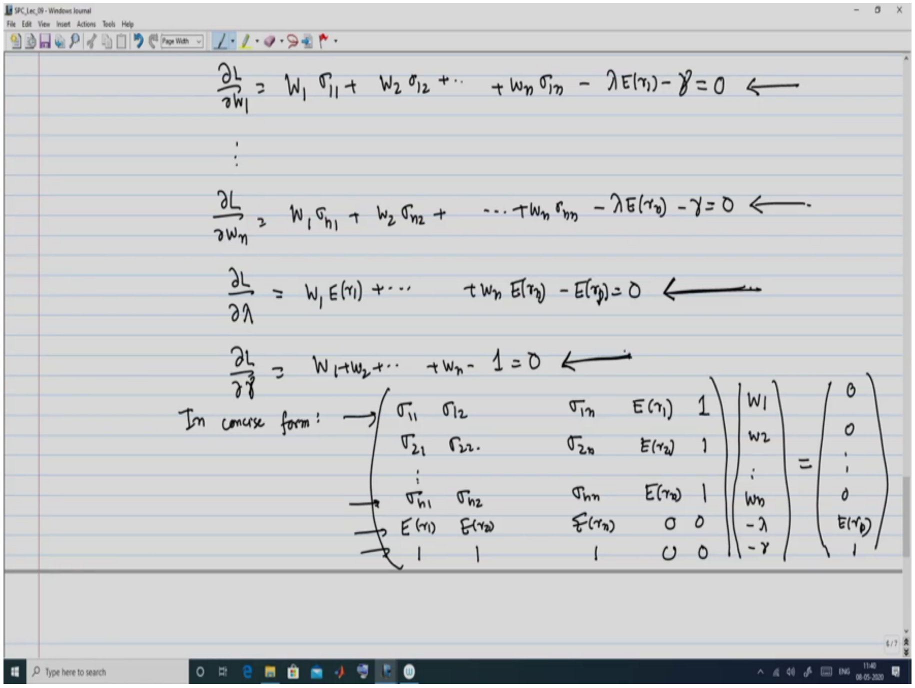
$$\frac{\partial L}{\partial w_1} = w_1\sigma_{11} + w_2\sigma_{12} + \cdots + w_n\sigma_{1n} - \lambda E(r_1) - \gamma = 0$$

Likewise,

$$\frac{\partial L}{\partial w_n} = w_1\sigma_{n1} + w_2\sigma_{n2} + \cdots + w_n\sigma_{nn} - \lambda E(r_n) - \gamma = 0$$

$$\frac{\partial L}{\partial \lambda} = w_1 E(r_1) + \cdots + w_n E(r_n) - E(r_p) = 0$$

$$\frac{\partial L}{\partial \gamma} = w_1 + w_2 + \cdots + w_n - 1 = 0$$



and these basically recovers the constraint that we had here. So, it recovers this constraint and you will see that the second derivative condition in terms of gamma will recover this condition, ok. So, this is satisfying the constraint that some of the weights is equal to 1, ok. So, in concise form, we have the following.

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} & E(r_1) & 1 \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} & E(r_2) & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} & E(r_n) & 1 \\ E(r_1) & E(r_2) & \cdots & E(r_n) & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ -\lambda \\ -\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E(r_P) \\ 1 \end{pmatrix}.$$

So, for example, if you take the matrix multiplication of the first row, then you obtain the first equation here and then, for and then this row here will give you this expression here and these last two rows here is if you multiply them, you will basically recover this constraint and this constraint. So, what you can do now is we can write this. So, this can be written as.

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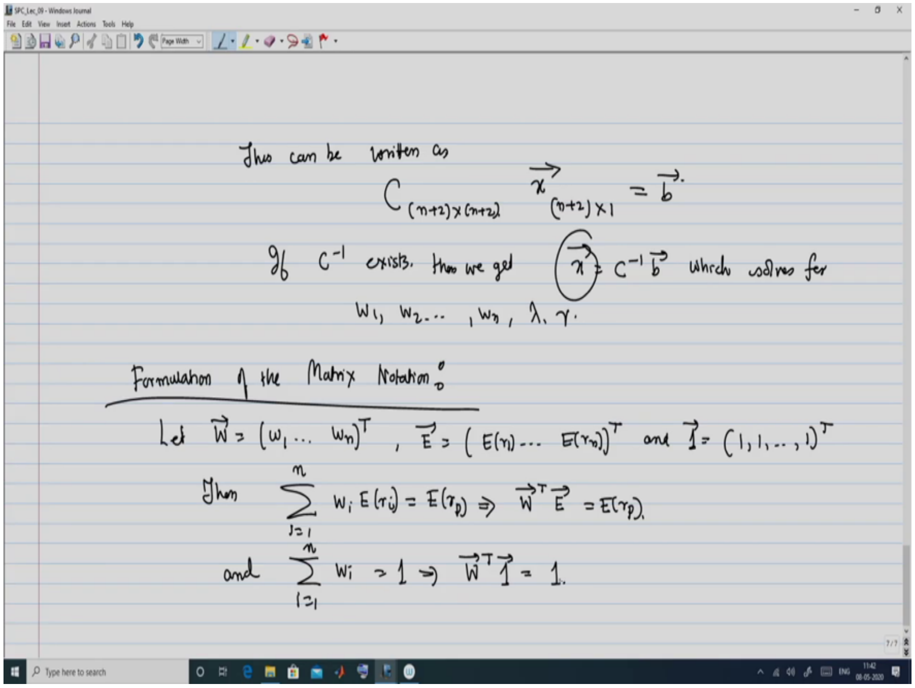
So, this is a $(n+2) \times (n+2)$ matrix. So, this is can be written as C which is a $(n+2) \times (n+2)$ matrix. Because this is these are n number of rows plus these two rows and I same thing with columns and this vector here, this vector is going to be let me call this as x as the unknown vector and which is of the size $n+2$ rows and 1 column and this is going to be equal to the vector b. So, if C^{-1} exists, then we get $\vec{x} = C^{-1}\vec{b}$ and what is vector $\vec{x}$? This is my vector $\vec{x}$. So, this solves for $w_1, \cdots, w_n$ and λ and γ, ok. Now, let us look at the same problem in a slightly different paradigm. So, we start off with the formulation of the matrix notation. So, accordingly, we introduce certain vector. So, let W be the vector of all the weights it is a column vector. So, this is $(w_1, \cdots, w_n)^T$. My $\vec{E} = (E(r_1), \cdots, E(r_n))^T$ and $\vec{1} = (1, 1, \cdots, 1)^T$. So, it is a vector of all 1's of with transpose taken. Then, how will the two constraints look like? So, let us look at the first constraint. Then,

$$\sum_{i=1}^{n} w_i E(r_i) = E(r_P) \implies W^T \vec{E} = E(r_P).$$

Likewise, the other constraint that

$$\sum_{i=1}^{n} w_i = 1 \implies \vec{W}^T \vec{1} = 1.$$



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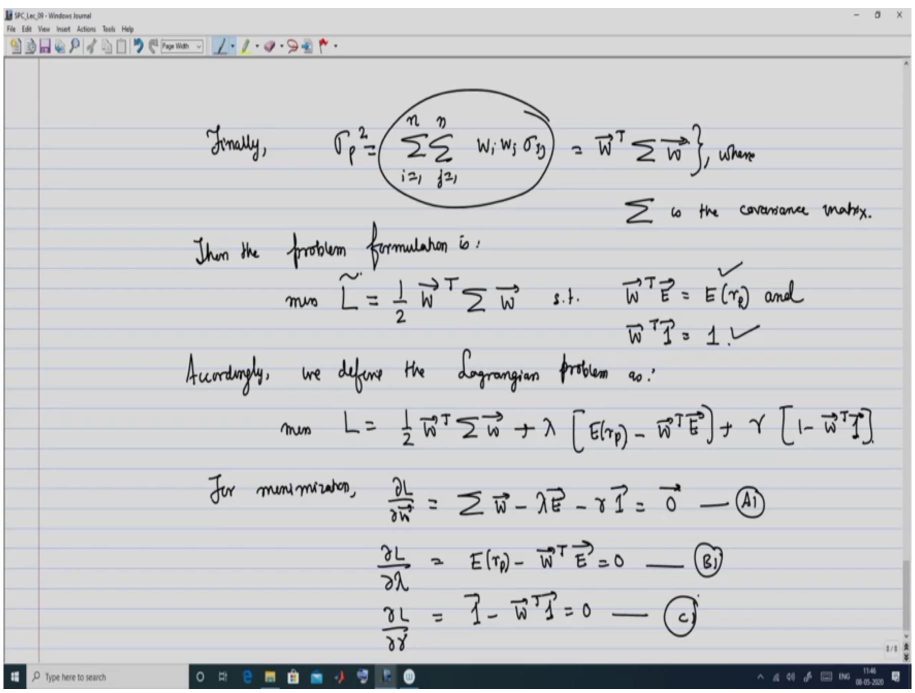

So, finally, we also introduce

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = \vec{W}^T \sum \vec{W},$$

where Σ is the covariance matrix. So, once I have this set up that means, the vector formulation of the constraint and the objective function here; then, the problem formulation is that

$$Min\ \tilde{L} = \frac{1}{2}\vec{W}^T \sum \vec{W}$$

So, this such that these two constraints are satisfied. So, this will mean $\vec{W}^T\vec{E} = E(r_P)$ and $\vec{W}^T\vec{1} = 1$. So, accordingly, we define the Lagrangian problem as

$$Min\ \ L = \frac{1}{2}\vec{W}^T\sum\vec{W} + \lambda[E(r_P) - \vec{W}^T\vec{E}] + \gamma[1 - \vec{W}^T\vec{1}]$$

So, for minimization, as before we will take the derivative with respect to the $\vec{W}$, $\vec{\lambda}$ and $\vec{\gamma}$. These are

$$\frac{\partial L}{\partial \vec{W}} = \sum\vec{W} - \lambda\vec{E} - \gamma\vec{1} = 0$$

Let me call this equation $A1$.

$$\frac{\partial L}{\partial \vec{\lambda}} = E(r_P) - \vec{W}^T\vec{E} = 0$$

Let me call this equation $B1$ and

$$\frac{\partial L}{\partial \vec{\gamma}} = \vec{1} - \vec{W}\vec{1} = 0$$

Let me call this $C1$, alright.

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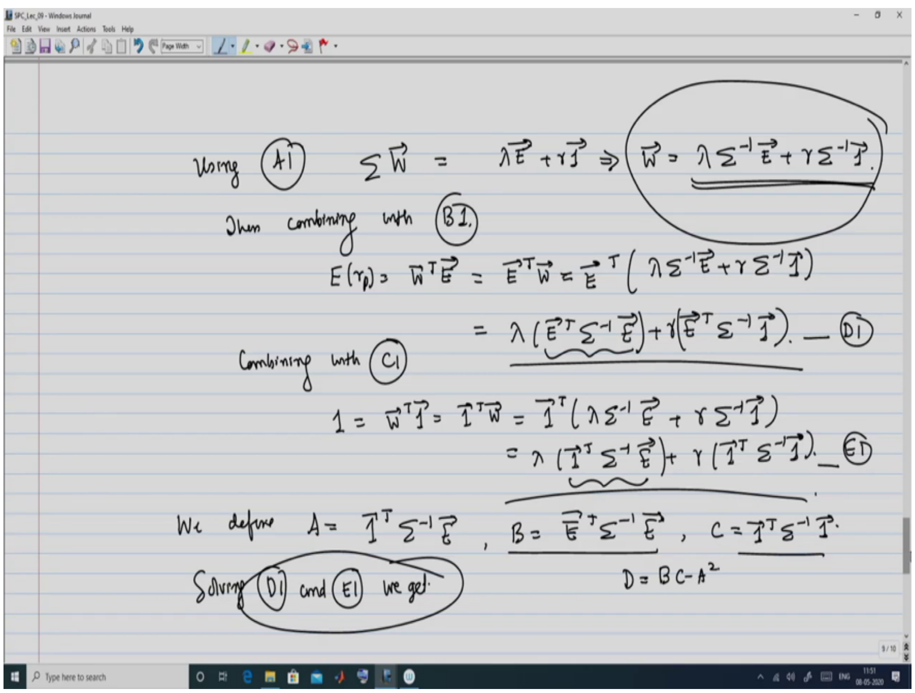

So, using $A1$, what do we get from $A1$? So, here I can solve this for W. So, first of all, I will take

$$\Sigma\vec{W} = \lambda\vec{E} + \gamma\vec{1} \implies \vec{W} = \lambda\Sigma^{-1}\vec{E} + \gamma\Sigma^{-1}\vec{1}.$$

Then, combining with $B1$; so, what I am going to do is that I am going to replace this W in $B1$ now. So, I get

$$E(r_P) = \vec{W}^T\vec{E} = \vec{E}^T\vec{W} = \vec{E}^T(\lambda\Sigma^{-1}\vec{E} + \gamma\Sigma^{-1}\vec{1}) = \lambda(\vec{E}^T\Sigma^{-1}\vec{E}) + \gamma(\vec{E}^T\Sigma^{-1}\vec{1}).$$

Again, combining $A1$ with $C1$, what do we get? So, we carry out the same exercise with $C1$ now. So, this gives me

$$1 = \vec{W}^T\vec{1} = \vec{1}^T\vec{W} = \vec{1}^T(\lambda\Sigma^{-1}\vec{E} + \gamma\Sigma^{-1}\vec{1}) = \lambda(\vec{1}^T\Sigma^{-1}\vec{E}) + \gamma(\vec{1}^T\Sigma^{-1}\vec{1}).$$

So, we have these two expressions that if you hear. Now, what you need to do is we need to solve for this for λ and γ. So, we define

$$A = \vec{1}^T\Sigma^{-1}\vec{E},\ \ B = \vec{E}^T\Sigma^{-1}\vec{E},\ \ C = \vec{1}^T\Sigma^{-1}\vec{1},\ \ D = BC - A^2.$$

Then, I can solve this equation. So, let me call this $D1$ and $E1$. So, a solving $D1$ and $E1$, we get; so, we are solving for a λ and γ.

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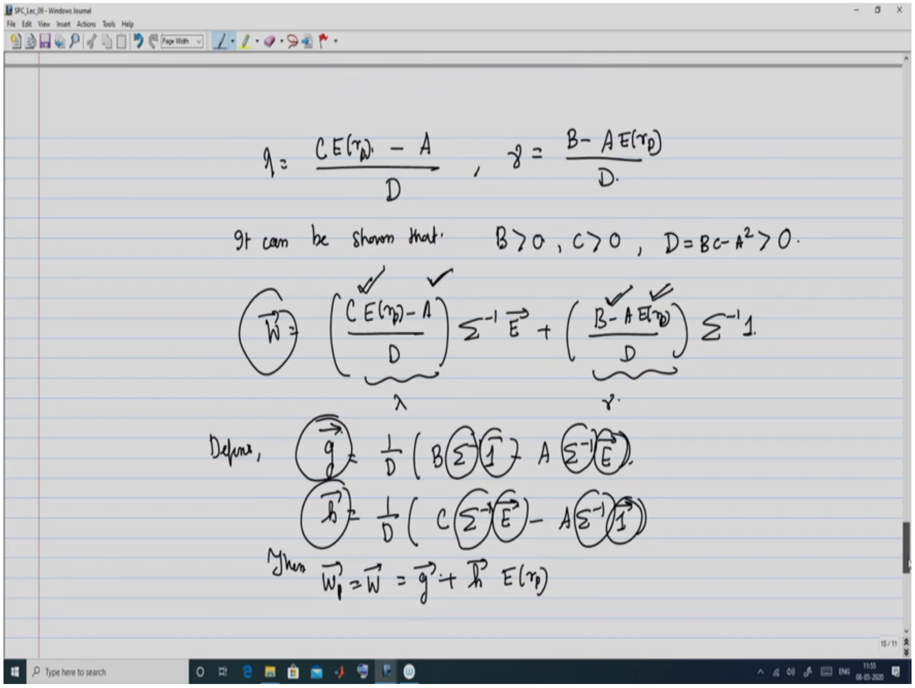

So, we will get by solving $D1$ and $E1$

$$\lambda = \frac{CE(r_P) - A}{D}, \quad \gamma = \frac{B - AE(r_P)}{D}.$$

Also there is one more thing, I forgot to add that is I will define $D = BC - A^2$. So, it can be shown that this term $B > 0$ and $C > 0$ and as well as $D = BC - A^2 > 0$. Now, let us go back to our original expression for $\vec{W}$. This was my original expression for $\vec{W}$. So, I substitute λ and γ in this expression for $\vec{W}$ to obtain that

$$\vec{W} = \left(\frac{CE(r_P) - A}{D}\right)\Sigma^{-1}\vec{E} + \left(\frac{B - AE(r_P)}{D}\right)\Sigma^{-1}\vec{1}.$$

Now, what do you do is the following. Now, we can simplify this weight $\vec{W}$ further. So, we define another set of vector say g. So, I will define this to be

$$\vec{g} = \frac{1}{D}\left(B\Sigma^{-1}\vec{1} - A\Sigma^{-1}\vec{E}\right)$$

and the remaining terms, I will define as h, where

$$\vec{h} = \frac{1}{D}\left(C\Sigma^{-1}\vec{E} - A\Sigma^{-1}\vec{1}\right)$$

So, then this term and this term combined. So, this will become g and this term and this term combined will become

$$\vec{W}_P = \vec{W} = g + hE(r_P).$$

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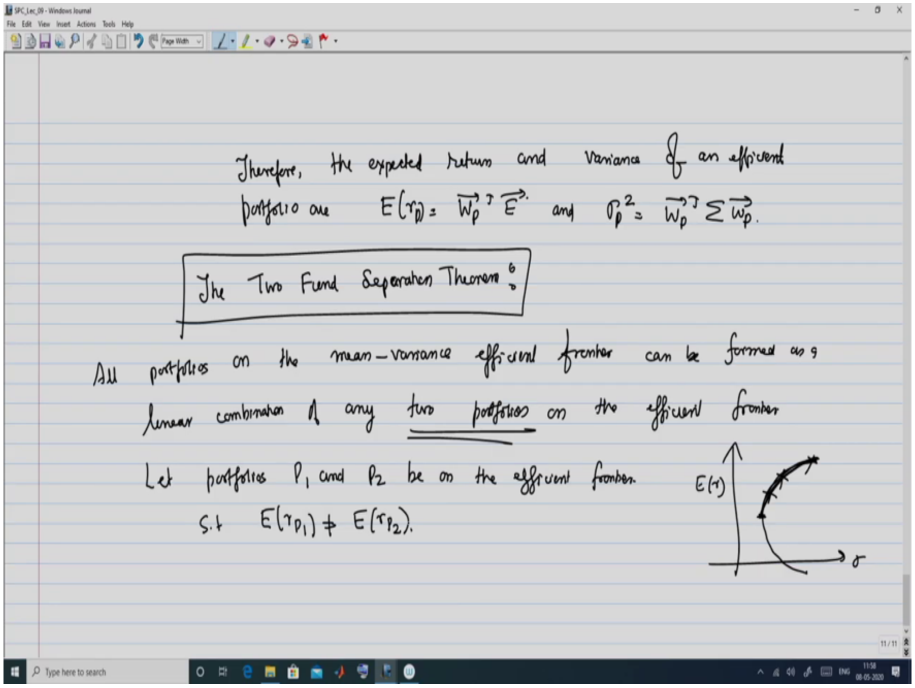
So, in conclusion, we can say that therefore, the expected return and variance of an efficient portfolio are $E(r_P) = \vec{W}_P^T\vec{E}$ and $\sigma_P^2 = \vec{W}_P^T\Sigma\vec{W}_P$. Now, observe very carefully, here what does this term g comprise of? The term g comprises of $\Sigma^{-1}\vec{1}\Sigma^{-1}\vec{E}$ and again, this term h comprised of $\Sigma^{-1}\vec{E}\Sigma^{-1}\vec{E}$. So, these are



same for since these are basically related to the expected return and the covariance matrix of the entire portfolio and this A, B, C and D are again given in terms of this expected return vector and the vector of once and the covariance vector or the covariance matrix. So, accordingly, the g and h that we have here, these are basically going to be fixed irrespective of the portfolio that you are taking. So, for and the reason, I am saying this is that once you have decided to make the asset picking, you already have made up your mind as far as the composition of the portfolio is concerned in terms of the assets that you are going to include. And so that means, you have essentially frozen on the $\vec{E}$ and the covariance matrix Σ which does not change. So, once this is fixed, this is a known input; so that means, that any portfolio that rise on the efficient frontier will have the weights $W_P = g + hE(r_P)$. So, the weights will simply be now a function of $E(r_P)$. So, you can move to allow, you can choose the different values of portfolios on the efficient frontier by choosing different values of $E(r_P)$ which specifies your level of expected return that you want; but recognizing that your g and h are going to remain fixed, ok. So, this brings us to the last theorem which is known as the two fund separation theorem which is basically a direct application of the efficient frontier. So, I can make the statement that all portfolios on the mean variance efficient frontier can be formed as a linear combination of any two portfolios on the efficient frontier. So, this means that if you are looking at the $\Sigma - E(r)$ diagram and this is an efficient frontier, then essentially it means that so this part is going to be an efficient frontier. So, it says that any portfolio on this efficient frontier, say this portfolio can be written as a linear combination of any two portfolios. So, this means that any two portfolios on the efficient frontier is able to generate the entire efficient frontier. So, accordingly, suppose that we take any two portfolios. So, let these portfolios. So, let us consider this portfolios P_1 and P_2 on the efficient frontier such that $E(r_{P_1})$ that is the expected return of the two portfolios are distinct from each other.

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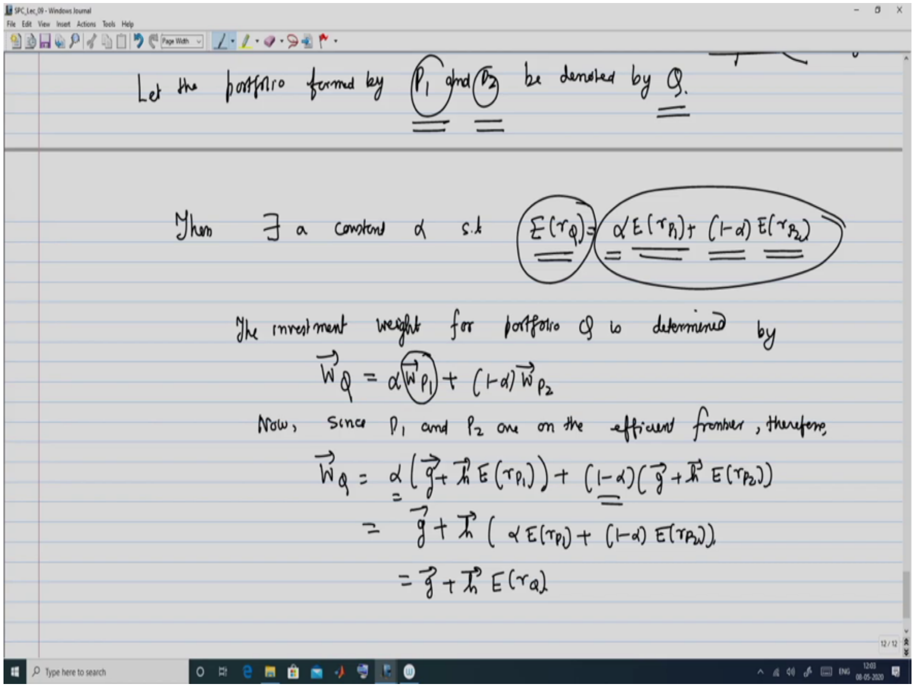
Now, let the portfolio formed by P_1 and P_2 be denoted by Q. Now, if it is formed with P_1 and P_2; then obviously, there exists a constant α such that $E(r_Q) = \alpha E(r_{P_1}) + (1-\alpha)E(r_{P_2})$. So, this means that these two portfolios P_1 and P_2 on the efficient frontier, they can be combined to form a new portfolio Q such that the weight of the portfolio P_1 is α and consequently the weight of the portfolio P_2 is going to be $1-\alpha$. So that means, that you already have two portfolios P_1 and P_2 and then, you create a new portfolio Q, which is essentially just like a two asset portfolio or in this case, it is a portfolio of just two portfolios with the respective weights be given by α and $(1-\alpha)$. Now, you can see that if you want to create any portfolio Q out of this portfolio P_1 and P_2, then you simply need to put in a weight alpha in the portfolio P_1 and a



weight $(1-\alpha)$ in the portfolio P_2. And how do you determine those α and $(1-\alpha)$? You do it simply by setting this to be equal to $E(r_Q)$. So, you can solve this for α in terms of $E(r_Q)$ which is what you specify is an investor and $E(r_{P_1})$ and $E(r_{P_2})$ which is already given to you. So, now, we want to show that now as I said as there as a part of the efficient frontier theorem, we need that any portfolio that you are creating the optimist portfolio is going to lie on the efficient frontier. So, all you need to do is essentially now prove that this portfolio Q will lie on the efficient frontier and then, you are done with the statement that any combination of two portfolios in the efficient frontier P_1 and P_2 in this case will result in a portfolio that also lies on the efficient frontier. So, which means that that is a desirable portfolio from the point of view of a risk averse investor. So, accordingly so that means, that the investment weight. So, what is going to be this particular portfolio Q? So, when I say portfolio, I have to specify the weight. So, the investment wait for portfolio Q is determined by $\vec{W}_Q = \alpha\vec{W}_{P_1} + (1-\alpha)W_{P_2}$. Now, since we began with P_1 and P_2 being on the efficient frontier; therefore,

$$W_Q = \alpha(\vec{g} + \vec{h}E(r_{P_1})) + (1-\alpha)(\vec{g} + \vec{h}E(r_{P_2}))$$

$$= \vec{g} + \vec{h}(\alpha E(r_{P_1}) + (1-\alpha)E(r_{P_2}))$$

It is going to be

$$\vec{g} + \vec{h}E(r_Q).$$

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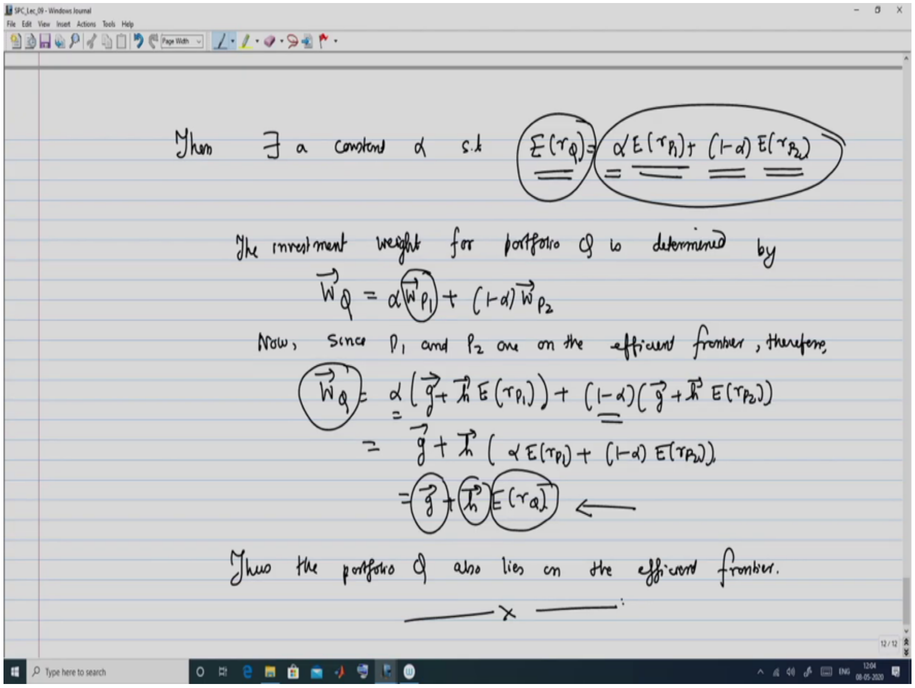
So, this means that the weight of this portfolio Q is of the form $\vec{g} + \vec{h}E(r_Q)$. So, therefore, since it is in this particular form; thus, we can conclude that the portfolio Q also lies on the efficient frontier alright. So, this brings us to the end of this lecture. So, just to do a recap, what I did is that we looked at the problem of minimizing the variance in case of an n asset portfolio. We talked about the opportunities set and visualize this in case, where no short sells are allowed and in case when short selling is allowed. And then, we talked about the efficient frontier which essentially is those cases where a portfolio at a certain, it is a portfolio which at a certain given level of return in the is has the minimum risk or which at a certain given level of risk has the highest expected return or simply, it is the minimum various portfolio. And accordingly, we formulated the problem in terms of a Lagrange multiplier problem. And then, we determine what those weights are going to be and we showed that the weights that lie on this efficient frontier is a linear combination of or a linear function of the expected return of their particular portfolio. And that as an



application to this, we showed that for a rational investor, it is always reasonable to hold a portfolio on the efficient frontier and for that, you do not need to look at all the portfolios in the efficient frontier. But you can determine the weight of the portfolio once you have specified your given level of risk. You can basically determine the weight of the portfolio must taking appropriate linear combination or an appropriate two portfolio; in this case which is denoted by P_1 and P_2 to represent another portfolio lying on the efficient frontier which exactly matches your given level of expected return. So, in the next class we will start a little more elaborate discussion on this and we will also over the course of next one week we will also focus on the situation, where in addition to all the risky assets, we can also take into account a risk free asset in my portfolio. So, this concludes our session for today.

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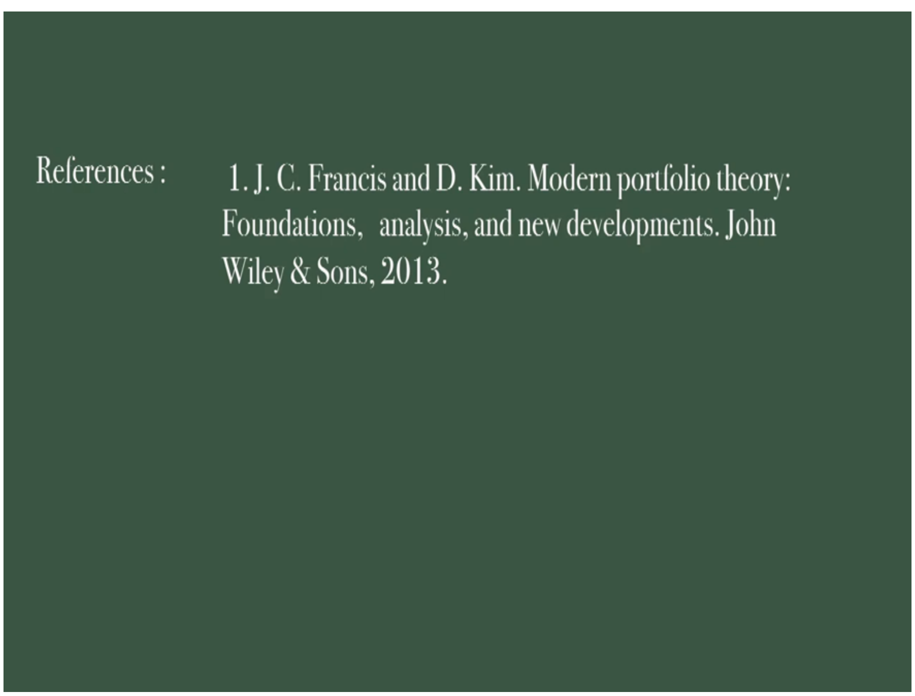

Thank you for watching.