Mathematical Portfolio Theory

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Module 03: Mean-Variance Portfolio Theory
Lecture 02: Expected return and risk of a portfolio; Minimum variance portfolio

Hello viewers. Welcome to the next lecture on this MOOC course on Mathematical Portfolio Theory. You would recall that in the last lecture, we started our discussion on the modern portfolio theory. And, we identified that the two main pillars of modern portfolio theory which characterize it are the expected return and risk, and then we discussed that in the context of the definition of the expected value, and variance of a random variable that we had studied earlier. So, we will continue the discussion on this topic and we will now move on to a discussion, where we consider a portfolio of several assets. And, then look at what is going to be the expected return and risk of that particular portfolio. So, just to begin with a portfolio essentially is a collection of different kinds of assets, such as stocks, bonds and other financial derivatives. And, the key question that we want to answer in portfolio theory is that what should be the appropriate and optimal allocation of our money that we want to invest in different assets. And, you recall that this process involves first figuring out, which assets that we would like to invest in amongst the thousands of different alternative choices that we have. And, then we look at different portfolios that can be created out of it. And, finally, we need to look at, what is going to be the best choice amongst all those different portfolios that we would like to make our eventual investment in. So, we begin first with identifying the key inputs that are required in this particular process of portfolio optimization in the mean variance framework.

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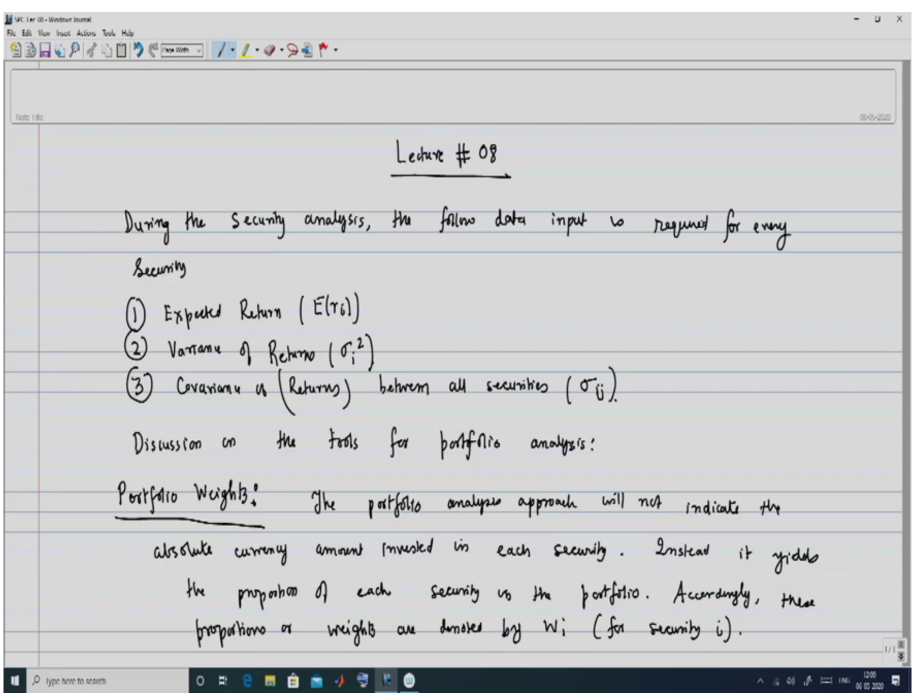

So, accordingly we studied start todays lecture. So, during the security analysis; that means, the analysis of each of the assets, that you are considering for investment. The following data input is required for every security. And, let us identify them one by one. First of all we have what is known as the expected return and, we have already seen what the definition of this, this is $E(r_i)$. Next is we will have variance of returns, that is σ_i^2, something that you have already seen. And, finally, covariance of returns between all securities and which we denote by σ_{ij}. So, we will begin with a discussion on the tools for portfolio analysis. And, the first topic that we look at is what are known as portfolio weights. So, the portfolio analysis approach will not indicate the absolute currency amount invested in each security. Rather we instead it yields the proportion of each security in the portfolio. And so, accordingly these proportions or weights are denoted by w_i for security i. So, let me explain this in a little more detail. So, what happens is suppose that, we want to invest an amount of 1000. And, then we decide to invest 500 each into different stocks. So, when you talk about a portfolio, we do not identify that by the statement that we have invested 500 in the first stock and 500 in the second stock. Rather it is customary and the standard approaches to identify by the proportion of the amount of in total investment that is invested in each asset. So, instead of making a statement that the investment is a 500 in each of the two stocks, we will instead say that a fraction of half is invested in the first stock and the fraction of half is invested in the second stock. And, this number are determined by the amount invested in each stock, divided by the total investment. So, I arrived at the amounts half and half byte 500 divided by 1000. So, there are of course, you know another way of how to identify a portfolio would be looking at the number of assets. But, again you know the number of assets will be merged and multiplied with the price of the unit of that asset and divided by the total number total amount of money that is invested to give the weights for a particular asset.

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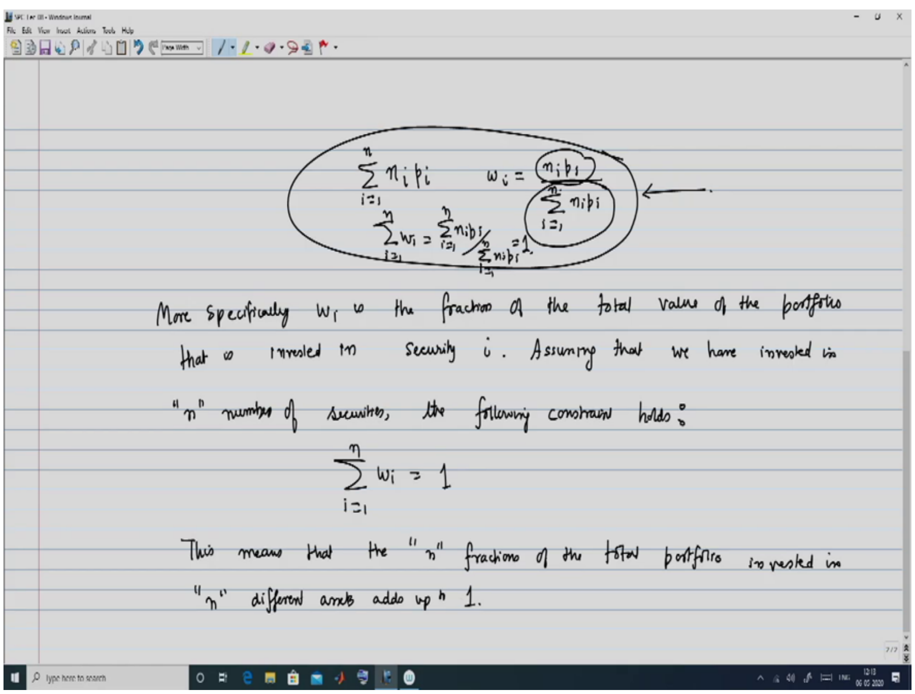

So, for example, if I invest in n assets, rather in n number of units of the i-th asset and the price of the asset is p_i. And, the total investment then is going to be $\sum_{i=1}^{n} w_i = 1$ to say n number of assets. And, consequently the weight w_i is going to be number of units of the asset into it is price, that is the total amount of money that you have invested in the ith asset divided by $\sum_{i=1}^{n} n_i p_i$. So, this is a statement that I am making slightly informally before again going back to my formal statements. So, more specifically, so if I can now view weights in the paradigm of this formulation. So, more specifically w i is the fraction of the total value of the portfolio that is invested in security i. So; that means, it is the fraction of total investment in security i out of a of the total investment, that is made in all the n number of securities. So, assuming that we have invested in n number of securities, the following constraint holds. And, this constraint is that $\sum_{i=1}^{n} w_i = 1$. So, if you observe here this is $\sum_{i=1}^{n} w_i$ is simply going to be $\sum_{i=1}^{n} n_i p_i$, divided by $\sum_{i=1}^{n} n_i p_i$, which is equal to 1. So, essentially this means that all the fractions they have to add up to 1. So, this means, that the n fractions; that means, $w_1, \cdots, w_n$ of the total portfolio invested in n different assets adds up to 1, ok. So, now, now that we have defined; what is the weight of a portfolio or rather the weight of a particular asset in a portfolio. The next thing we need to do is that we need to look at the two characteristic factors that are used to define a portfolio namely the expected return and risk. So, accordingly we first begin with what is the expected return of a portfolio comprising of s number of assets as just defined.

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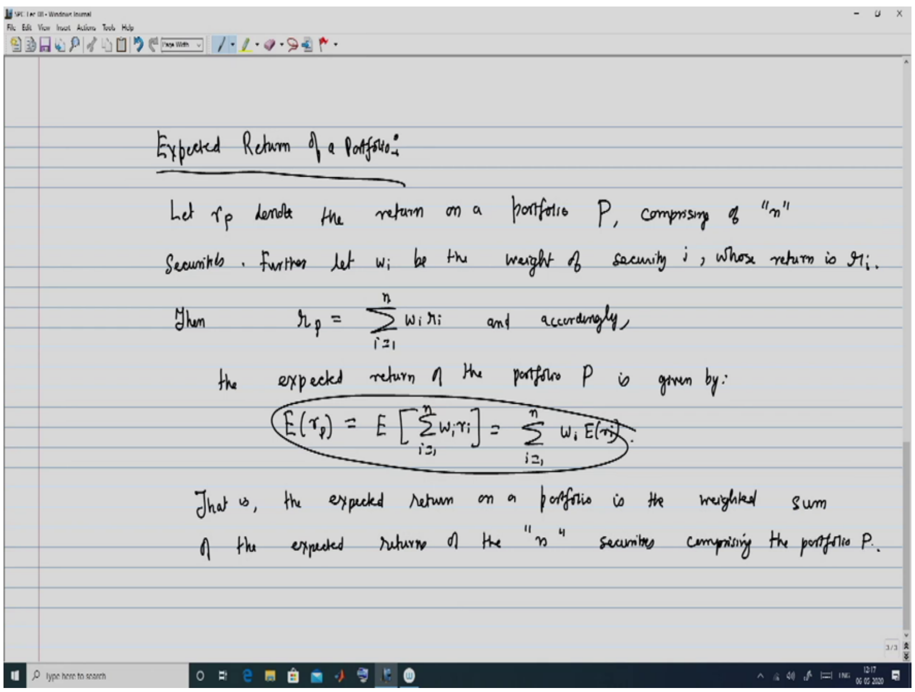

So, we start off with expected return of a portfolio. So, let r_P denote the return on a portfolio. And, I denote the portfolio by P, comprising of n securities. Further, let wi be the weight of security i, whose return is r_i as defined before. Then, we can show that

$$r_p = \sum_{i=1}^{n} w_i r_i$$

and accordingly, the expected return of the portfolio P is given by

$$E(r_P) = E\left[\sum_{i=1}^{n} w_i r_i\right] = \sum_{i=1}^{n} w_i E(r_i).$$

So, that is we can now interpret this as the expected return on a portfolio is the weighted average of the expected returns of the n securities Actually, I should say a weighted sum of the n securities comprising the portfolio, ok. Now, that we have defined the expected return of the portfolio. The next thing we are going to do is make use of that, in order to define and then give a formulation for the risk of this particular portfolio P.

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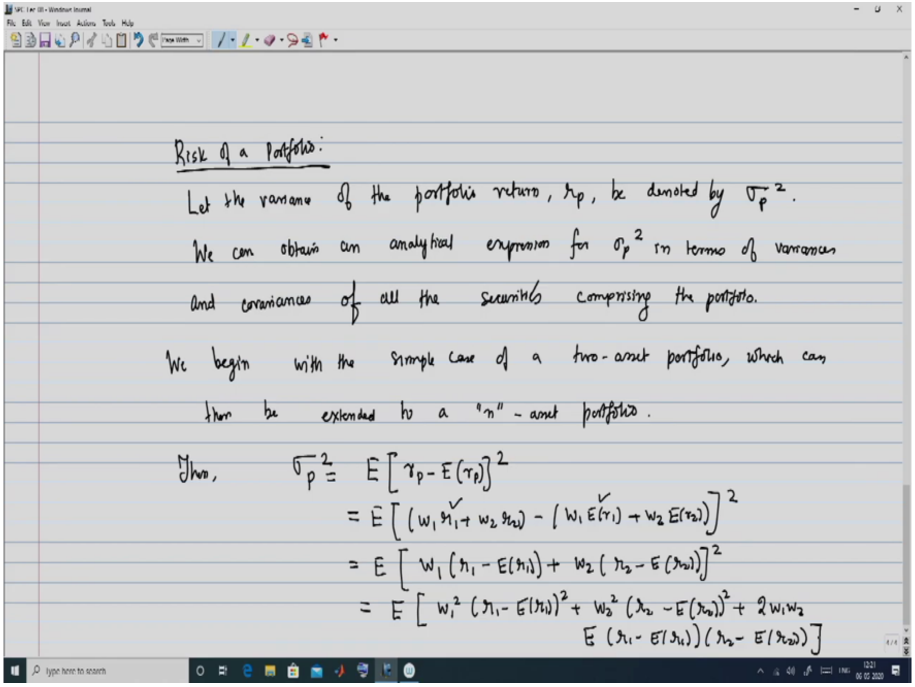
So, then we start off with the risk of a portfolio. So, let the variance and I will use the variance as a measure of risk, or equivalently the square root of that, which is the standard deviation as a measure of risk. So, let the variance of the portfolio return, r_P, be denoted by σ_P^2. Now, we can obtain an analytical



expression for σ_P^2 in terms of variances and covariances of all the securities comprising the portfolio. So, we begin with a simple case and then build upon it of a portfolio comprising of two assets, which can then be extended to a n asset portfolio. So, then so, we will first look at the two asset case, then what is σ_P^2 going to be? By definition this is going to be the $E[r_P - E(r_P^2)]^2$. Now, we recall the form for r_P and $E(r_P)$ and we substitute those here. So, accordingly we will get $E\,[(wr_P]$will be replaced by $w_1r_1 + w_2r_2$ and $E(r_P)$ will be represented by $[w_1E(r_1) + w_2E(r_2)]^2$. Now, I combine the terms involving w_1 and w_2 separately. So, this is going to be so, I will take this term and this term. So, I will get

$$E[w_1(r_1 - E(r_1)) + w_2(r_2 - E(r_2))]^2.$$

So, now this can be rewritten. As, so this is going to be

$$E[w_1^2(r_1 - E(r_1))^2 + w_2^2(r_2 - E(r_2))^2 + 2w_1w_2(r_1 - E(r_1))(r_2 - E(r_2))].$$

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So, I can now make use of the linearity property of expectation. So, this becomes

$$w_1^2E[(r_1 - E(r_1))]^2 + w_2^2E[(r_2 - E(r_2))]^2 + 2w_1w_2E[(r_1 - E(r_1))(r_2 - E(r_2))].$$

Now, observe that this is

$$w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}.$$

Now, we give a slightly different representation and it will be slightly more general in nature to this expression that we have obtained just now. So, another way to express the σ_P^2 is follows. So,

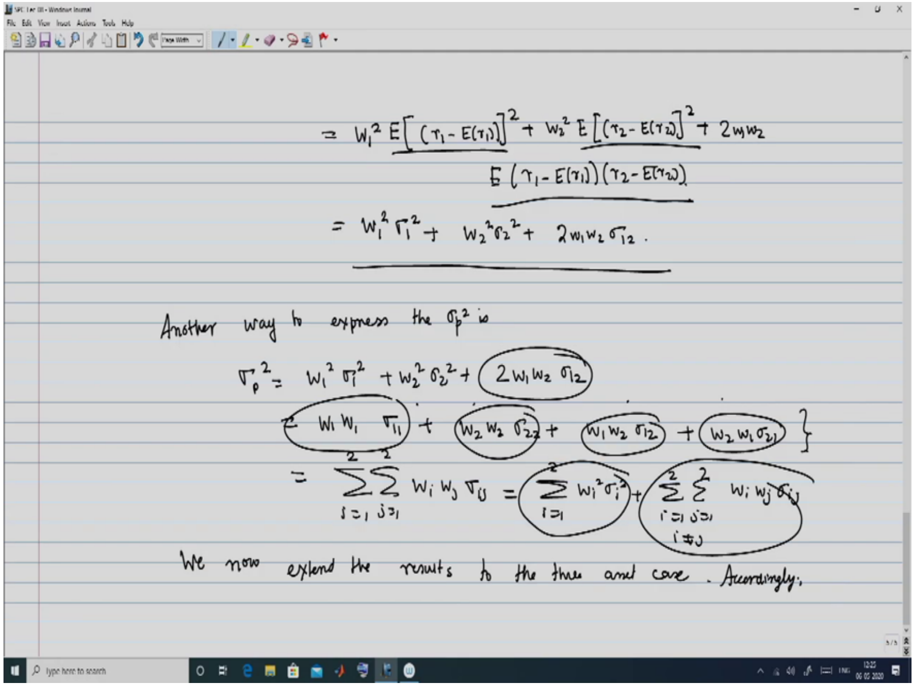
$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

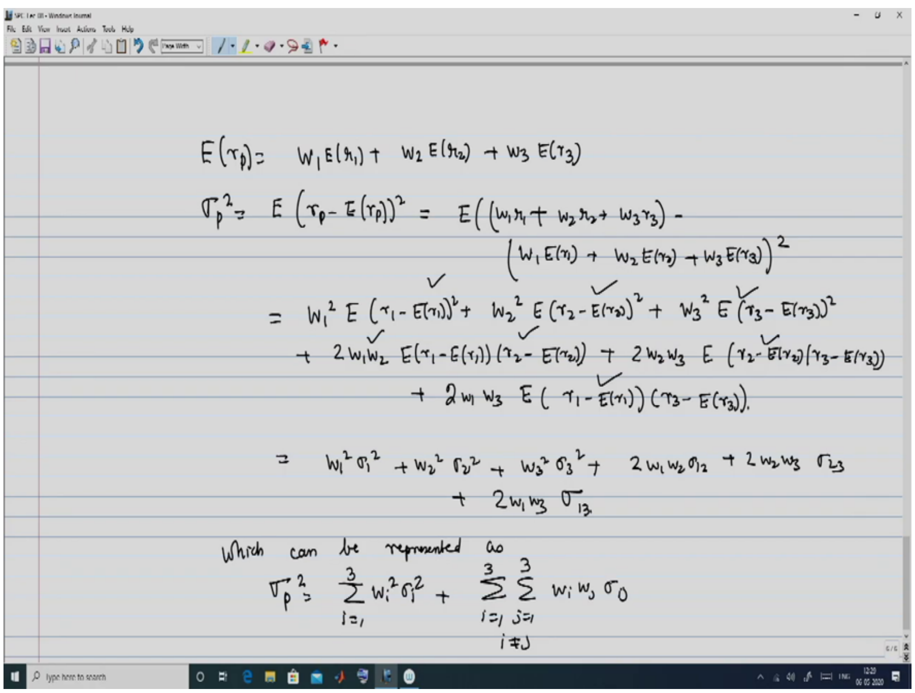
Now, I can write this as

$$w_1w_1\sigma_{11} + w_2w_2\sigma_{22} + w_1w_2\sigma_{12} + w_2w_1\sigma_{21}$$

This can be written as

$$\sum_{i=1}^{2}\sum_{j=1}^{2} w_iw_j\sigma_{ij} = \sum_{i=1}^{2} w_i^2\sigma_i^2 + \sum_{i=1}^{2}\sum_{j=1}^{2} w_iw_j\sigma_{ij},\ i \neq j.$$





So, this basically will collate to these two terms and this will correspond to this and this terms, all right. So, now that we are done with the case of two assets. So, we now extend, the results for to the three asset case.

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So, accordingly the expected return and variance are going to be given by

$$E(r_P) = w_1E(r_1) + w_2E(r_2) + w_3E(r_3)$$

$$\sigma_P^2 = E(r_P - E(r_P))^2 = E[(w_1r_1 + w_2r_2 + w_3r_3) - (w_1E(r_1) + w_2E(r_2) + w_3E(r_3))]^2$$

And, this again can be rewritten as

$$w_1^2E(r_1 - E(r_1))^2 + w_2^2E(r_2 - E(r_2))^2 + w_3^2E(r_3 - E(r_3))^2 + 2w_1w_2E[(r_1 - E(r_1))(r_2 - E(r_2))$$

$$+2w_2w_3E[(r_2 - E(r_2))(r_3 - E(r_3)) + 2w_1w_3E[(r_1 - E(r_1))(r_3 - E(r_3))$$

So, this is going to be

$$w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{12} + 2w_2w_3\sigma_{23} + 2w_1w_3\sigma_{13}$$

So, which can be represented as

$$\sigma_P^2 = \sum_{i=1}^{3} w_i^2\sigma_j^2 + \sum_{i=1}^{3}\sum_{i=1}^{3} w_iw_j\sigma_{ij},\ i \neq j.$$

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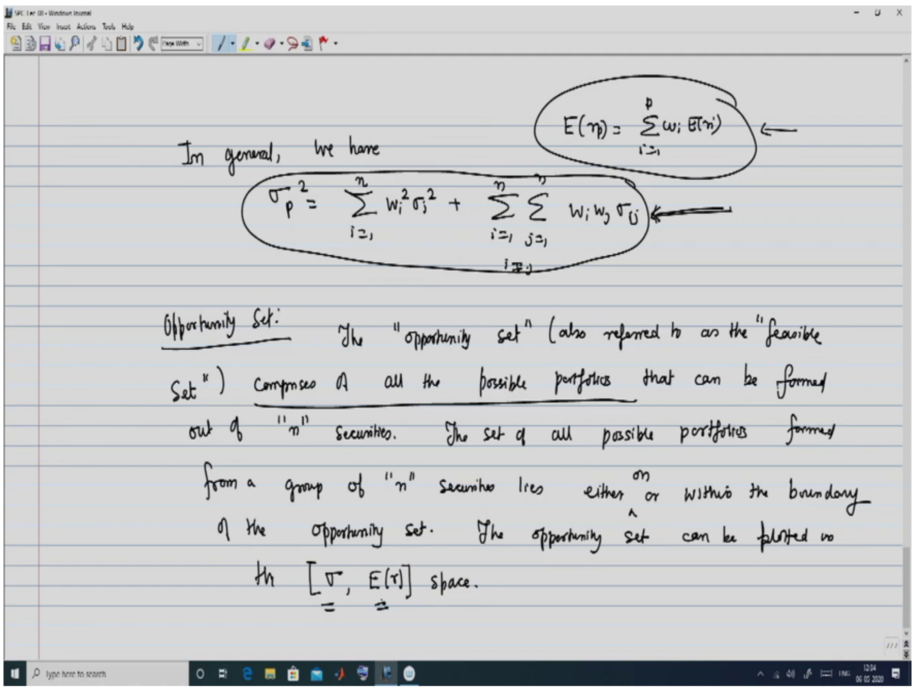

So, in conclusion so, in general we have

$$\sigma_P^2 = \sum_{i=1}^{n} w_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} w_iw_j\sigma_{ij}$$

So, this is nothing, but i am just rewriting the formula for the variance of the linear combination of n number of random variables that we had seen earlier, ok. So, we next come to something which is known as the opportunity set and we will discuss briefly about that. So, let me begin with opportunity set. The opportunity set also referred to as the feasible set comprises of all the possible portfolios, that can be formed out of n securities. The set of all possible portfolios formed from a group of n securities lies either or within the boundary, either on or within the boundary of the opportunity set. The opportunity set can be plotted in the sigma E r space. So, just to do a recap of the opportunity set I have made two observations; the first is that the opportunity set which is sometimes is called the feasible set. It comprises of all the possible portfolios that can be formed out of the n securities. So, essentially this means that if you have n number of securities, you can choose different combinations of weights for each of the securities in the portfolio of course, provided that the sum of the weights is always equal to 1. So; that means, it is going to be a combination of some $w_1, \cdots, w_n$, such that $\sum_{i=1}^{n} w_i = 1$. So, this effectively basically gives an infinitely many different possibilities of such portfolios and the collection of all these portfolios is what is known as the feasible set. Now, in particular what we can do is that. Now, if we consider n number of assets, then for large number of ns it is very difficult for us to visualize, this vector $w_1, \cdots, w_n$ that is the vector of the weights. So, equivalently what we can do is that for each combination of $w_1, \cdots, w_n$, we can use this formula and previous formula the, that is

$$E(r_P) = \sum_{i=1}^{P} w_i E(r_i).$$

So, for each combination of $w_1, \cdots, w_n$, what I will have is I will have a pair of the $[\sigma, E(r)]$. So, instead of so, the feasible set instead of being represented by this vectors $w_1, \cdots, w_n$ which; obviously, cannot be drawn on n dimensional plane. We equivalently represent this by their corresponding sigma as given here and the corresponding $E(r_p)$ as given here.

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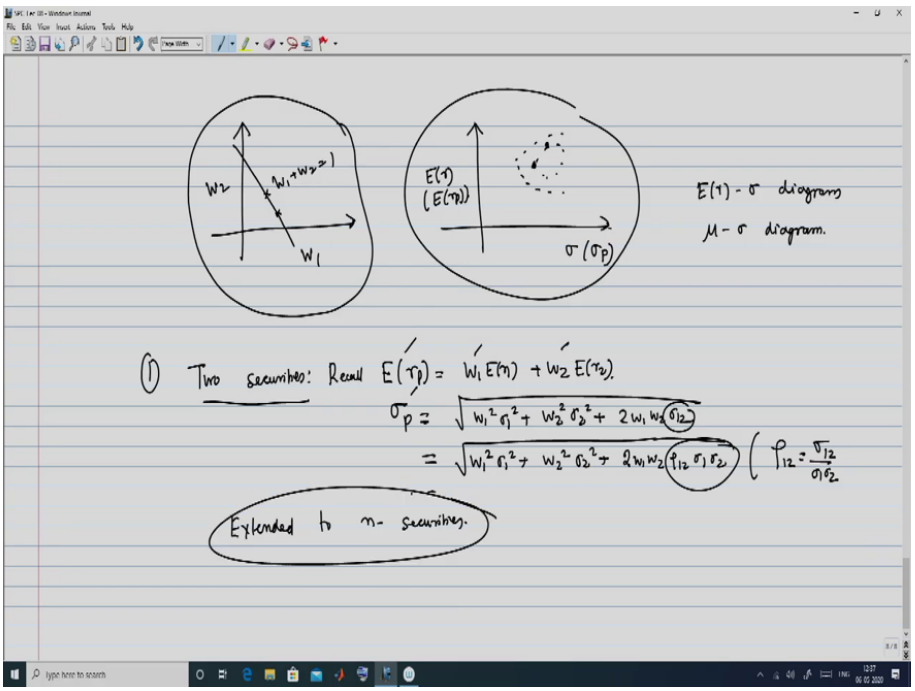

So, graphically this will look something like this. So, say if you have for example. So, you can actually see this in the context of a two-asset portfolio. So, if you have the weights w_1 and w_2, satisfying the conditions say $w_1 + w_2 = 1$. Then, corresponding to each pair w_1 and w_2, we can use the formula to calculate $E(r_p$ and $\sigma(r_p$ and they will basically comprise of these particular points. So, for example, if I choose $w_1 w_2$ from here, the corresponding sigma, which actually is σ_P and the corresponding $E(r)$ which is basically $E(r_p)$, this might be this point here. Likewise, if I choose this pair of $w_1 w_2$, the corresponding value of E_r and σ in the $E_r - \sigma$ diagram this could be here. So, essentially the set of feasible portfolios actually is the combination of all these w_1 and w_2, but because it is going to be very difficult to visualize them on a n dimensional plane. So, they are equivalently visualized by their corresponding values of σ and E_r. And, lot of times this is what is known as the $E_r - \sigma$ diagram or sometimes it is called the $\mu - \sigma$ diagram. So, we will discuss more of this later on when we talk about efficient frontier, So, let us now come back to the case of two securities. Now, for two securities what did you have? So, we recall that, $E(r_p) = w_1 E(r_1) + w_2 E(r_2)$. And,

$$\sigma_P = \sqrt{Var} = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12}}$$

And, this can be rewritten as

$$\sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho_{12} \sigma_1 \sigma_2}.$$

So, what I have done here is we have made use of that $\rho 12 = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$ and I replace σ_{12} with $\rho_{12}\sigma_1\sigma_2$, ok. So, basically the set of feasible set is not represented by this w_1 and w_2, but rather it is represented by this $E(r_P)$ and σ_P, ok. So, now so, this can be then can be extended to n securities.

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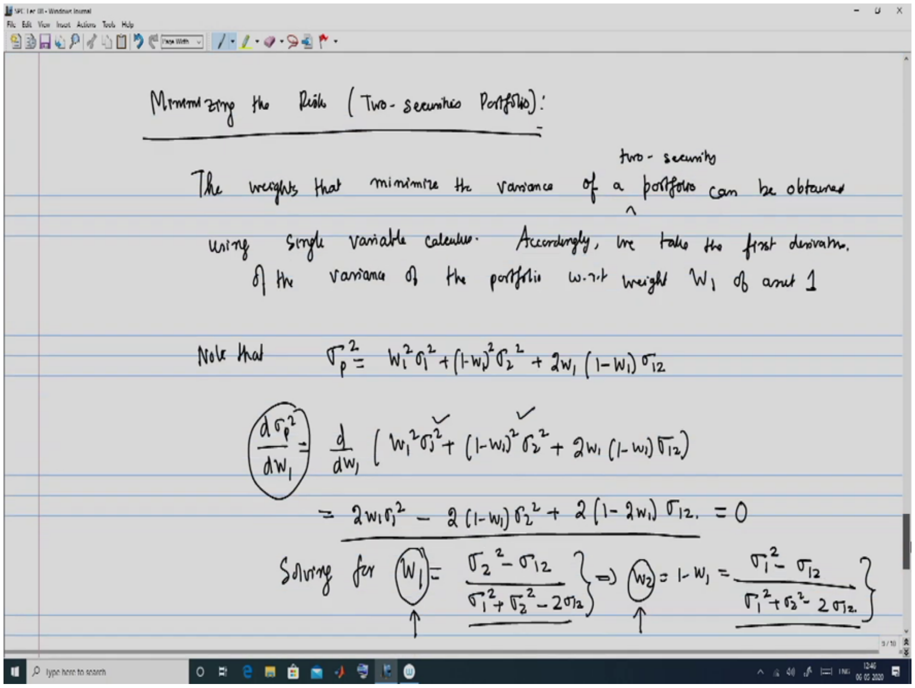

So, next we come to minimizing the risk. And, this will be again for two securities portfolio. So, this topic of minimizing the risk is going to be the first topic, on portfolio optimization and how you determine the portfolio once we have decided, what is going to be the asset that will constitute the portfolio. And, for the sake of brevity I will start off with considering a two asset portfolio. And, the goal is that I want to look at the situation that amongst the different assets. That means, amongst all the feasible assets, that I have just now defined in the case of these two asset portfolios, which is the one that is going to give me the minimum risk. So, later on we look at a more generalization of this that it is going to be no longer just the minimization of the risk. But, we will also talk about minimizing the risk, for a given set of return or maximizing the return for a for a certain level of risk. And, all of this will be collated to form what a what is known as the concept of efficient frontier. So, coming back to this very elementary and the first step in determining what is going to be the best choice of my portfolio. In the case of a portfolio comprising of two assets, we will just focus on minimizing the risk. So, while you are doing the minimization of the risk we will essentially make use of single variable calculus. So, let us come to the minimization of the risk. So, the weight, that rather weights because there are two assets. So, the weights that minimize the variance so, please remember the minimization of the standard deviation is the effectively the same as minimization of the variance. So, that minimizes the variance of a portfolio can be obtained. So, I should specify that this is a two-security portfolio. So, this can be obtained using single variable calculus. Accordingly, we take the first derivative of the variance, of the portfolio with respect to weight w_1 of asset 1. Now, please remember that the minimization of this variance will be to determine the values of w_1 and w_2, which is going to give me the minimum value of σ_P^2. Now, we note that here I have to determine two values w_1 and w_2. So, it is effectively minimization of a function in terms of two variables w_1 and w_2, but I am saying that I will basically make use of single variable calculus here. Now, for that purpose we need to recall that, we had the condition that $w_1 + w_2 = 1$. So, this means that I can replace w_2 with $1 - w_1$ and put that in the expression for σ_P^2. So, this will result in the expression for sigma p square no longer being a function of two variables w_1 and w_2, but rather it is going to be a function of one variable namely w_1. And, once I have found out what is the optimal value of w_1 obviously, the corresponding w_2 given by $1-w_1$ is going to be the minimum variance weight for the second asset. So, accordingly we do the following. So, we start off with that, then note that what is σ_P^2? σ_P^2 was $w_1^2,\ \sigma_1^2$, then I had $w_2^2,\ \sigma_2^2$, so which are replaced by $1 - w_1^2$. Then, I had plus $2w_1$ and then I will multiply this replace w_2 with $(1 - w_1)\sigma_{12}$. So, now I take the partial derivative of this with respect to w_1. So, this is going to be the derivative of w_1. So, actually I can just take the ordinary derivative, since it is a function of single variable now. So,

$$\frac{d\sigma_P^2}{dw_1} = \frac{d}{dw_1}[w_1^2\sigma_1^2 + (1 - w_1)^2\sigma_2^2 + 2w_1(1 - w_1)\sigma_{12}].$$

So, this is going to be

$$2w_1\sigma_1^2 - 2(1 - w_1)\sigma_2^2 + 2(1 - 2w_1)\sigma_{112} = 0.$$

And, in order to determine what is the maxima and the minima, or the value of w_1 at which this will be a minima, we will have solving for w_1. We can get that

$$w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}.$$

Which implies that

$$w_2 = 1 - w_1 = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}.$$

And, you can do the second derivative test also that here, you can take the second derivative. And, you can show that, the second derivative of this expression this is going to be greater than 0, by making use of the properties of the first and second moments as well as the properties of covariance ok. So, essentially what you have obtained is that, the portfolio so, if I have two assets and I am in a dilemma as to choose what is the proportion or the weights that I should assign to each of those assets. Then, if I set my goal that I want the asset with the minimum risk as given by the variance or equivalently standard deviation, then it turns out that the weights for that minimum variance portfolio is going to be w_1 and w_2 as given by this expression here and this expression here, respectively. And, you notice that both this expression what are the; what are the quantities? It involves it involves $\sigma_1^2,\ \sigma_2^2$ and σ_{12}, which are the variances of the two assets as well as the covariance of the returns of those two assets. And, these are the numbers that we have already seen how you can determine from the historical data. So, you basically once you decide on two assets, then you can make the historical use of the historical data. And, you can just substitute those numbers here and decide what is going to be your optimized portfolio. So, this is a very simple way in which you can actually download the data that is available publicly. And, make a decision on how you are going to set up your portfolio based on this optimization? Ok. So, now let us move on to minimizing the risk in the case of a three asset portfolio and eventually we will look at an n asset portfolio.

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So, I look at minimizing the risk for three asset portfolio. So, for the three asset portfolio, you recall that $E(r_P) = \sum_{i=1}^{3} w_i E(r_i)$. Now, we need to figure out what is going to be the σ_P^2? So, here we recall that

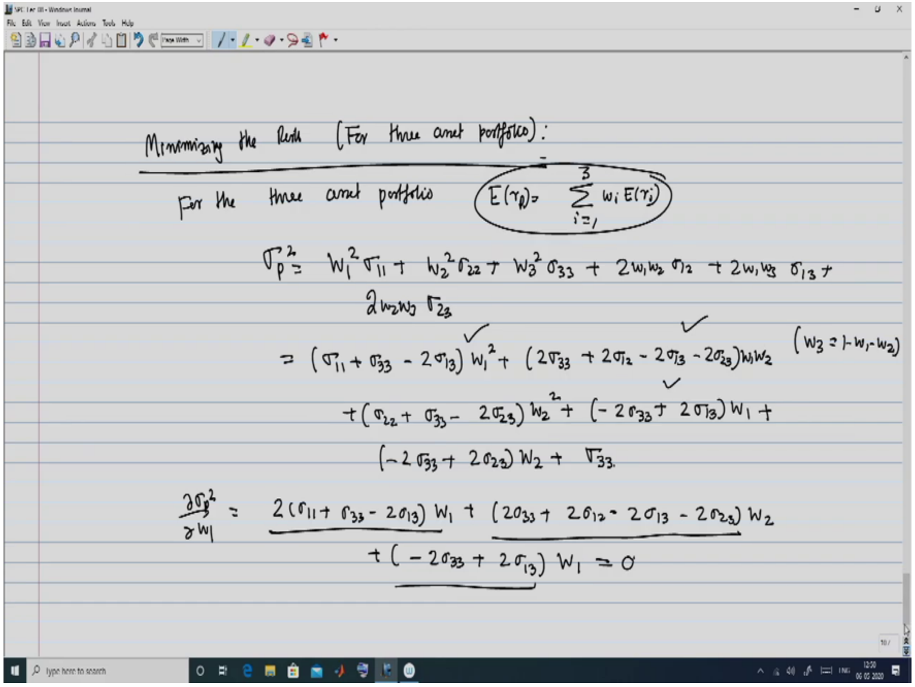
$$\sigma_P^2 = w_1^2\sigma_{11} + w_2^2\sigma_{22} + w_1^3\sigma_{33} + 2w_1w_2\sigma_{12} + 2w_2w_3\sigma_{23} + 2w_1w_3\sigma_{13}.$$

Now, what I can do is that, I can make a substitution by replacing $w_3 = 1 - w_1 - w_2$. So, accordingly what this will become is this is going to be

$$(\sigma_{11} + \sigma 33 - 2\sigma 13)w_1^2 + (2\sigma_{33} + 2\sigma_{12} - 2\sigma_{13} - 2\sigma_{23})w_1w_2 + (\sigma_{22} + \sigma_{33} - 2\sigma_{23})w_2^2$$

$$+(-2\sigma_{33} + 2\sigma_{13})w_1 + (-2\sigma_{33} + 2\sigma_{23})w_2 + \sigma_{33}.$$

Now, what is my goal? My goal here is to basically figure out what is going to be my values of $w_1,\ w_2$ and w_3, that is going to result in getting the minimum variance σ_P^2. And, now I have reduced this instead of being it a function of three variables $w_1,\ w_2$ and w_3. By making the substitution that $w_3 = 1 - w_1 - w_2$.



I have reduced this σ_P^2 to be a function of two variables namely $w_1,\ w_2$. So, in order to determine what is going to be the optimized value of $w_1,\ w_2$ and; obviously, consequently w_3, I take the partial derivative of the various with respect to $w_1,\ w_2$ and set it equal to 0. So, accordingly I get

$$\frac{\delta\sigma_P^2}{\delta w_1} = 2(\sigma_{11} + \sigma 33 - 2\sigma 13)w_1 + (2\sigma_{33} + 2\sigma_{12} - 2\sigma_{13} - 2\sigma_{23})w_2 + (-2\sigma_{33} + 2\sigma_{13}) = 0.$$

So, this coefficient comes from here and this one it comes from this term and this one comes from these particular term and I set it equal to 0.

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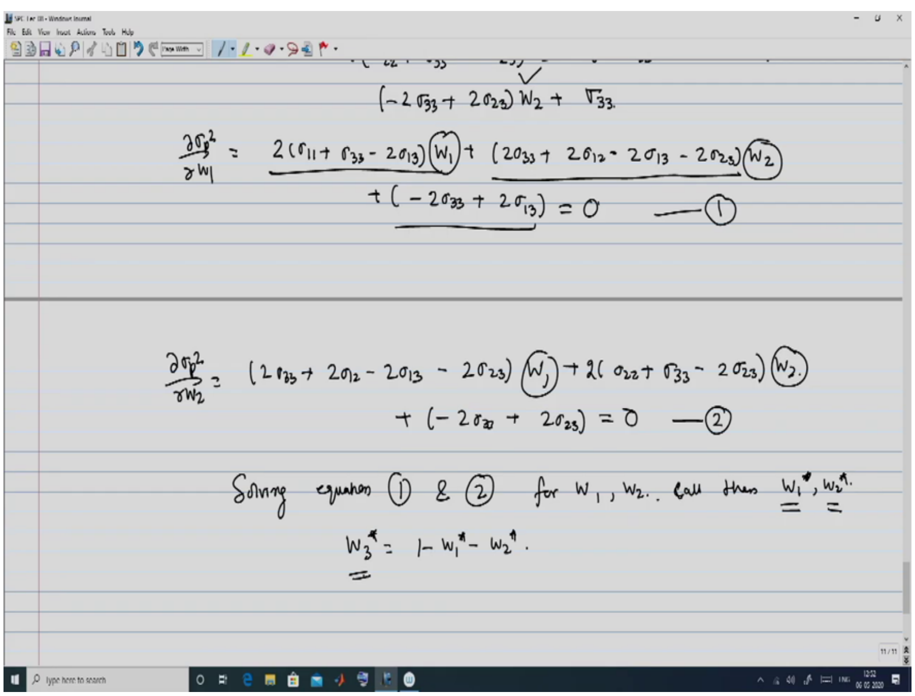

Likewise, I will set

$$\frac{\delta\sigma_P^2}{\delta w_2} = (2\sigma_{33} + 2\sigma_{12} - 2\sigma_{13} - 2\sigma_{23})w_1 + 2(\sigma_{22} + \sigma_{33} - 2\sigma_{23})w_2^2 + (-2\sigma_{33} + 2\sigma_{23}) = 0.$$

So, I call say this equation 1 and I will call this as equation 2. So, if I solve equation 1 and 2, remember that these are linear equations in w_1 and w_2. Actually, there is no w_1 here so, please make this correction and here I have w_1 and w_2. So, I have solved this equation for w_1 and w_2 and call them say w_1^* and w_2^*. Then, $w_3^* = 1 - w_1^* - w_2^*$. So, this w_1^* and w_2^* and consequently w_3^* are the weights in a three asset portfolio, that minimizes the variance of the three asset portfolio. So, in general if you want to extend this to an n dimensional or an n asset portfolio then; obviously, you can guess that the calculations are going to be much more cumbersome. And, for that point of time we need to look at some alternative methods to figure out, how we are going to determine $w_1^*, w_2^*, \cdots, w_n^*$, in the case of a n-asset portfolio, all right. So, just to conclude whatever we have discussed today. In todays class we started looking at the by talking about the three key data input that is required, when you are doing a portfolio optimization. And, then we talked about what is going to be the return. First of all we talked about what is the weights and this consequently gave us what is the definition of the return of a portfolio in terms of weights. and the returns of the individual assets that constitute the portfolio. And, then we took the expectation and the variance for both of them to calculate what is going to be the expected return. And, variance for a two asset portfolio, for three asset portfolio and in general for a n asset portfolio. And, then we talked about what is the feasible set. And, then we will started our discussion on how among a feasible set of portfolios. We can choose the one that is the best and by best we could basically mean have different meanings. To begin with we define that the best portfolio amongst the feasible set. We are choosing the 1 that will minimize the risk. And, accordingly we have determined the weights of portfolio, such that the risk of that particular portfolio comprising of those assets, are obtained in the case of a two asset portfolio and in the case of a three asset portfolio using single and multi-variable calculus respectively. So, we will extend this in the next class and discuss this in a lot more detail.

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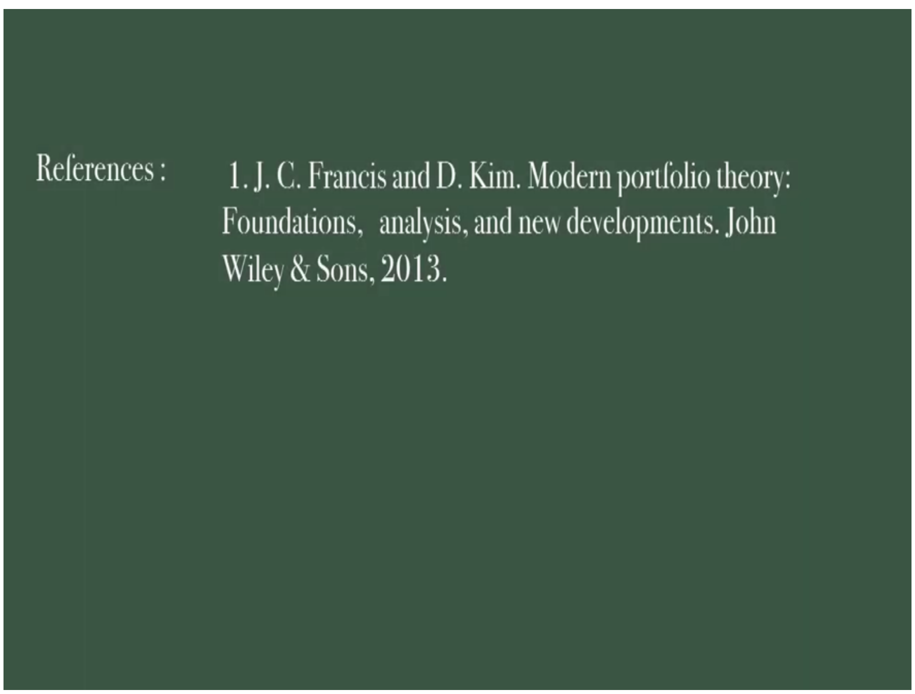

So, this concludes this lecture for today.

Thank you for watching.