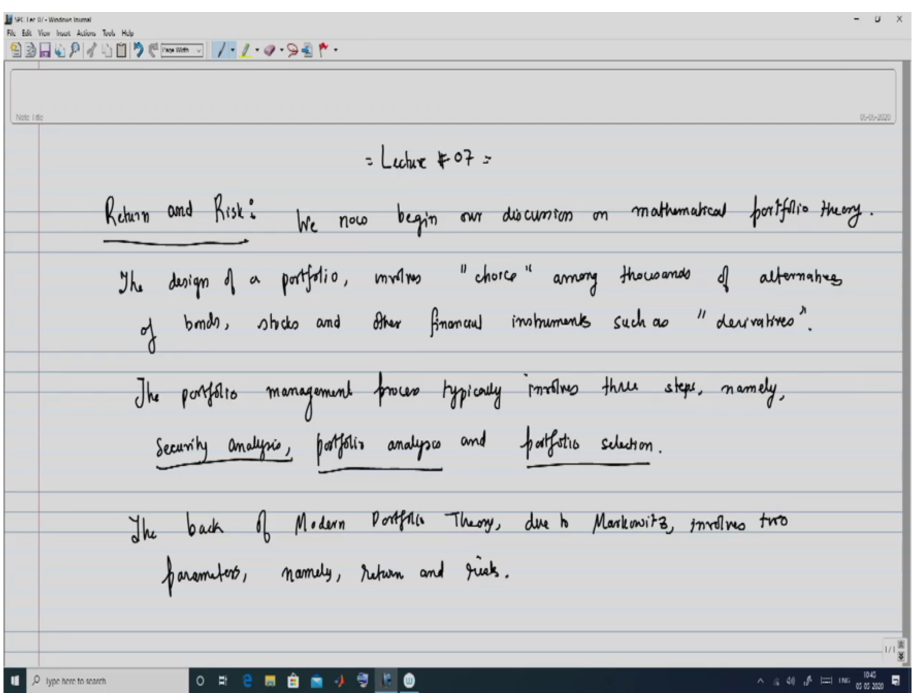
Mathematical Portfolio Theory

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

Module 03: Mean-Variance Portfolio Theory
Lecture 01: Expected return, risk and covariance of returns

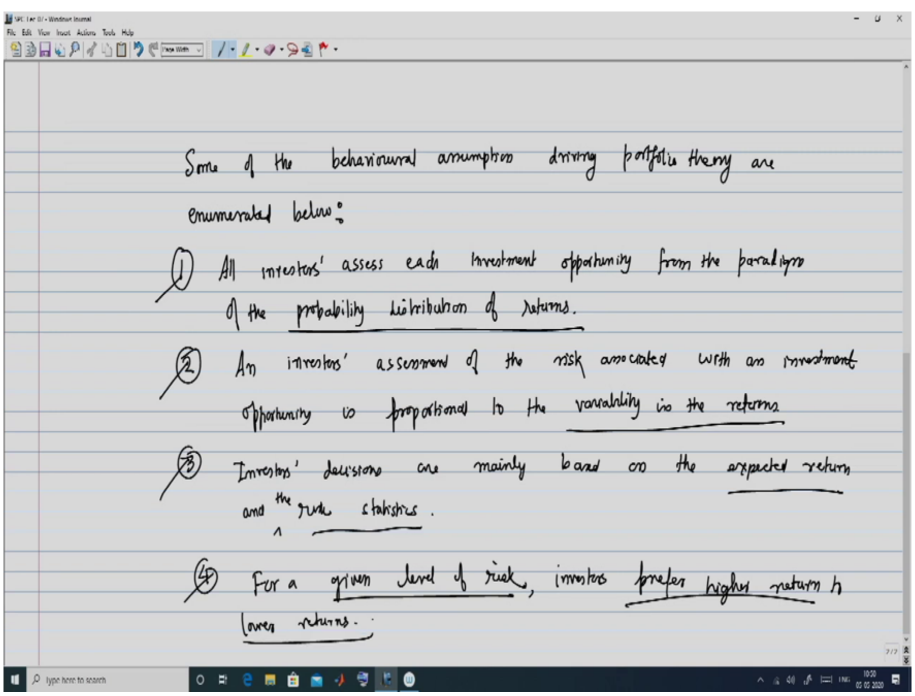
Hello viewers. Welcome to this next lecture on the MOOC video course on Mathematical Portfolio Theory. So far we have looked at two important aspects as far as the course is concerned in general: the first thing is about the probability theory and the second is about financial markets and asset pricing model. From todays class we will start the main topic and focus of this course namely portfolio theory. And we will begin with by talking about the modern portfolio theory, which is also known as the mean variance analysis So, the mean variance analysis owes its origin to the seminal work of Harry Markowitz in 1952, wherein for the first time Markowitz basically gave a proper structure to how a judicious decision that can be made in terms of an investment decision. So, prior to Markowitz theory coming into existence there was no clear and well defined manner in which investment decisions were made, and this led to a huge amount of losses in a lot of cases for the investors. But, Markowitz for the first time identified that one cannot simply judge the potential of an investment in a risky asset by the return it gives, because very often you observed that assets which offer a very high return are also associated with a high amount of risk. And he identified that both return and risk should be taken as the two pillars when deciding on an investment strategy. And accordingly, the foundations for the modern portfolio theory or what is equivalently known as the mean variance theory was laid down.

(Refer Slide Time: 02:12)



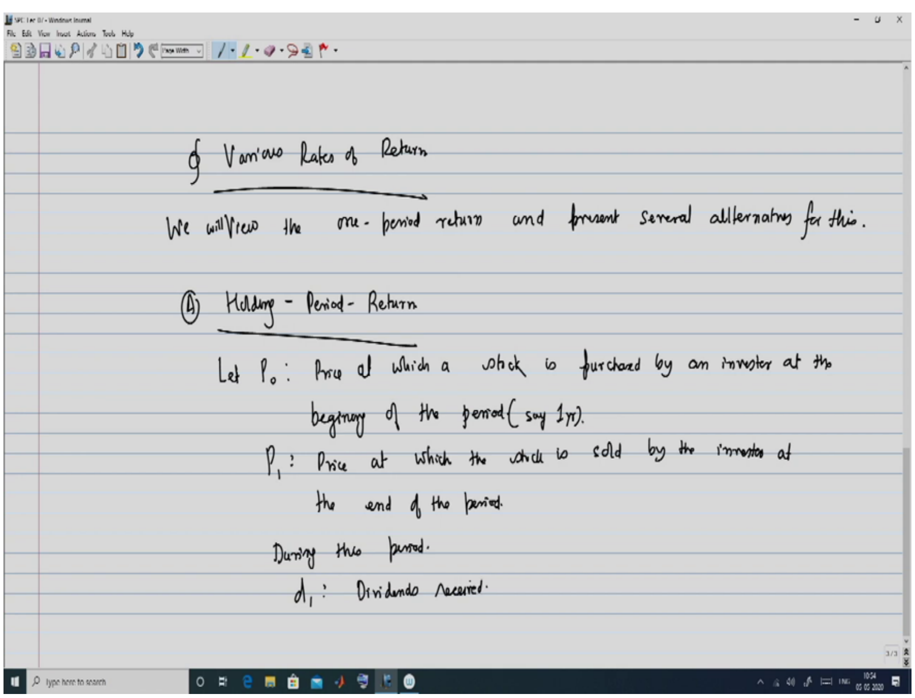
So, we start off with this lecture, so this is lecture number 7. So, if as I pointed out the Markowitz theory is based on two pillars namely, return and risk. So, we now begin our discussion on the mathematical portfolio theory with initial stages being devoted to the Markowitz theory. So, I can say that the design of a portfolio and I will explain later what is a portfolio. So, on a very informal level a portfolio is a collection of assets both risky and risk free. So, the design of a portfolio involves something called a “choice” among thousands of alternatives of bonds, stocks and other financial instruments such as derivatives. Now, the portfolio management process typically involves three steps and I will briefly describe about these three steps. So, these three steps are: security analysis, portfolio analysis and portfolio selection. So, let us go through this one by one. So, the first one is the security analysis. So, basically a portfolio manager, what he or she does is that the security analysis means that they will look at the securities particularly risky securities such as stocks, and they will analyze those stocks and then they will rank them in order of the desirability. So, basically amongst the thousands of choices, what they will do is, that they will pick up certain amount of stocks which in their opinion are better investments in terms of return balanced against the risk. So, once these stocks are chosen the next question that arises is: how are you going to distribute your wealth among the different stocks. So, suppose you have decided that you want to invest in 10 stocks and you start off with an amount of say 10000. So, then the next question is that how about what should be the distribution of this 10000 that you want to invest in stocks; that means, how much money should you allocate at to each of the stock, so that the sum adds up to 10000. And there are many different ways this can be done, and the final step would be basically figuring out what is the best way to make this allocation amongst different portfolio choices. So, for example, I have a certain amount of money and that money can be used to buy these 10 different stocks in different proportions. And, then it becomes a question of optimizing that which amongst those different ways of creating a portfolio is going to be the best choice. So, that is why I say that the first step is security analysis, look at the level of the risky asset. The next is portfolio analysis, while basically we analyze the different portfolios that you can construct out of those chosen securities and finally we make a portfolio selection out of them, ok. So, now next; so, as I said as a prelude to this discussion, so the backbone of modern portfolio theory which is due to Markowitz involves two parameters, and these two parameters these are return and risk.

(Refer Slide Time: 07:18)



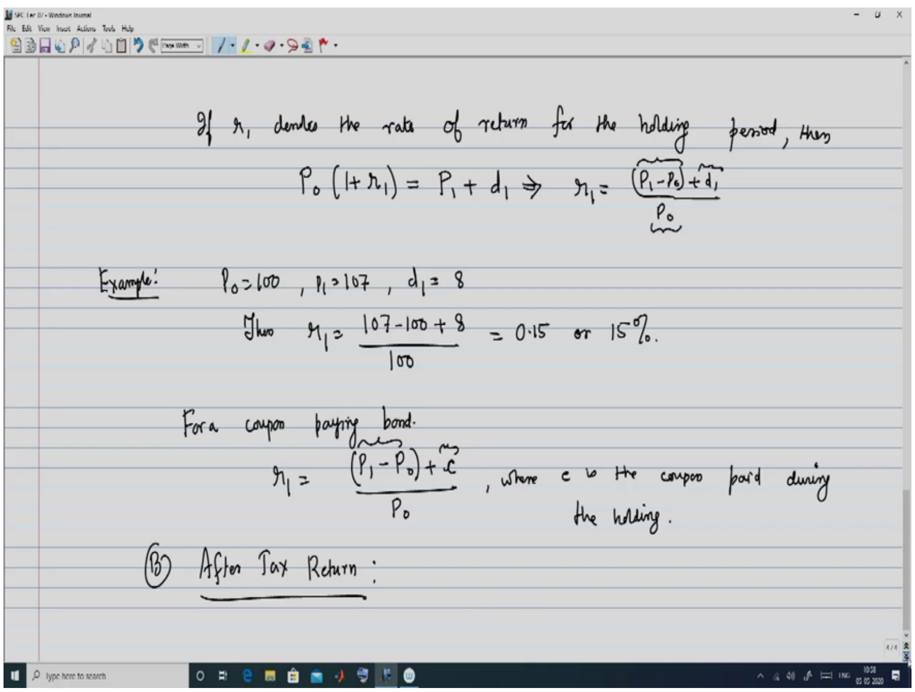
So, before we start the discussion on return on risk. Let me enumerate some of the behavioural assumptions. And these behavioural assumptions are the ones which drive the portfolio theory and these are enumerated below. So, by behavioural assumption I basically mean that what is the thought process that goes on in the minds of a rational investor while deciding on how to make an appropriate choice of the portfolio. So, accordingly let me enumerate all of them. So, the 1st behavioural assumption that I take into account here is that; all investors assess each investment opportunity from the point of view or the paradigm of the probability distribution of returns. Secondly, an investors assessment of the risk associated with an investment opportunity is proportional to the variability in the returns. Thirdly, investors decision are mainly based on the expected risk and; sorry this is expected return just make this correction and risk statistics. And 4, for a given level of risk the investors prefer higher return to lower returns. So, let me explain this one by one. So, let us look at point number 1. The point number 1 says that all investors they will basically make an assessment of each of the investment opportunity from the point of view of the probability distribution of returns. So, as you can see that, whenever you are looking at returns a return is basically the percentage gain that you make over your initial investment and that is going to be a random variable, and consequently obviously it is going to have a probability distribution. So, the first thing that the investors take into account is the probability distribution of the returns. Secondly, they also need to make an assessment of the risk that is associated and the risk is seen as the variability. That means, how much fluctuation does there exist in terms of the returns about its mean value and this is what is known as the volatility or the variability of the returns. So, that is another factor that needs to be taken into account; whether the particularly asset returns are extremely volatile or less volatile. And consequently as a result of the first two assumptions the investor decisions are mainly based on what is the expected return and what is the risk statistics. And as you in later in todays discussion we will talk in a little more detail about what we mean by expected return and the risk statistic. And finally, and this is an obvious rational way of looking at it that for a given level of risk; so if you have two assets so the identical amount of risk the investor is obviously going to go for the one which offers a higher return as compared to the one that has lower returns, ok. So, let us now formally introduce the concepts of returns and risk. So, we have already seen in the last class, we have already some idea about what a return is about and we had looked at returns earlier also when we were talking about bonds, but now let us put it in terms of a formalized notational structure.

(Refer Slide Time: 12:40)



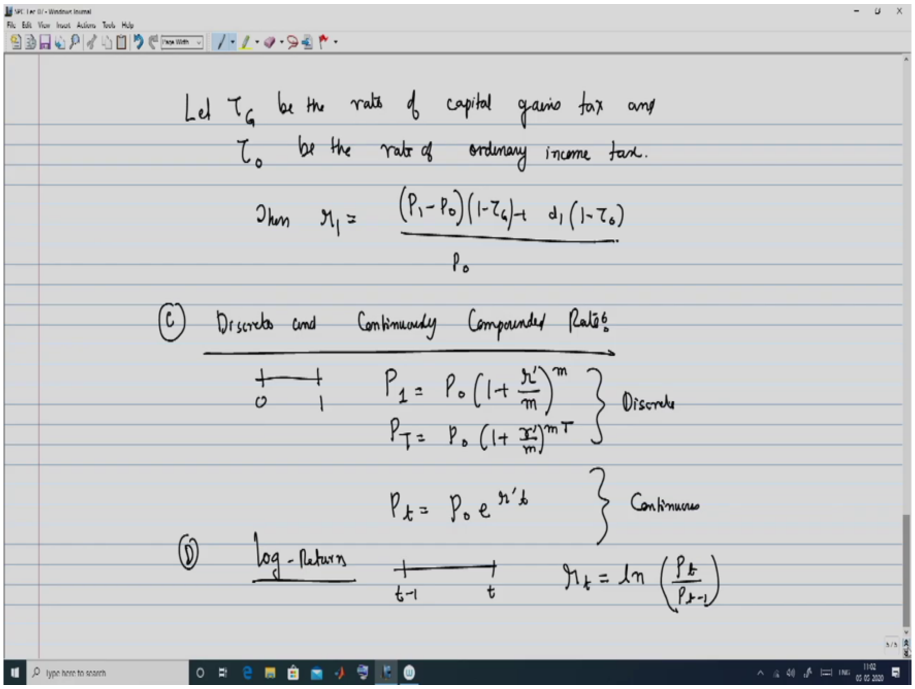
So accordingly, we will talk first about the various rates of return. And please understand that we will view this, view the one-period return and present several alternatives for this. So, for the time being what we will do is that we will look at a one-period setup; remember that in the binomial model we looked at a multi-step binomial model. So, if you remember that we started off with the case where there is just a single step. So, we look at the single step in terms of returns which then can be extended in case of multi-step and there are ways of optimizing this such as dynamic programming in the multi-step setup. But for the point of view of the mean variance theory that is the Markowitz framework we will primarily restrict ourselves to what happens in a single period, and we can repeatedly make use of that one at a time. So, if you start at some time $t = 0$ and at the end of the first period we will reassess and then again use the theory for a single period model up to time $t = 2$ and so on. So, to begin with as I have mentioned here that we will talk first about what is the different concepts of returns in a more formal way and we will introduce the notation for the same. So accordingly, we first of all we will talk about what is known as the holding-period-return; that means, the period for which you are in ownership of thus the asset. So, accordingly let me introduce some notations. So, I will say let P_0 this be the price at which a stock and by stock I basically mean a risky asset in general. So, this is the price for which a stock is purchased by an investor at the beginning of the holding period. Then let P_1 be the price at which the stock is sold by the investor at the end of the period. And this period could be for example, say 1 year. Now during this period what is d_1 going to be? It is going to be dividends received, ok. So, basically these are the cash transaction that happens during the holding period.

(Refer Slide Time: 16:00)



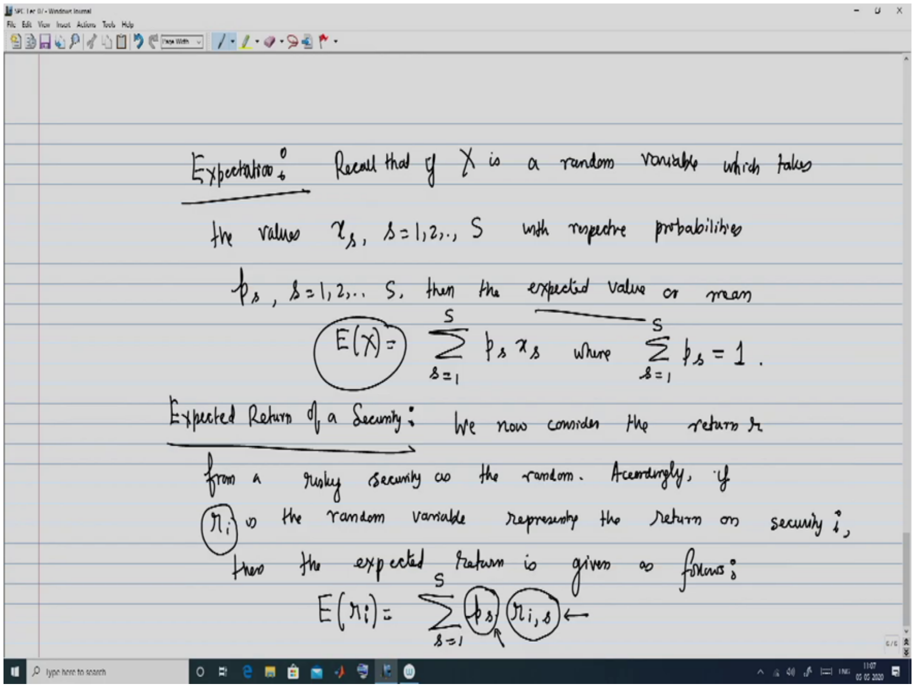
Then, if the notation r_1 denotes the rate of return for the holding period that is from 0 to 1; then what does it mean? So, this means that if you make an investment of P_0 at the end of the period it will grow by a factor of $1 + r_1$ and this is the same as the final amount of money P_1 that you have received plus d_1 which is your dividend return. And this implies that $r_1 = (P_1 - P_0)/P_0$ and on the numerator I have a plus d_1. So, this means that it is that $P_1 - P_0$ is the excess amount of money that you get as a result of selling the stock and d_1 is the dividend. So, this is the total net (Refer Time: 17:01) that you have during the entire period and this as a proportion of the initial investment P_0 this is what is known as the return. And this return money that the amount that you get it is if you multiply it by 100 then you basically get the equivalent return in terms of percentage. So, just a little example sort of very straightforward example; that if you have purchased the stock for 100 and at the end of 1 year you sell it for 107 and during this 1 year period you have received a dividend of 8. Then, the return is going to be $P_1 - P_0$, so that is 107 minus 100 plus the dividend which is 8 divided by the original amount of 100 which is equal to 0.15 or if you multiply it with 100 this becomes equal to 15%, ok. So, what we have now talked about the return for a stock. So, in case of a bond say you have a coupon paying bond. So again the same thing, suppose that you purchase the bond for an amount of P_0 at the end of 1 year you sell it for P_1 and in the intervening period you receive a coupon of c, then your gain is going to be $(P_1 - P_0 + c)/P_0$. Where, c is the coupon paid during the holding period. So, we have given the basic definition for return in case of a stock and a bond for a single holding period, ok. So, now let us now look at the next thing that we look at is what is known as the after tax return. So, we will now just we now generalize the definition of returns for the holding period to include tax liabilities that arises.

(Refer Slide Time: 19:12)



So, for the after tax return I will introduce two notations. So, let τ_G be the rate of capital gains tax and τ_0 be the rate of ordinary income tax. So, just make this correction this is τ_G, so G state for capital gains. So, capital gains in the context of the discussion is basically the amount or the difference in the amount of money that you get as a result of selling the asset. So, if you purchase this asset for P_0 and sell the asset for P_1 then your capital gains income is going to be the difference between the two and this is taxable in many countries. And τ_0 is the rate of ordinary income tax. So, basically this is the tax rate that is applicable to your regular income and dividends are considered as regular incomes. So accordingly, we now need to modify the rate of return to incorporate these two tax rates. So, this gives us that the rate of return r. So, as before this is going to be $(P_1 - P_0 + d_1)/P_0$; but please remember that on the amount of $P_1 - P_0$ you have to pay at capital gains tax at the rate of τ_G so that means, the amount of money that remains with you is $(P_1 - P_0)(1 - \tau_G)$. And also for the dividend you have to pay the income tax at the ordinary rate. So that means, on the amount of d_1 after having paid the ordinary income tax at rate τ_0 you are left with an amount of $d_1(1 - \tau_0)$, ok. And the last thing that I want to talk about is this is discrete and continuously compounded rate. And this is something that I had broached upon in the last class, but I will just recall this. That if we have two time periods 0 and 1 then your P at 1 is going to be P_0 and if the r is the annual interest rate then this is and it is paid m times year, so this is going to be $P_0(1 + r/m)^m$. And then in general if it is 40 number of years this is going to be $P_0(1 + r/m)^{mT}$. And the continuous version of this is going to be that. So, that this is the discrete and the continuous version for this as you had done in the previous class is going to be that generically $P(t)$ this is going to be $P_0 e^{rt}$ or some $r't$. So, denote this by say r' just to distinguish this, ok. And there is just one more return that I want to talk about and this is what is known as the log return. So, if you for example, have consider a time window say $t - 1$ and t two consecutive time point, then the return rt which is the log return is given by the natural log of price of the asset at time t over price of the asset at time $t - 1$, alright. So, now, that we have defined what a return is now and we recognize the fact that the return is a random variable, because you know the initial price, but you do not know what the price of the asset is going to be at the time you sell it off.

(Refer Slide Time: 23:41)



So accordingly, we need to talk about not just returns, but we need a single number which indicates what you should expect from a particular asset in terms of returns. And this brings us to the concept of what is the expected return. So, accordingly we first recall the definition in the context of this discussion. So, recall that if X is a random variable which takes the values $x_s,\ s = 1, 2, \cdots, S$ with respective probabilities $p_s,\ s = 1, 2, \cdots, S$. Then the expected value or mean of the random variable $E(X)$; that means, the expected values

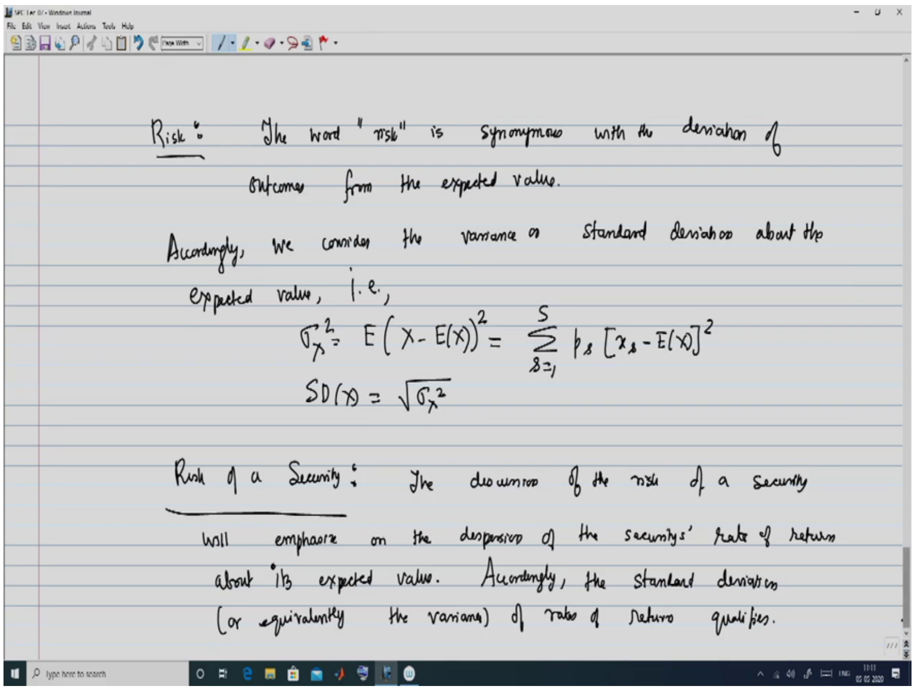
$$E(X) = \sum_{s=1}^{S} p_s x_s, \quad \sum_{s=1}^{S} p_s = 1.$$

So, now let us look at extend this notion to the expected return of a security; and obviously, I mean mostly in the point of view of risky security. So, we now consider the return r from a risky security as the random variable. So, here I am choosing the candidate for x to be the return from any particular asset. So, accordingly what do you do is that. So, this so, now, I am I will now extend this definition of $E(X)$ here to that of the return of a security. So accordingly, if r_i is the random variable representing the return on security i. So, I choose some generic security which I call the i of security, then the expected return is given as follows. And this is given by expectation of the random variable which is r_i for the i of security this will be given by

$$E(r_i) = \sum_{s=1}^{S} p_s r_{i,s}, \quad s = 1, 2, \cdots, S.$$

So, note that here that this random variable r_i will take possible values of r_i's with the corresponding probability of p_s. So, this is something like that this is going to be this is the random variable and this random variable is the return of a stock price. And, I am saying that under S number of different scenarios; that means, so if I choose a generic scenario say s, then for that the random variable r_i will take the value of $r_{i,s}$ with the corresponding probability of p_s. So, an example for this could be that this $s,\ s = 1, 2, \cdots, S$; so, that this S number of things could be the state of the economy. So, in particular if you go back to the binomial model you can actually see this p_s these are going to be p and $1 - p$. And the return $r_{i,s}$ there are two possible values, that the namely your u and d given by the upward and the downward movement. So likewise, you could have another model where you basically have a much larger number of such possible values of $r_{i,s}$. However, as we will later see that from the practical implementation point of view it is very difficult to figure out what this p_s are going to be. So, in practice we will see towards the end of this lecture that the an alternative way of calculating the expectation is going to be using the historical data, ok. Now, coming back to this general discussion; so, here I have calculated the expected data. So, now, once we have talked about expectation here and the expected return of the security.

(Refer Slide Time: 28:59)



So now, we move on to the next pillar of the Markowitz model namely the risk. So, here in the context of the discussion the word "risk" is synonymous with the deviation of outcomes from the expected value. So accordingly, if I again look at the counter part of this definition of expectation and then I will extend this specifically to risky security. So accordingly, we consider the variance or standard deviation about the expected value; so, that is

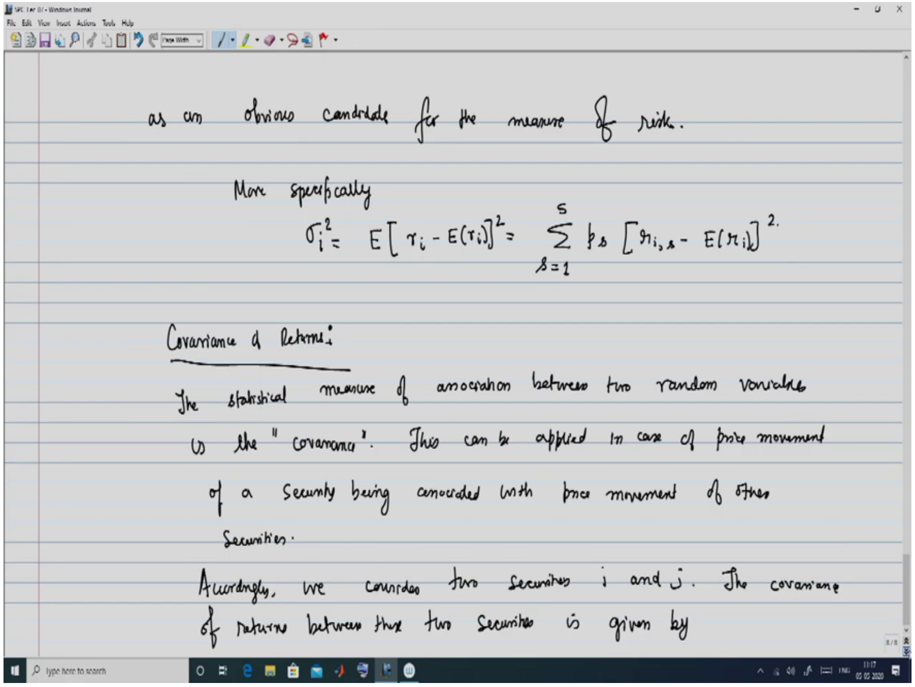
$$\sigma_s^2 = E(X - E(X))^2 = \sum_{s=1}^{S} p_s[x_s - E(X)^2.$$

$$SD(X) = \sqrt{\sigma_X^2}.$$

And now we are in a position to talk about the risk of a security. So, the discussion of the risk of a security will emphasize on the dispersion of the securities rate of return about its expected value. Accordingly, the standard deviation or equivalently of the variance of rates of return qualifies as an obvious candidate for the measure of risk.

(Refer Slide Time: 32:25)

So, just to sort of elaborate a little bit on this; what I mean is that as we have earlier identified that the deviation from the expected value which is the average value is an indicator of how volatile the movement of the random variable is. And in this context we are worried about the volatility of the return. So accordingly, the obvious choice for this is that we look at the deviation of the return from its expected value which we have just now calculated and take the square of it. So, that is and take its expectation which gives us the variance. So, what we will do is that we will mostly be using the variance or equivalently the standard deviation as the measure of risk. So, please take into account that we will use both the standard deviation



and the variance depending on the context as the measure of risk. And it will become obvious from the discussion, from the context of the specific discussion as to which of these two are being used as the measure of risk at that point of time, ok. So, then this gives me that. So, more specifically, so now, again I will now put a formula for the risk. So, more specifically the variance of the i return of the i-th asset which are denote by σ_i^2. This is going to be by definition of variance $E(r_i - E(r_i))^2$. And this in terms of probability this is going to be $\sum_{s=1}^{S} p_s[r_{i,s} - E(r_i)]^2$, alright. So, far we have looked at the first two moments of the returns of the assets and accordingly we have defined what is the expected value of r_i; that is $E(r_i)$ and the variance that is σ_i^2. Now, notion of using the variance as a measure of risk is based on the premise that the risk of an asset is primarily or essentially driven by the characteristic of that particular asset. However, in practice the risk of a particular asset is not exclusively driven by factors that are unique to that particular asset, but also they are driven by the behaviour of the market and by extension by the behaviour of the other assets that are in the market. So, we also need to take into account the behaviour of a particular asset vis-a-vis the behaviour of the other assets in the market or equivalently we need to look at the behaviour of the returns of the i-th asset with every other j-th asset that are available in the market. So, accordingly we are now in a position to (Refer Time: 35:51) start talking about what is going to be the covariance of the returns of assets. So, we start off with this definition of covariance of returns. And I will just briefly note down the motivation that I have discussed just now. So, the statistical measure of association between two random variables is the covariance. This can be applied in case of price movement of a security being associated with price movement of other securities. So accordingly, we consider two securities which will identify as the i-th and j-th security.

(Refer Slide Time: 37:54)

So, the covariance of returns between these two securities is given by

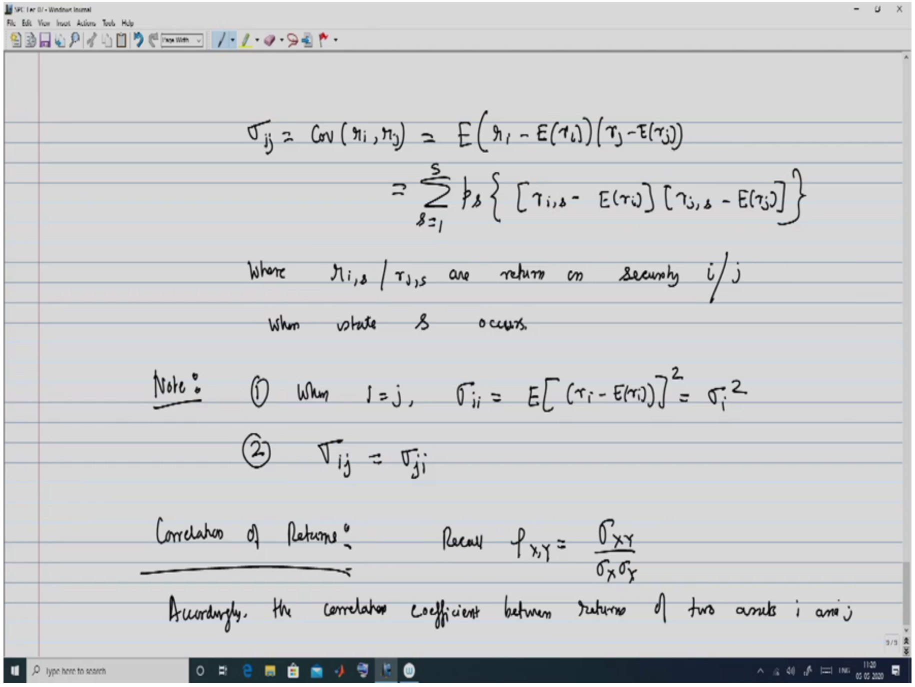
$$\sigma_{ij} = Cov(r_i, r_j) = E(r_i - E(r_i))(r_j - E(r_j)).$$

And, this in terms of the probabilities is going to be

$$\sum_{s=1}^{S} p_s[r_{i,s} - E(r_i)][r_{j,s} - E(r_j)].$$

So, just a couple of observation that follows. First of all is that, when $i = j$, then obviously,

$$\sigma_{ii} = E(r_i - E(r_i))^2 = \sigma_i^2.$$

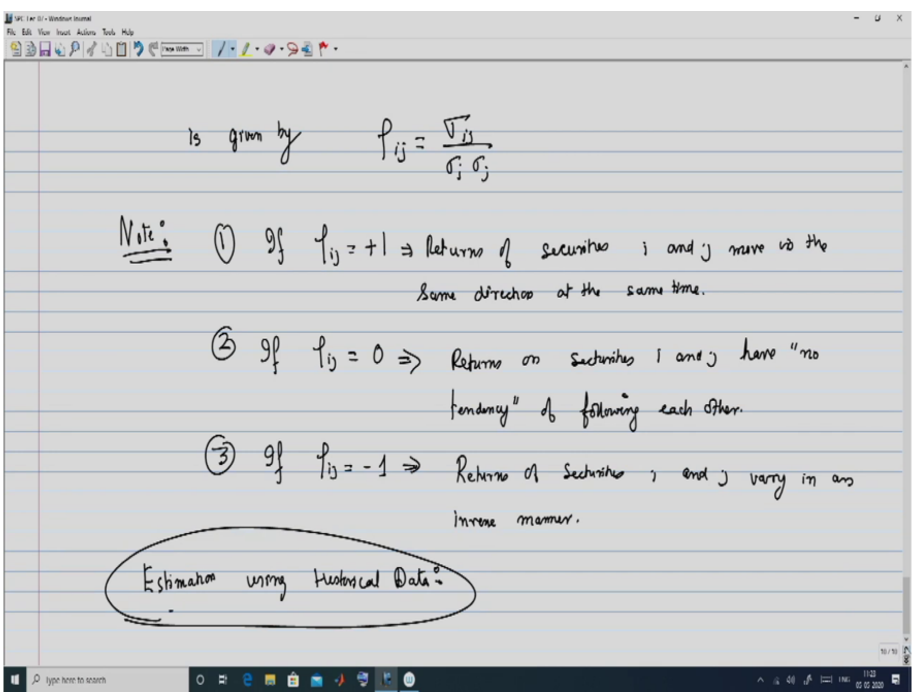


And secondly, using the symmetric property of covariance

$$\sigma_{ij} = \sigma_{ji}.$$

That means, the covariance of return of the i and the j-th asset is the same as the covariance return of the j-th and the i-th asset, alright. And so, we now come to the last concept which and the one that is a fall out of the covariance of returns and that is the correlation coefficient. So, the correlation of returns; so, recall that the correlation coefficient ρ_{xy} or between two of random variables x and y is $\frac{\sigma_{xy}}{\sigma_x \sigma_y}$.

(Refer Slide Time: 40:39)

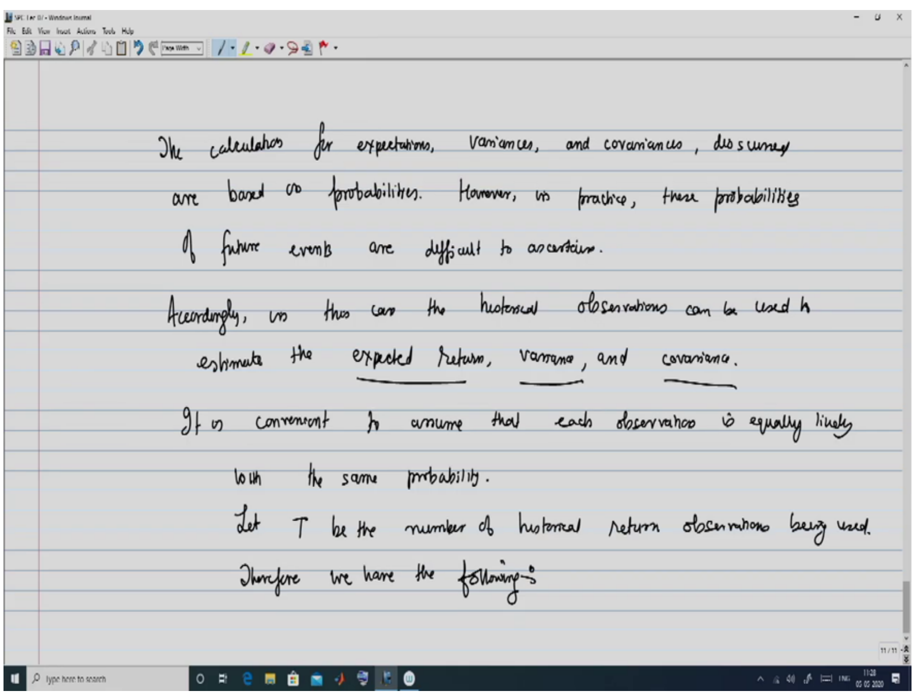


So accordingly, the correlation coefficient between returns of two assets i and j is given by. So, using this I will denote this by notation

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}.$$

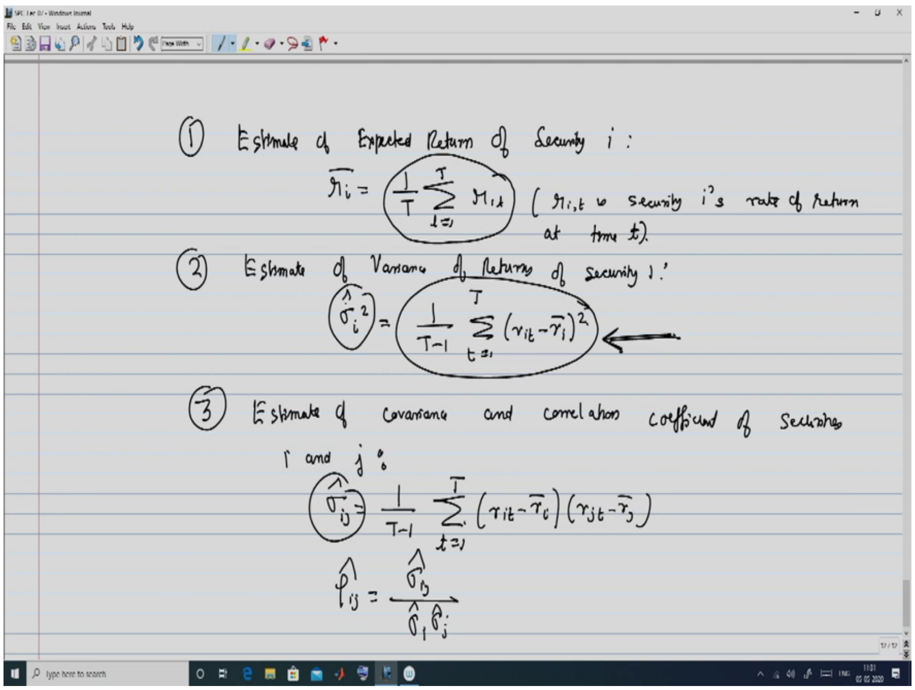
So, if $\rho_{ij} = +1$, then returns; so, this implies that returns of securities i and j move in the same direction at the same time. The 2nd observation; and again these are similar to the observations that I had made earlier. So, when $\rho_{ij} = 0$, this implies that returns on securities i and j have no tendency at all of following each other. And thirdly, if $\rho_{ij} = -1$, this implies that returns of securities i and j vary in an opposite manner or a inverse manner. So, these three observations are similar to the interpretation of $\rho = \pm 1$ and $\rho = 0$, when we were discussing and when we are introduced earlier the definition of a correlation coefficient between two random variables, ok. So, now we come to the last topic for today's lecture that is calculating the expected returns and the variance the covariance and the correlation coefficient based on historical data. So, we talk about estimation using historical data. So, let us see how we can make the historical data. So, by historical data I basically mean that the stock prices or the asset price movements from the past and you can view this as the closing prices at the end of each day. And the return is going to be given by the definition of the return that we had introduced earlier in that class without the dividends we taken into account. And later on this can be expanded to take the dividends into account. So, now you see that the reason why you have to do this is that the definition of expectation involves certain probabilities, and those probabilities are in practice very difficult to ascertain and so are the random variables associated with those probabilities. So, historical data makes the assumption that the possible returns over the say next one day is going to have possible values amongst the returns that are observed in the past say preceding 500 days of specific 200 days. So that means, that the random variable $r_{i,s}$ for r_i, this potentially are values that were observed in the past. And the other assumption is that the corresponding probabilities for each of those values occurring again is taken to be all identical. Of course, in a more generalized setup sometimes these probabilities can be designed in a way that more weightage is given in the calculation of the expectation, more weightage is given to the returns of more recent times as compared to the more distant times.

(Refer Slide Time: 44:59)



So, just to note down what this is a motivation. So, the calculations for expectations of this assets, variances and obviously equivalently standard deviation and covariances and by extension correlation coefficients; already discussed are based on probabilities. However, as I have already mentioned from a practical point of view these probabilities; that means, the probability p_s of future events are difficult to ascertain. So accordingly, in this case the historical observations can be used to estimate the expected return, variance, and covariance. So, it is convenient to assume that each observation from the past is equally likely with the same probability. Let T be the number of historical return observations being used. So, this means that they are actually $T+1$ number of historical data points and which will give us T number of historical observations. So, this means that the probability if all the such observations are likely to be repeated and are taken as the random variable $r_{i,s}$, then the assumption of them have likely to happen with equal probability. Means that, each of those probabilities is going to be 1 divided by the number of observations of return that is T; so that means, each of the probabilities is going to be $1/T$. So accordingly, based on the historical data we can calculate. So therefore, we can calculate expected return variance and covariance. So accordingly, we have the following three things or actually four things.

(Refer Slide Time: 48:29)



The first one is the estimate of expected return of security i, which i will denote by

$$\bar{r_i} = \frac{1}{T}\sum_{t=1}^{T} r_{i,t},$$

where $r_{i,t}$ is security i's rate of return at time t. Secondly, we look at estimate of variance of returns of security i. This i will denote by

$$\hat{\sigma}_i^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r_i})^2.$$

So, here I want to briefly pause and just point out that this expression that is used to estimate the expected return, this is an unbiased estimator of the mean. And this estimator that is being used to estimate the variance, where we have a factor of $1/(T-1)$ instead of $1/T$, that is also an unbiased estimator of the variance. So, if you look at if you for this you can look up any statistics text on unbiased estimators. So thirdly, the estimate of covariance and of course, we should not forget the correlation coefficients of securities i and j. This is given by $\hat{\sigma}_{ij}$ for the estimator, again 1 divided by $T-1$ summation. So, I will follow the same pattern as they are unbiased estimator for variance. So, this will be

$$\hat{\rho_{ij}} = \sum_{t=1}^{T}(r_{it} - \bar{r_i})(r_{jt} - \bar{r_j}).$$

and

$$\hat{\rho}_{ij} = \frac{\hat{\sigma}_{ij}}{\hat{\sigma}_i\hat{\sigma}_j},$$

which have been estimated here, ok. So, just so, this gives us basically a historical weight the way to make use of the historical data to estimate the basic data sets that are needed in order to move on to a portfolio optimization process; namely, an estimate of the expected return, estimate of the variance, and the estimate of the covariance on equivalently of the correlation coefficients. So, just to sum up what we have discussed in today's class. We started our discussion on the modern portfolio theory and we identified that the two important characteristic of the modern portfolio theory are return and risk. And accordingly we have defined what is the or return and what is the risk in case of the returns of risky securities. And then we also defined what is going to be the covariance of return between the two risky securities and the correlation coefficient, which are necessary to identify the behavioural pattern of a risky security. That is in terms of the risk of the particular security i in the paradigm of movement of the risky securities. The other risky securities that are available in the market. And finally, identifying the difficulty in using the basic definition of these three parameters namely, the expectation, the variance, and the covariance slash correlation coefficient; we have given three approaches to estimate all this all of them making use of the historical data of all the securities. So, in the next class we will extend more of the discussion and we will make use of the properties of expectation and variance to introduce our discussion of determining what is going to be a best portfolio. And first we start looking at a two asset portfolio and then we will look at a more generalized case.

Thank you for watching.

(Refer Slide Time: 53:26)