Mathematical Portfolio Theory

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Module – 08
Applications with market data
Lecture – 03
Portfolio optimization with constraints, Value-at-Risk: Estimation and backtesting

Hello viewers, welcome to the last lecture of this NPTEL MOCC course on Mathematical Portfolio Theory. In today's lecture we will continue our discussion on the usage of MATLAB in portfolio theory.

And we will look at two main topics: one on Portfolio optimization using constraints. So, in particular we will look at the two key constraints; one which will put a constraint on the minimum weight that has to be assigned to each asset being considered for inclusion in the optimized portfolio.

And the other one that is going to impose constraints in terms of cardinality; that means on the number of assets that has that can be assigned to the portfolio, in particular what is going to be the maximum number of assets and what is the minimum number of assets that are allowable for inclusion in the portfolio.

And the second problem, we will talk about a recently discussed topic of value at risk and we will look at three approaches, namely the normal distribution approach, historical simulation and the estimated weight averaging method in order to estimate the value at risk. And then we will talk about how we are going to do a back testing to ascertain the validity and the effectivity of the model.

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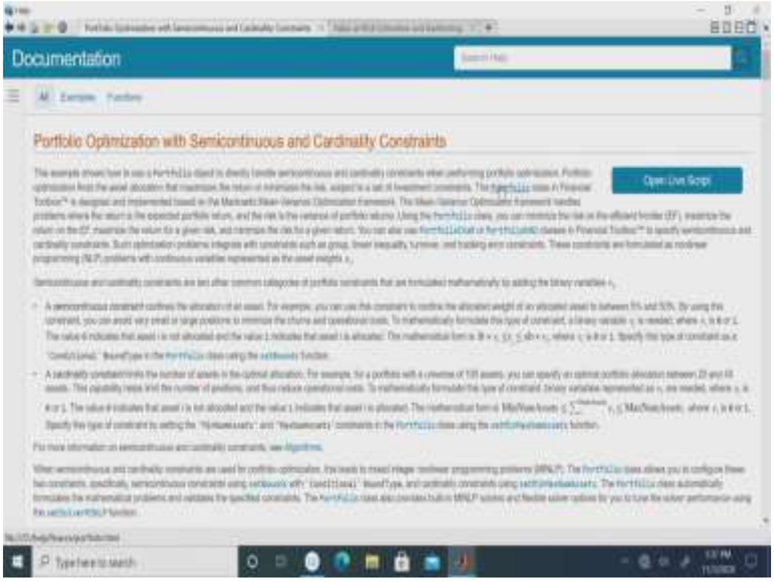

So, we begin with the first of the two topics for today's class. So, the first topic that we will do is on portfolio optimization with semicontinuous and cardinality constraints.

So, here what we will do is, we will look at an illustrative example in order to handle the constraints that can be imposed on a portfolio optimization problem formulation. And the two constraints that we are going to look at, essentially is one will be a constraint on the weights and the other is going to be a constraint on the number of assets that are being included in the portfolio.

So, here of course, we will again go back to the portfolio class, where we will look at the asset allocation with the goal of maximizing the return and or minimizing the risk. And of course, now along with the basics the setup of maximizing return or minimizing risk, we will now subject these two certain investment constraint.

So, let us now look at this in the paradigm of these two problems. So, the two categories that we have is the first one is the semi continuous constraint. So, what you do is this in this constraint is that, we will confine the allocation of an asset. So, to mathematically formulate this, we basically say that, we will have a binary variable v i which is going to take the value of 0 or 1.

So, when v i is equal to 0; this means the asset has not been allocated and when v i is equal to 1, that means the asset has been allocated. So, x i is essentially the weight of the ith asset. Now, if your v i is equal to 0; that means x i will lie between both 0 and 0 both inclusive.

So; that means, x i is anyway going to be equal to 0. And when your v i is equal to 1; that means the asset has been included. So, a i will lie between lower bound l b into v i into upper bound into less than or equal to upper bound into v i. So, that means, the weight of the ith asset in case v i is equal to 1, that is going to lie between the lower bound l b and upper bound l v that you have decided, upper bound u b that you have decided to permit for this particular asset.

And remember that this is the cash asset; this is the case when your v i is equal to 1 and of course, when v i equal to 0 as I have already mentioned, your x i is going to be equal to 0.

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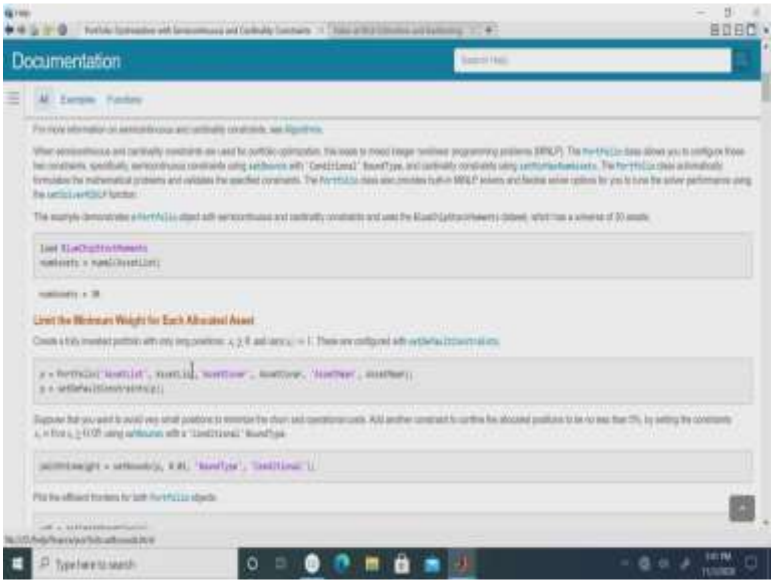

Also the other constraint that we have is the cardinality constraint. So, as the name suggest, it puts the limit on the number of assets in the allocation. So, what happens is that, if you are considering a universal set of assets.

So, for example, 100 assets or so and in that case, it is very difficult to actually make an allocation of assets in those 100 assets. Given the resources that, it will take up, both in terms of the computational cost as well as the transaction cost. So, accordingly you would ideally like to have a portfolio with a lesser number of assets.

So, you can specify your optimal allocation between a certain number of assets. So, for example, it is customary to have the number of assets, where you have made the allocation to lie between somewhere between 20 and 40. So, mathematically this can be written as summation of v i's i equal to 1 to number of assets.

So, here as you have pointed out v i's either going to be 0 or 1; so that means, the summation of v i i is equal to 1 to number of assets this will be less than equal to the maximum number of assets and this is going to be greater than or equal to the minimum number of assets, which I will denote by max num assets and min num assets.

So, let us now start our discussion on this. So, remember that in the earlier lecture, we talked about the BlueChipStockMoment. So, we again recall and load that BlueChipStockMoments. So, here a num assets will give me the number of assets, which in this turns out to be equal to 30. So, this is essentially a universe of 30 assets. Now, we first begin with a scenario where we limit the minimum weight for each allocated assets.

So, we carefully invest in a portfolio and have only long position. So, this means that, I will have only long positions; that means I will disallow short selling. So, in this case my x i is going to be greater than or equal to 0 and of course, the customary constraint, that the sum of the weights that is sum of x i is going to be equal to 0 and these are already included in the configured with the set default constraints.

So, I start off with my p as my portfolio, where I have the asset list along with the moments, namely the asset covariance and the asset means; so that means I have the expected vector and the I have the covariance matrix as well as I have the AssetList and to which I set the default constraints.

Now, on top of that, now I want to limit the weights. So, the first constraint in the context of this lecture will happen now. So, suppose that you want to avoid very small positions. So, one illustration of this could be that, you should confine yourself that to the constraint that there should not be less than 5 percent allocation for each asset.

So, this means that you set the constraint that, either your x 1 equal to 0 or x i is equal to 0; which means that the asset is not being included in the portfolio. And in case it is included in the portfolio, then your x i has to be greater than or equal to 0.05 or 5 percent. So, accordingly you now then set the bounds on p to be 0.05.

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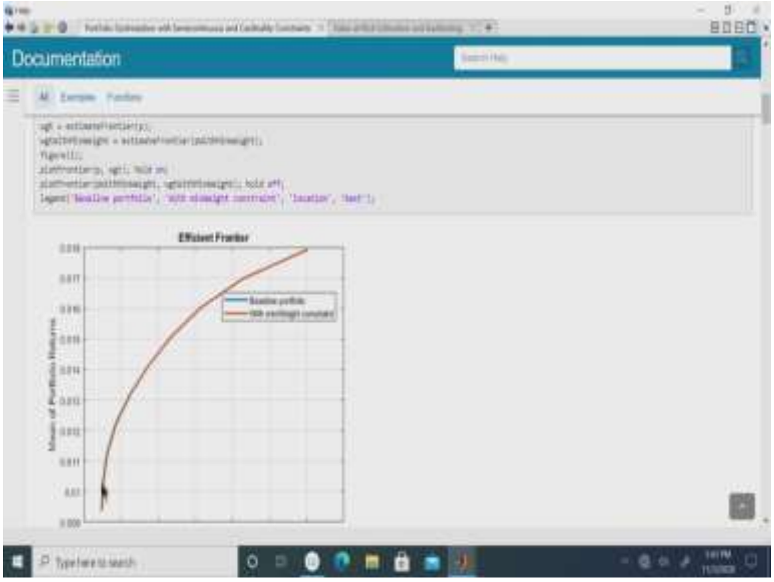

So, having this done this; so, what you do is that, we now calculate the efficient frontier. So, the efficient frontier for the original portfolio p with the default constraint; I will designate that the efficient frontier with wgt.

And then the efficient frontier with the minimum weight; that means the weight which has the one with the constraint of the weights being at least 5 percent, that is that efficient frontier in that particular case is carried out on p with mean weight which has set the bounds on it.

And the corresponding w with mean weight is basically going to give the efficient frontier. So, if you actually saw the first of this, which is basically the unconstrained efficient frontier I will call this as the baseline portfolio and the second one we will call that the minimum weight constraint portfolio.

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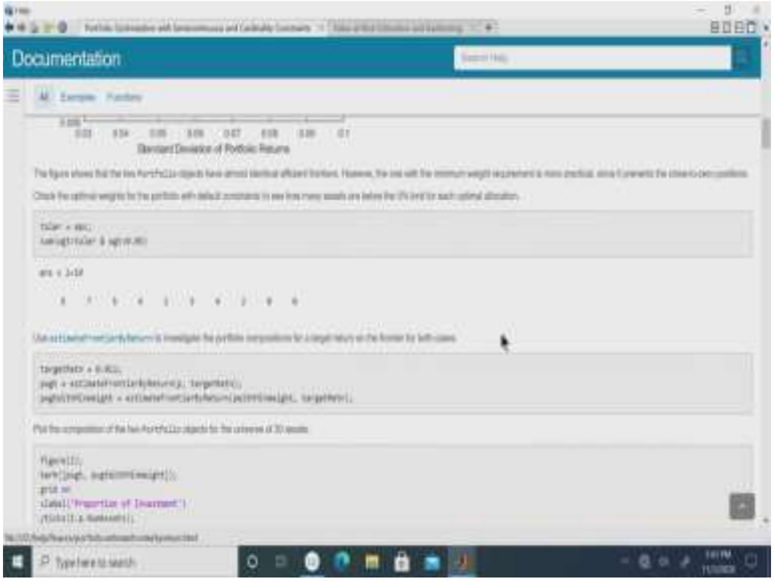

See if you look at the efficient frontier, you will observe very carefully that the baseline portfolio and the one with the minimum weight constraint portfolio; they are almost identical to each other, except at this part where the blue line is slightly above the line for the blue line for the baseline portfolio is slightly above the minimum weight constraints.

Now, with this you can what you can do is that, you can also test for those optimal words of the portfolio default constraints and you can then ascertain also the how many assets are actually below the 5 percent limit for each optimal allocation. So, that means for each of this optimal allocation, this is going to indicate the number of assets that actually fall below the 5 percent limit, ok.

So, now what you do is that, now once you have done this efficient frontier by accounting for the minimum weight constraint; we then now move on to a scenario, where we set a target written on the portfolio for both the cases.

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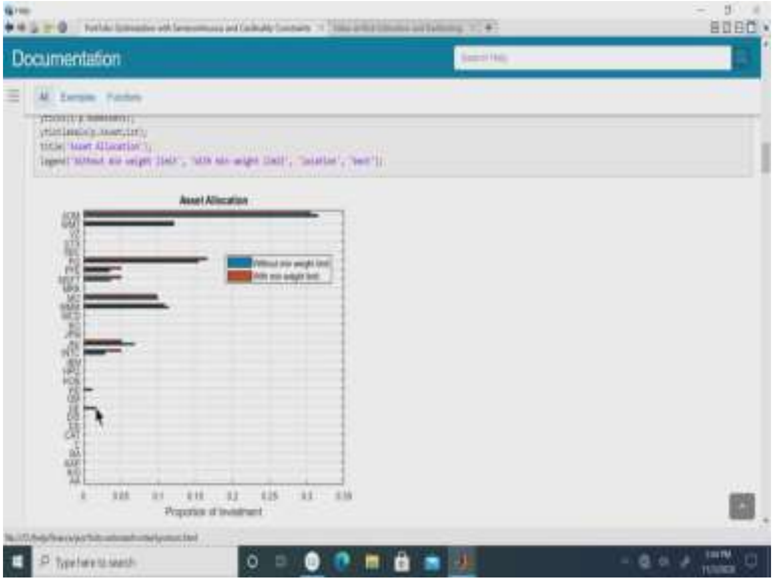

So, accordingly suppose that we set the target return to be 0.011 or 11 percent target return. So, with that target return you can calculate the efficient frontier and then you can calculate the weights. And so, accordingly what happens is that, you can then now look at the asset allocation.

Now, the asset allocation will then be given can be visualized in this particular picture, where for each of the assets which you identified in the previous class; the weight without the limit being imposed that is just only with the default constraints, that is marked in blue and the one with minimum weight limit is marked in orange.

So, accordingly you see that these are the two assets that is the one of which is G; you observe here carefully that these are the two word we have the blue line, but we do not have the orange line, so that means that this does not satisfy the criteria for the minimum weight limit.

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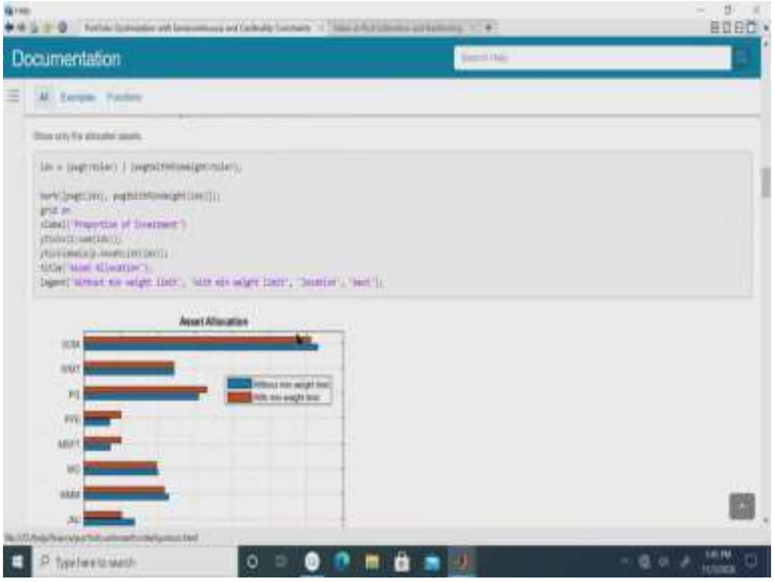

So, now once we have done this; so this basically gives the asset allocation for all the assets that are included in the original data set or the universal set of assets. And now among this we observe that, there are several assets for which that there is no allocation. So, the next thing we can do is that, we can just display the allocated assets. So, accordingly we have done the asset. So, we now enumerate the allocated assets here.

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So, here you observe that we have 1, 2, 3, 4, 5, 6, 7, 8, 9, 1o and 11. So, you observe carefully here that, without the minimum weight limit, you find that we have 11 number of assets; that means 11 cases where we have the blue line, where the asset allocation has been done. And once you put the minimum weight limit, then it reduces from 11 number of assets to 9 number of assets.

So, that means that, in your optimal portfolio without the minimum weight constraint; you originally started off with 30 assets and after the execution of these commands, you observe that you are only left with 11 assets for inclusion in the portfolio.

And upon further imposition of the minimum weight limit; you conclude that there are only 9 assets, which are now eligible for inclusion in the portfolio and where the eligibility is being determined by the minimum weight being at least 5 percent, ok.

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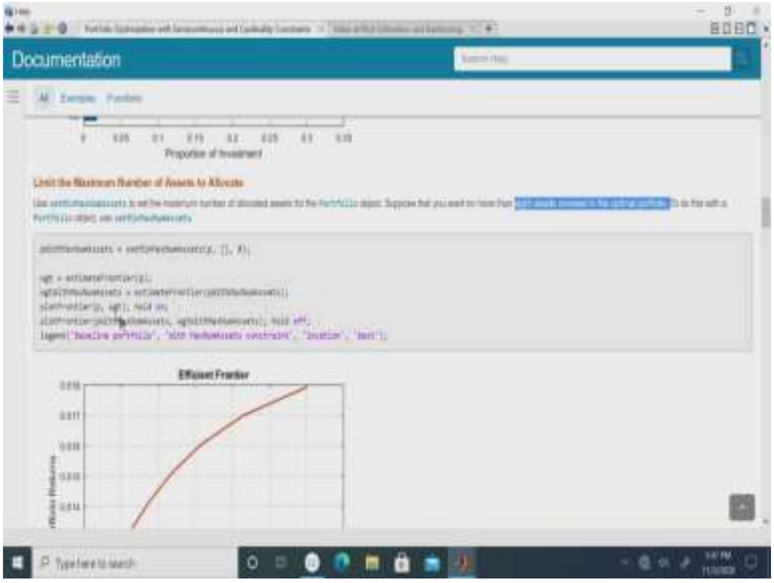

So, this takes care of the first constraint that I had mentioned and are enumerated at the beginning itself. For the second one, we now talk about the maximum number of assets to allocate. So, for this, we will make use of the commands set min set min max num assets. So, what it does is that, it will set the maximum number of allocated assets for the portfolio object.

Now, suppose that for illustrative purpose, we assume that you want no more than eight assets invested in the optimal portfolio. So, this means that the optimal portfolio that you are determining, that is going to comprise of no more than eight assets. So, it is going to be eight or less than or equal to asset.

So, this is accomplished with the portfolio objects inbuilt command of set min max num asset. So, set the minimum and maximum of the number of assets. So, then what you do is that, we will set the min max number of assets on the portfolio object p and this is set to be equal to 8 and this is going to be defined as or the sign to the variable p with max number of assets.

So, what you do now is that, you now develop two efficient frontiers; the first efficient frontier is going to be the original efficient frontier on p, and the second efficient frontier is going to be the one which is going to be with the constraint of the maximum number of assets being eight being imposed on them. So, then what you do is that, then you plot both the efficient frontiers and accordingly you get the two efficient frontiers in both the cases.

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So, the one in the blue is the baseline portfolio and one in the orange is the optimized portfolio, which includes a maximum number of eight assets. So, now, what you do is that, now what you do is once you have this.

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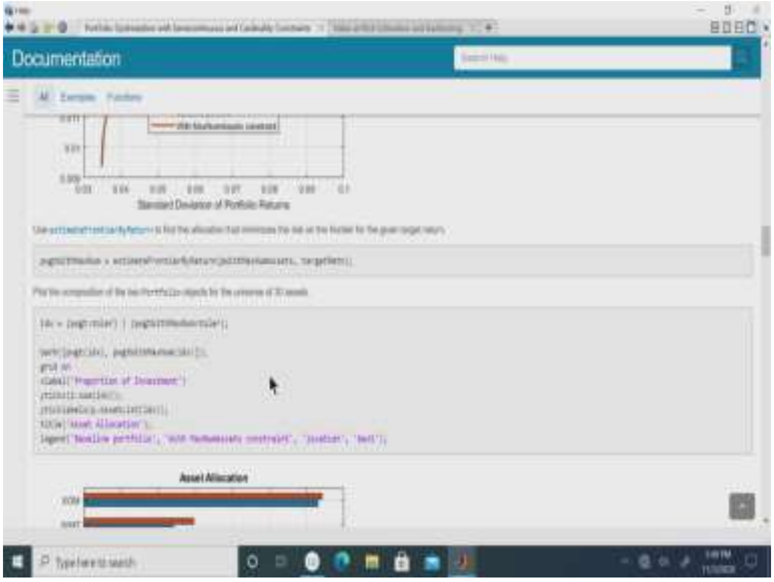

So, we now use the command estimate frontier by return and the goal of this is to add find an optimal allocation, which accomplishes the goal of minimizing the risk on the frontier for a given target return. So, once you have set a certain target return, it is going to essentially figure out what is going to be the portfolio with the minimum risk.

So, that means once I have set this horizontal value and that means, amongst the numerous portfolios that I have along this horizontal value; it is going to identify the one that is has the minimum variance, that means it is going to identify that value on the efficient frontier corresponding to your pre specified level of the expected return. So, accordingly what you do is the following that, you just set the composition of two objects in the universe of 30 assets.

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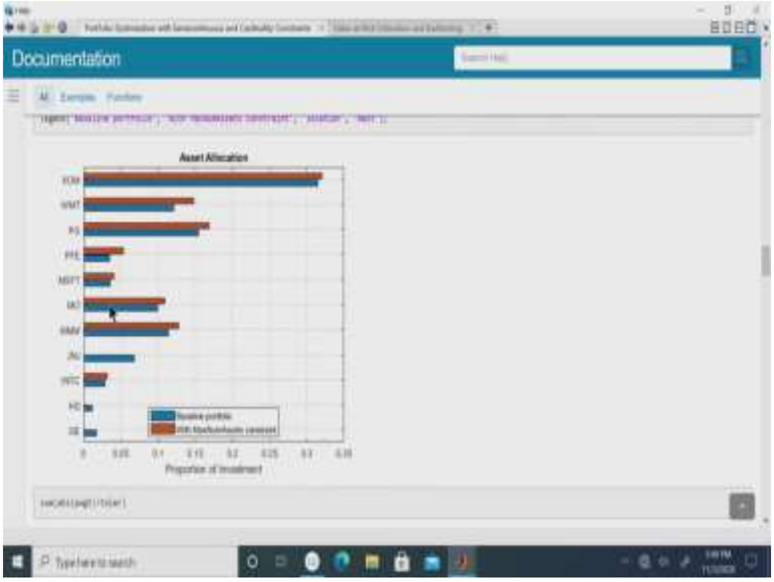

So, again we are doing this on the on 30 assets and the asset allocation now turns out to be of this particular form. So, you see here that a before we have 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11.

So, these are the assets that have been allocated through the 11 assets in the baseline portfolio; but after you have imposed this condition, where the maximum number of assets that can be included in the portfolio to be equal to 8, then you observe that we just have 1, 2, 3, 4, 5, 6, 7 and 8 number of assets which are going to be included in the portfolio.

And of course, you know the value that you have here on the x axis corresponding to this; these are going to give us the exact weight for the allocation, ok.

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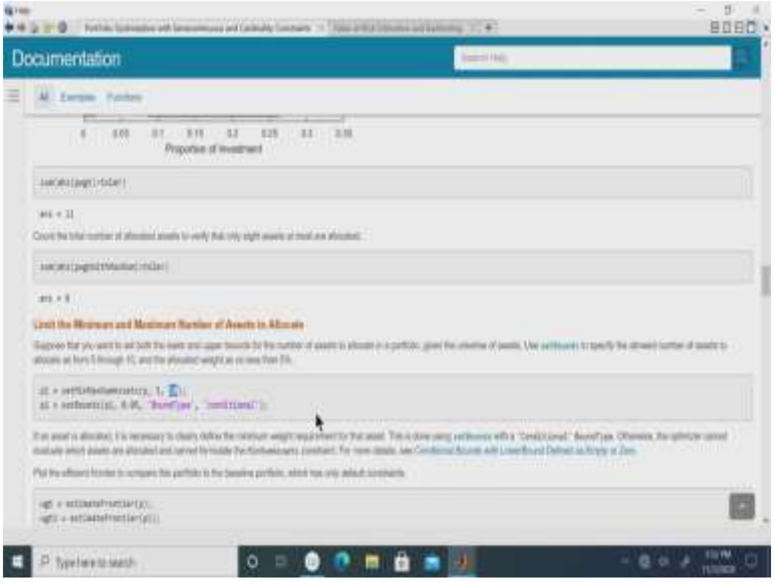

So, now, what you do is the following. So, far what we have done is, we looked at the constraint on the weights and we have set a limit on the maximum number of assets. Now, this has this means that, on the cardinality or the number of assets the bound that has been set is going to be an upper bound.

So, accordingly what you do is that, we limit the minimum and the maximum number of assets to allocate. So, we now set bounds to specify the allowed number of assets to allocate. So, as an illustrative purpose we say that, the number of assets which you can allocate is going to have be a minimum of 5 and is going to be maximum of 10 and of course, with the condition that the weights being no less than 5 percent.

So, what I am doing here is that, here I am taking the first constraint that the weight should not be less than 5 percent; the second constraint on the maximum number of assets. So, which in this case for illustrative purpose, as an example I set it to be 10. And on top of it now, I put an additional constraint on the minimum number of assets that has been allocated in the portfolio.

Now, as you can see that, this is necessary from the point of view that, if you do not have a minimum number of assets that are allocated; you might have a scenario where all your the optimal allocation might actually turn out to be just an allocation into a couple of assets which violates the fundamental principle of modern portfolio theory itself, namely diversification.

So, in this case accordingly. So, we decide that ok, we will have a minimum count on the number of assets to be 5. So, then what we do is that? We will define p 1 and will define p 1 by first setting the bounce on number of assets to be at a minimum level of 5 and a maximum level of 10.

And then on top of it, we will add the constraint that the no weight of, the weight of each of the individual assets which the number of which will lie between 5 and 10, that is not going to fall below 5 percent.

So, now so obviously, if an asset is allocated; it is necessary obviously to clearly define the minimum weight requirement for that asset. So, accordingly that is the reason why we have set this particular bounce.

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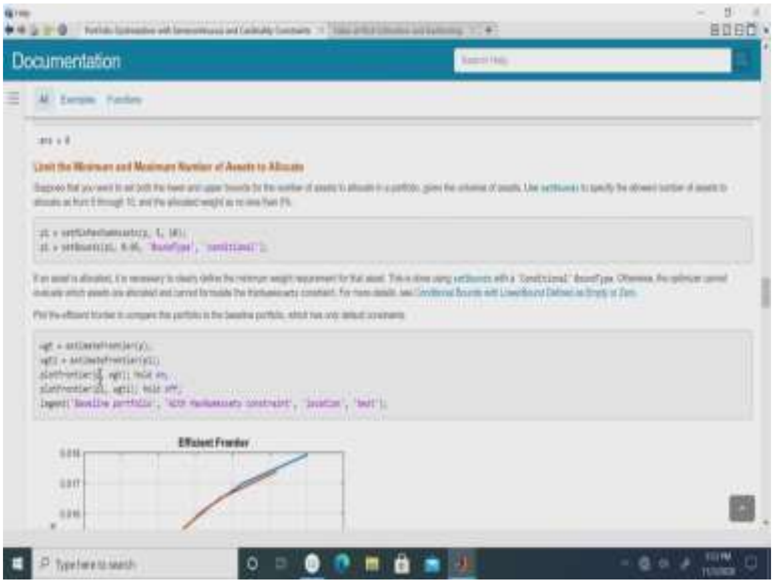

So, now, what you do is that, again we go ahead and we plot the efficient frontier. And we plot the efficient frontier and again make a comparison between a baseline portfolio, the baseline portfolio is the portfolio p. And then we compare that with the newly defined portfolio with these two bounds of 5 and 10 percent, 5 and 10 number of assets and a minimum weight of 5 percent.

So, I estimate the efficient frontier for p which is without any additional constraint, and p 1 which is with the min and max constraints as well as the weight constraints. And I will designate the weights corresponding to each of those efficient frontiers as wgt and wgt 1. And then I will plot those efficient frontiers, so that will be given by plot of p n, wgt and p 1, wgt.

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So, it turns out that, this is going to be the efficient frontier that we have; the blue one again is the baseline portfolio and the orange one is the one with max number of asset constraints. So, this means that, this is the one which is number of assets lying between 5 and 10 and the minimum weight is going to be 5 percent for each asset, ok.

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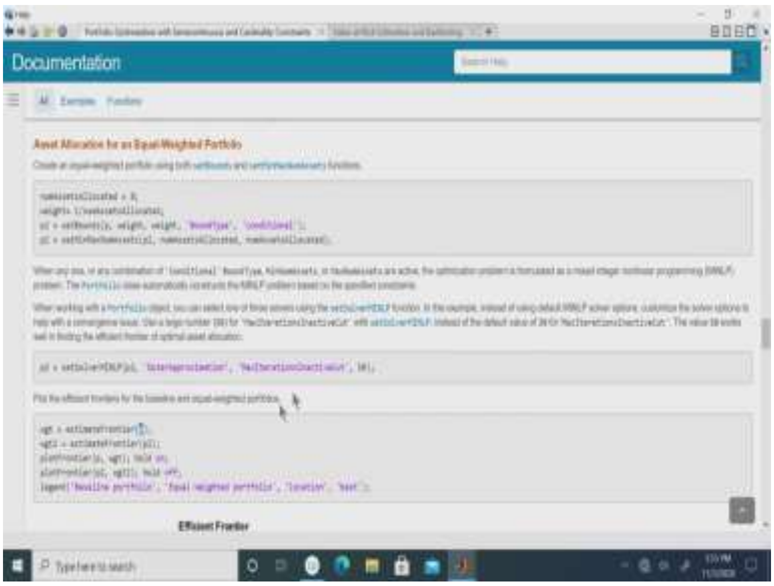

So, now let us look at an equal weighted portfolio. So, what you can do is that, we can create an equal weighted portfolio by using more the setting bounds as well as the min max number of asset functions. So, suppose that we have the number of assets to be equal to 8 and then the weights will be 1 over number of assets to be allocated; so that means each individual asset will have a weight of 1 over 8 and then we send set the bounds.

So, this bounds is set in interesting manner that, you put the bounds to be p weight. So, here if you observe here, we have set the bounds here p 1 0.05; so that means, the minimum weight is going to be 0.05. But here we are setting that the minimum as well as the maximum weight for each asset is simply going to be 1 over 8.

So, also what you do is that, we set the number of assets to be the; again you know is going to the minimum number of assets and maximum number of assets are identical, which is basically going to be equal to 8 that we have chosen. So, what you do is the following that, you know this is something, this is a mixed integer non-linear programming kind of a problem.

So, you have to then solve this using the set solver MINLP; that is for MINLP is for mixed integer non-linear programming problem. So, what you do is that, you set the solver for this and then you again you know estimate the efficient frontier. So, of course, your p remains unchanged, that is the baseline one; but your p 2 now is going to be the solution of this equal weighted portfolio by usage of this method of MINLP.

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So, that is going to give you the second efficient frontier. And so, once you put plot both of them, you get the efficient frontier. So, you observe here again I have the blue line, which is the baseline portfolio and the orange line which is going to be the equal weighted portfolio, which as you would expect is going to be obviously, lies significantly below the baseline portfolio, given extremely restricted nature of this equivalated portfolio, namely that it is essentially just one portfolio where you have all the weights to be equal identically equal to 1 over 8, ok.

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So, now what you do is that, now we have already looked at the problem of specifying a specific return level. So, we now estimate the efficient frontier by this. So, recall that we have encountered this earlier, where we can actually estimate the efficient frontier by specifying the given level of risk.

So, suppose that for illustrative purposes, we set the risk level to be 5 percent. So, in this case, just as we have done in case of target return; so we do the target risk and set it to be equal to 5 percent. So, in this case we estimate the efficient frontier; original efficient frontier with the target risk of 5 percent. And here remember, what is p 2? My p 2 is going to be the solver which makes use of the conditions that have been imposed on p.

So, again we now estimate the efficient frontier for both the baseline portfolio p and this equivalated portfolio p 2; both at the target is that has been set, namely this 0.05 or 5 percent.

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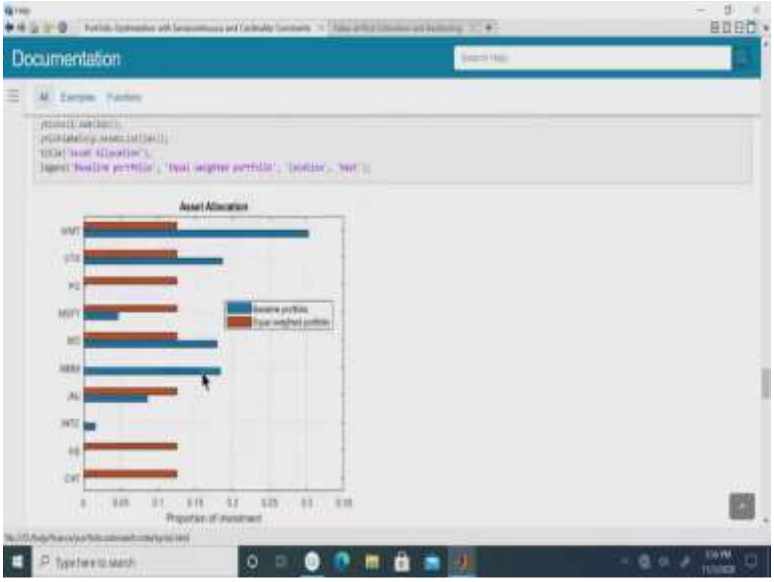

So, as a result of this what you have is that, we have the corresponding asset allocation. So, if you observe carefully; so again I will have 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. So, we have this 10 number of assets that, that actually show up here. And if you observe carefully here that, the blue ones which are the baseline values; so there is 1 2, 3, 4, 5, 6 and 7 baseline value and the equal weighted portfolio is the one which is in the orange portfolio.

So, obviously you know all the orange lines are or the orange bars are identical length; because it is going to be the equal weighted portfolio, but the blue lines of course are of a different length. So, this is going to be the asset allocation in terms of the weights for the baseline portfolio as well as the equal weighted portfolio.

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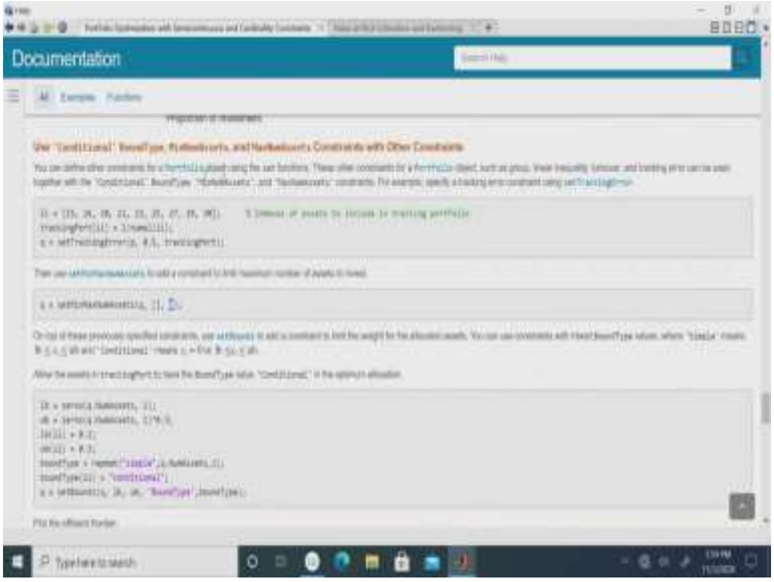

So, now, I just want to talk about this conditional bound type min num as min number of assets and max number of assets constraint with other constraints. So, what you do here is that, you can define other constraints for a portfolio object using the set function.

So, what you do is, these are the constraints that we are now talking about; this could involve things like a group or linear inequality, turnover, tracking error and so on which can be used. So, for this you know you can actually look at, you can look at some of the earlier examples which you have done, particularly in terms of turnover and tracking.

So, for example, let us just now bring about this new constraint or the conditional constraint, which is the set tracking error. So, what you do is that, we have a tracking index and for this tracking index what we do is that, we identify the following assets to be included in the tracking portfolio.

So, there are these 9 number of assets that are in the tracking portfolio and we assign identical weights to each of the tracking portfolio. And we basically set these two q and then we set the number of assets to be equal to 8.

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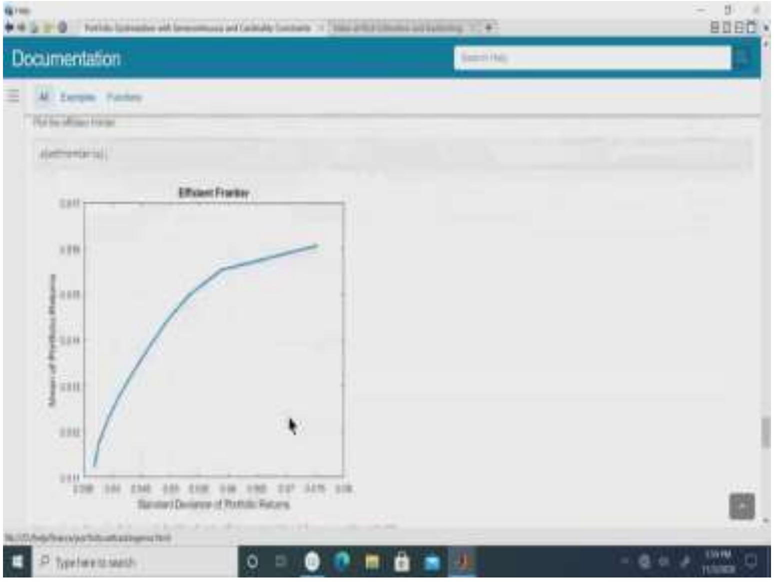

So, and so, here we set the lower bound and upper bound. So, in case an asset is included; so here what we will do is that, we set the lower bound to be 0.1 or 10 percent and the upper bound we set this to be 0.3 or 30 percent. And in this case we get the efficient frontier, so the efficient frontier turns out to be of this particular form.

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Now, you can actually now set the target return to at 12.5 percent. So, you can set the efficient frontier by return on q. So, remember that this is going to be you are setting now if. So, you might wonder that why you are setting the target return at 12.5 percent again, when you have already seen a case of this target return.

So, here I just want to point out that, here I am putting a target return on q resulting from this scenario of where you are considering a tracking portfolio. So, you set the target return of 12.5 percent tracking portfolio.

And what you do is that, once you have included that particular condition; you can again do the allocation, it turns out that these are the weights that are going to give you the optimal allocation in case of this target return being imposed on the problem, where you on a problem for of a tracking of a portfolio that is used for the tracking error purposes.

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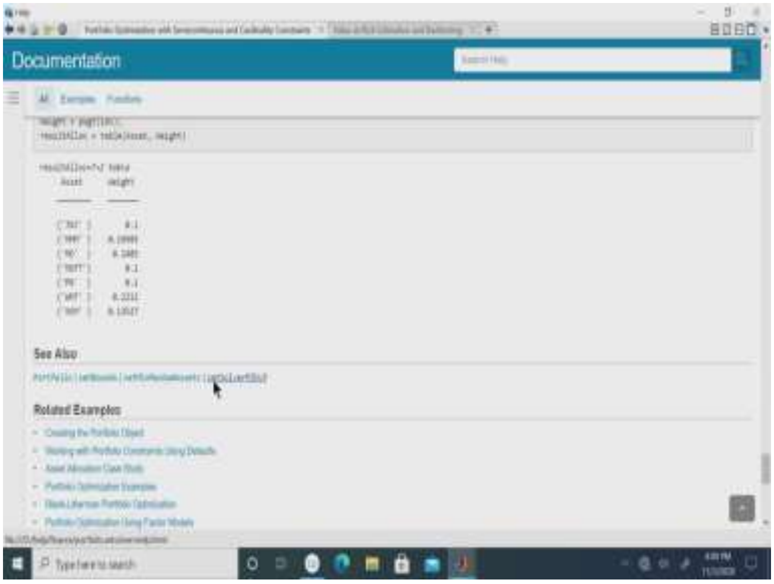

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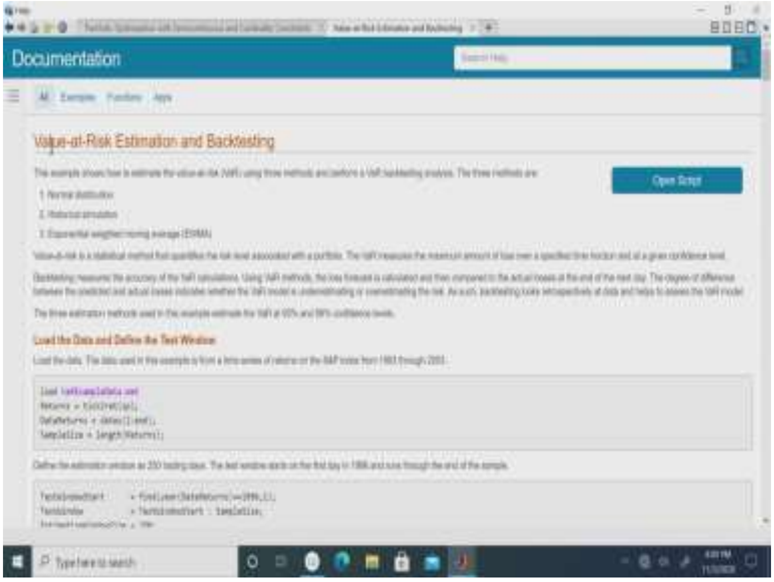

So, now we come to the second example that we are going to do today. So, that is going to be on value at risk estimation and back testing. So, you recall that the value at risk essentially is defined for a certain time window into the future for a certain confidence level.

So, if I say that the value or at risk at 95 percent confidence level for the next one day is some little x. So, this means that, 95 percent chances are that over the next one day, my losses will not exceed an amount of x and there is only one 5; the only 5 percent chance that over the next one day, my losses are going to exceed the amount of x.

So, likewise you can have analogous definition in case of 99 percent bar and as well as you know looking at a time interval of more than one day. So, it is customary to consider 95 percent and 99 percent for over a one day or a ten day window.

So, we will just look at some examples on how to estimate the value of risk using three methods; three these are very well established methods and these are based on normal distribution, historical simulation and exponential weighted moving average.

So, I will explain each of these in some amount of detail. So, now, just you know just to reiterate the fact that, value at risk is a well widely used technique in financial risk management with a statistical method that quantifies the risk level associated with the portfolio. So, what it does?

So, what you do is that, the three estimation level that will consider in this the three estimation methods which will be used for illustrative purposes here, will be done to estimate the value at risk at 95 percent and 99 percent confidence level.

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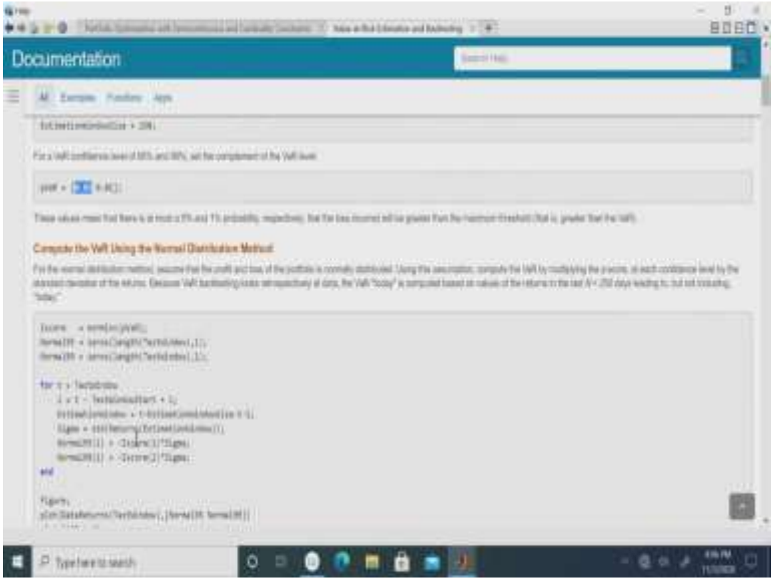

So, accordingly we start off with by loading the data. So, there is this data VaR example data dot mat that actually comes with the package. So, accordingly what you do is that, we take the data returns and we set the sample size. So, the sample size, it is basically the length of returns; that means the number of return points. So, now, what we do is that, we define the estimation window as 250 day trading day.

So; that means that, we will make an estimate of the value at risk for say one day into the future making use of the data for the preceding 250 trading days. So, the data that is being used here, this starts on the first day of 1996 and runs through the end of the sample. So, in this example, they are really looking at a very large data set points.

Now, we will take the estimated window size to be 250. And the reason why it is chosen to be 250 is that, roughly there are about 250 trading days in a year, which you can estimate from the fact that from the 52 weeks and each week having 5 working days. So, that is roughly about 250 trading days in a year.

Now, we a priory or ahead of time, we set the VaR confidence level to be at 95 percent and 99 percent and this is set by assigning this variable p VaR to be 0.05 and 0.01. So, p VaR1 within bracket will pick up 0.05 and p VaR within bracket 2 which you see later is going to pick up 0.01.

So, this means that we are only considering the scenario of 5 percent at 1 percent probability that the losses will be greater than the value of the actual value of the VaR. So, now we start off with the first method of computing the VaR. So, in this case we say that we compute the VaR using the normal distribution method.

So, for the normal distribution method as the name suggests, we will begin with the assumption that the profit and loss of a portfolio is normally distributed or in other words the returns are normally distributed. So, what you do is that, under this assumption will compute the VaRVaR by multiplying by the z score at each confidence level by the standard deviation of returns.

So, what actually happens is that, the value at risk is going to be given by the product of the Z's minus of Z score into sigma which is the volatility that we have seen in a geometric Brownian motion. So, in order to estimate the VaR, we need to find out two things; one we need to find out this Z alpha this Z score and the other you have to find out sigma.

Now, observe carefully that, the Z score that you have is just something that is dependent on your choice of the percentage. So, we will have one Z score for 0.05 and the other Z score for 0.01.

So, once you have the Z score, you do not really need to worry; because you can keep using the same Z score depending on whether you want a 95 percent confidence level or whether you want a 99 percent confidence level. And also we need to account for the fact that you know this Z score is nothing, but a number which basically gives you the inverse of a normal cumulative normal distribution, the inverse of that of 0.05.

So, that means it is going to give that particular value, such that 5 percent of the area lies to the left of that particular value on the x axis. So, accordingly what you do is that, we first set the Z score and you know keep it ready; because we have to just do it for one time.

So, Z score will be given by norm in p VaR; so that means this p VaRwhich is 0.05 and 0.01. So, then this Z score will just be given by the inverse of the cumulative distribution of the normal random variate for 0.05 and 0.01.

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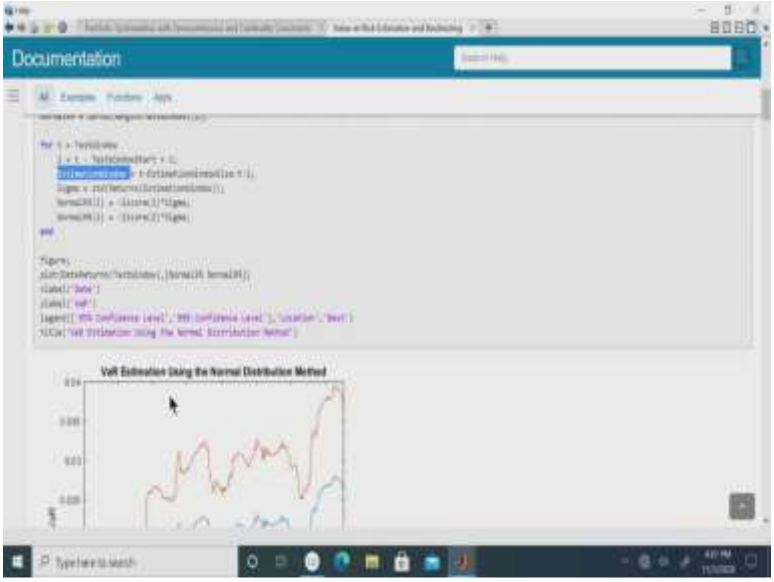

So, what you do is now, what do we need to now do? So, the only thing that is left in order to ascertain what is going to be our VaR is going to be the determination of the standard deviation.

So, accordingly we look at the estimated window; remember that this estimated window is going to be over 250 days. So, accordingly we look at the standard deviation of returns over the 250 days, the daily standard deviation. So, accordingly what you get is, we get Normal95 i and Normal99 i is going to be the corresponding value of this VaR at for the first and the second case.

So, accordingly what you do? So, we have done the variance estimation using the normal variable by and these are, these Normal95 and Normal99 that you obtain here; these are going to be nothing, but the 95 percent and 95, 99 percent confidence level value at risk that you have estimated by making use of the normal distribution method based on the preceding number of 250 days.

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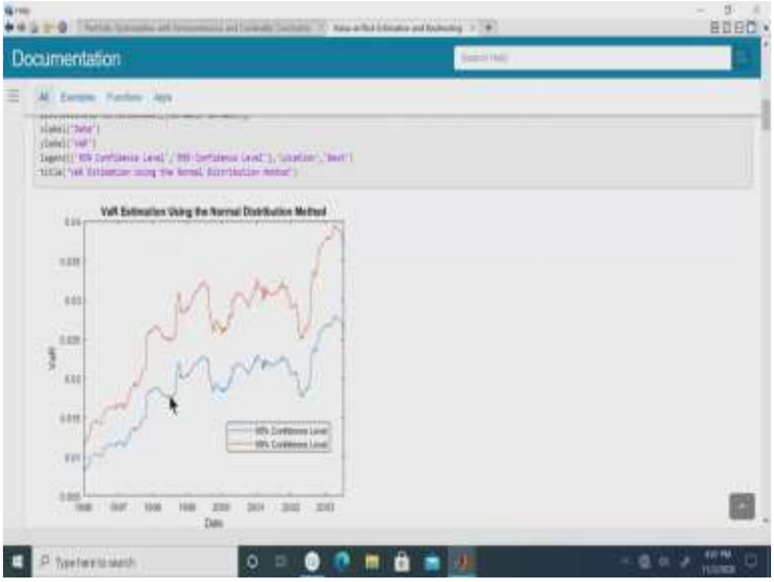

So, accordingly what happens is that suppose that on 1st January 1996 the for the entire period of 1 year, you are making use of the data and you are looking at the return and you are fitting it to the normal distribution and you are using this to estimate the value of the VaR on 1st January 1997 and so on.

So, accordingly what happens is, if you are making use of all these VaR using the normal distribution table; so basically you are going to have a VaR on a daily basis. And if you are plotting this VaR that is the value at risk for this entire period under consideration; then this the you can see here what you see here is going to be is the, is the movement or the progression of the estimate of VaR making use of the normal distribution method.

The one with the blue is at a 95 percent confidence level and the red, the orange graph is the one which is at 99 percent confidence level. And so obviously, when you are looking at 99 percent confidence level; so obviously the value at risk that you are estimating is going to be more than the corresponding value at risk has 95 percent confidence level. Because 95, 99 percent confidence level means that, you will essentially get a much more conservative estimate of the value that the value that is actually at rest.

So, this means that, you know if you look at the curve on the top which is the orange curve; these are the values which are only 1 percent likely to be breached, but if the one the bottom curve which is in blue, these are the values that are likely to be breached in 5 percent of the cases. So, obviously, these values are going to be less.

So, this is these values are less; so obviously, they are more likely to be breached; that means they are likely to be breached at 5 percent on 5 percent of occasions as compared to the one at the top, which are likely to be breached only in 1 percent of the scenarios.

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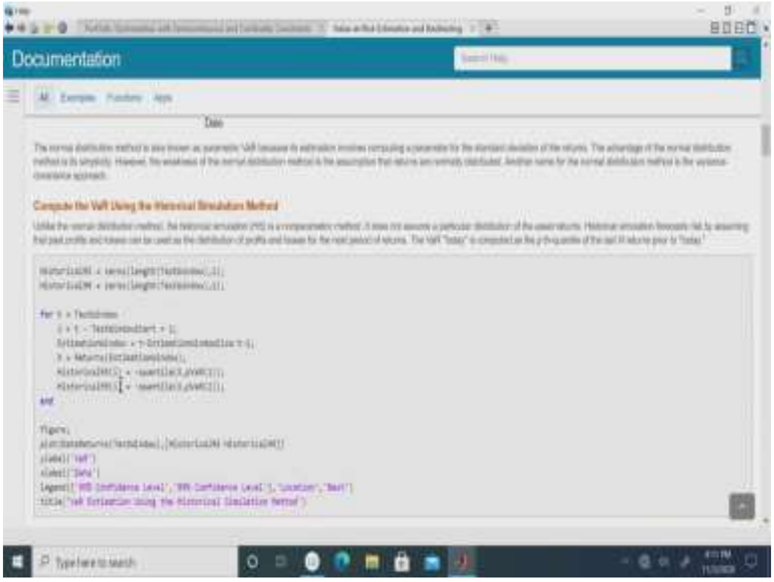

So, sometimes this normal distribution method is also known as parametric VaR. And the reason why it is called parametric VaR; because in the process of it is determination, you are actually estimating the standard deviation of the returns.

Now, the advantage of the normal distribution is very simplicity and the reason is that, you know as you can see the code is very easy; all you need to do is you need to look at the data and you need to look at the standard deviation of returns and the Z score that is something that is independent of the data set that you are considering.

So, you are able to estimate the VaR very easily. So, the usability and the user friendly nature of this method makes it very very appealing; but as is the case with most user friendly things, it also suffers from the weakness.

And the weakness is that, it is makes use of the assumption that, the normal the returns of the assets are normally distributed. Also sometimes this normal distribution method also has a different name that you will see in literature and that is what is known as the variance covariance approach.

So, let us now move on to the second method. So, this in this second method we calculate the VaR using the historical simulation method. So, unlike the normal distribution method, the historical simulation method is a non-parametric method. So, by non-parametric method I mean that, the historical simulation approach does not make use of an approximation for VaR which is given by this minus Z score into the sigma value that you have estimated.

The and because the sigma value here is a parameter, so that is a parametric method; but there is no such parameter involved in the historical simulation approach. And so, since there is a non-parametric method, so obviously it will not assume a particular distribution; because if you had a distribution, you would obviously expect that there is going to be some sort of a parameter that accompanies it.

Now, historical simulation forecast risk by assuming that past profits and losses can be used as the distribution of profits and losses for the next period of return. So, what you do is the following is that, you look at the historical returns; if you look at the historical returns, which you set as X. So, if we just look at the historical returns over a period of time; what you will get is that, you assume that, the historical return for you know several time points in the past.

You expect that the random variable which represents the return for the next one day that is going to take a several values, because it is the random variable. And the assumption is that, the random variable for the return for the next day that will take the values it; it will take a certain number of values and those values are simply going to be the returns from the past.

And so, accordingly what you do is that, you will take then the quantile of this returns of the past; remember that we talked about quantiles when we were talking about value at risk. So, it will take minus the quantiles which just straight away makes use of the definition of VaR. So, it takes the minus the quantile; and which quantile?.

So, in the first case it will take the quantile at 95 percent; so alpha equal to 0.05 and the second one it takes at 99 percent, that is alpha equal to 0.01. So, what it does is that, it takes the historical returns and it arranges them in a sequential order and then it sets the cut off at 5 percent or 1 percent.

So, suppose that you have 500 values of historical data and you want to calculate the 95 percent VaR; so that means, you will basically pick up the 25th value of this 500 values and that is going to be the value at risk.

And secondly, when you are looking at this 500 values of the historical returns and you are looking at a 99 percent VaR. So, you are going to pick up 1 percent of the 500 values; that means you are going to pick up the 5th value. And these are what I which are which can be related to the quantile p X, p VaR 1 and the quantile X, p VaR 2.

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So, here you can see that we are again. So, in the previous case we had the VaR as it progresses over time; one at the 95 percent level and the one at 99 percent level. So, we now have an identical graph here; again the blue graph is the 95 percent confidence level and the orange one is in the 99 percent confidence level as before.

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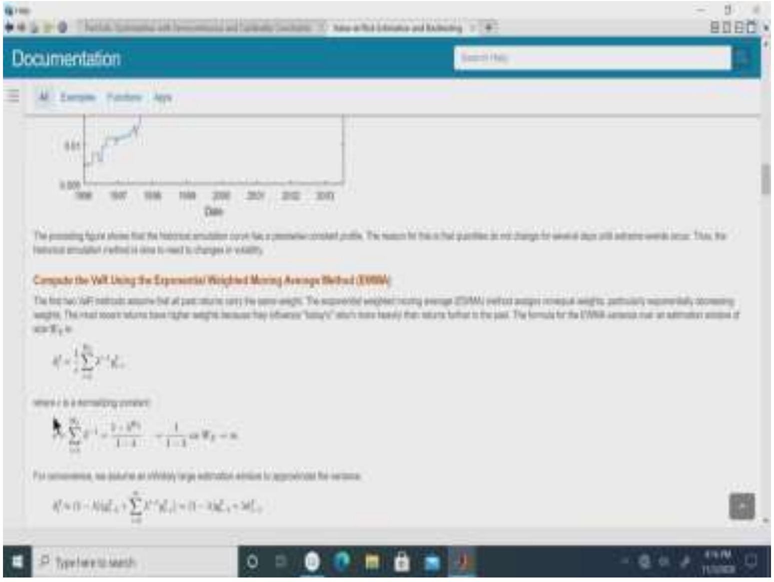

Just one thing that you need to observe here is that, you see that there are periods where this exhibits a constant behaviour. And the reason for this is that, quantiles does not change for several days until extreme events occur.

From this you see that this historical simulation method, while it is very easy to use and does not make use of any assumption about the distribution of returns; but it also suffers from the drawback that it reacts slowly to changes in the volatility that happens.

So, now we come to the computing of VaR, the last approach and this approach is. So, we will make use of this method which is known as the exponential weighted moving average method, which is commonly known as EWMA. Now, if you observe carefully the previous two methods of VaR, namely the normal distribution and the historical simulation; both of them assume that, whatever has happened in the past is equally likely in terms of happening again.

So, it assumes that, all the past returns that are being used either to calculate the value of sigma in the first approach or to calculate the possible value of the random variable of the return over the next day, they have the same weight. However, the exponential weighted moving average intuitively offers a more improved and realistic approach to this.

And the reason why I say that this is more realistic is because, the exponentially weighted moving average method, it assigns non-equal weights. And how do we assign these non-equal weights?

If we assign this non equal weights, but because exponentially decreasing weight what it does is that; it gives a more weightage or higher weights are assigned to more recent returns as compared to more remote returns, because it is based on the premise that the today's return is more likely to be influenced by more recent returns as compared to the returns from further in the past.

So, now what you do? So, again the estimated this EWMA method, it again goes back to the estimation of volatility. So, this is again a parametric method, like the normal distribution method. So, the formula for estimation of this EWMA variance over estimated window of size W E. So, if the time window; that means you have W subscript capital E is the number of time points; then what you have is that, the estimated weight at time t is going to be nothing but.

So, the estimated variance at time t is going to be 1 over c into summation of lambda raised to i minus 1 for using the y of t i minus 1; that means you are going to use the values starting for the preceding W E number of time points. And here c is a normalizing constant. So, here c is going to be nothing, but summation of i equal to 1 to W E; it was into lambda of i minus 1 over i.

So, the reason for this, so if you do the summation here; this turns out to be 1 minus lambda raise to W E by 1 minus lambda. And if we are essentially considering infinite number of points, this is going to tend to 1 over 1 minus lambda.

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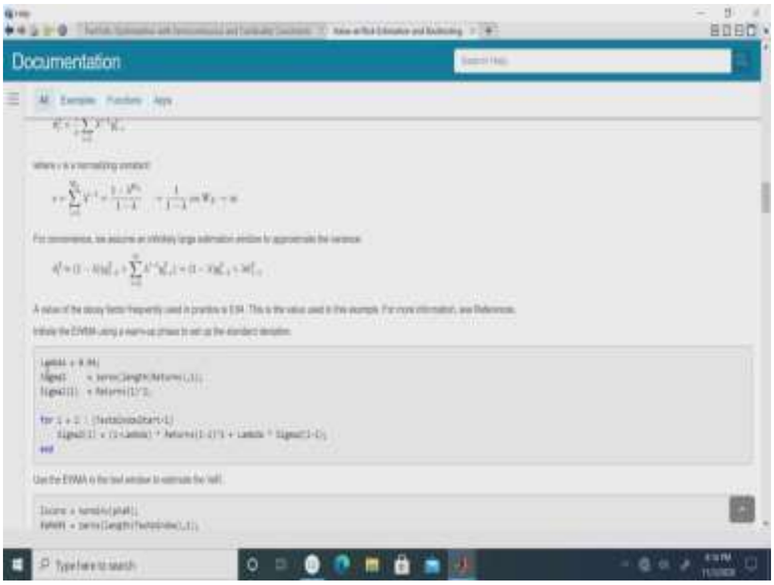

So, for convenience what you do is that, you know we estimate you assume that in very infinitely large estimation window. And accordingly what you do is that, if you observe carefully that; this estimation delta hat then can now be approximated as a recursive relation of 1 minus lambda into y t minus 1 square plus lambda into sigma hat t i minus square.

So, this means that you start off with a certain sigma hat estimation and then you can recursively generate all subsequent sigma hat t. So, for example, here you are using the sigma hat at t minus 1 that is the value and y hat t minus 1 to estimate the value at t. And, likewise once you have estimated the value of t, you can replace it here in order to estimate what is going to be the sigma hat square at t plus 1.

Now, in practice, this decay factor that we have here; that decay factor is taken to be 0.94. So, in this example, we will make use of this particular decay vector. So, decay vector here is the value of lambda. So, let us now just look at this illustrative example. So, in this example what we do is that we will initiate their EWMA.

So, first of all we will set the lambda to be 0.94 and we will take the sigma to be the zeros of the length of returns and 1. And then we will calculate the sigma 2 I; so that means, we are recursively calculating. So, what we do is that, we first we set the first value of the sigma hat to be returns of 1 square and then we recursively making use of this particular formulation. So, we have this particular relation in order to generate all the values of sigma.

So, start off with sigma 2 of 1 and like and sigma 2 of 1 is the initial value for this particular relation. And then we make use of this recursive relation for calculating all the sigma's for from the for i equal to 2 all the way to number of points in the test window.

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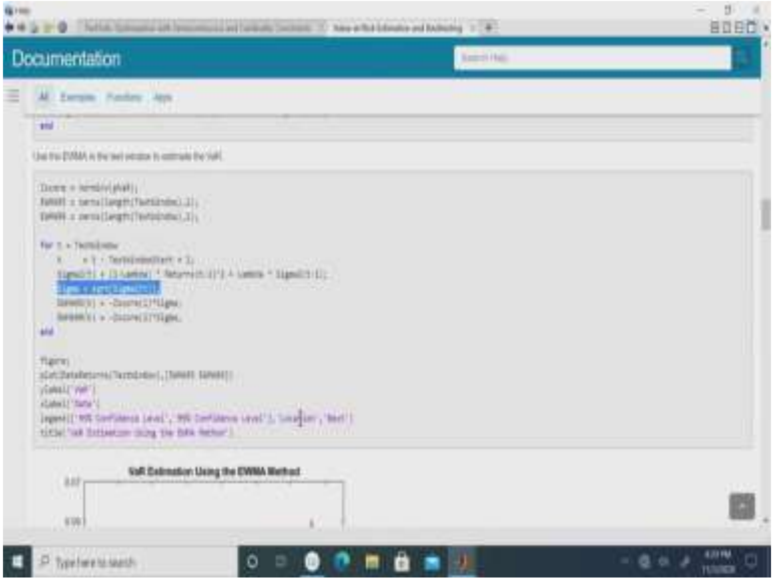

So, now what you do is, now we have to now we; how do we make use of the EWMA in order to estimate the VaR? So, as before of course, you know my Z score is going to be given by the inverse of p VaR; remember that p VaR were either 0.05 or 0.01.

And what you do is that, we will now calculate the sigma making use of this approach of EWMA and will assign the value of sigma. And then using the formula that, the value at risk which will using the EWMA method.

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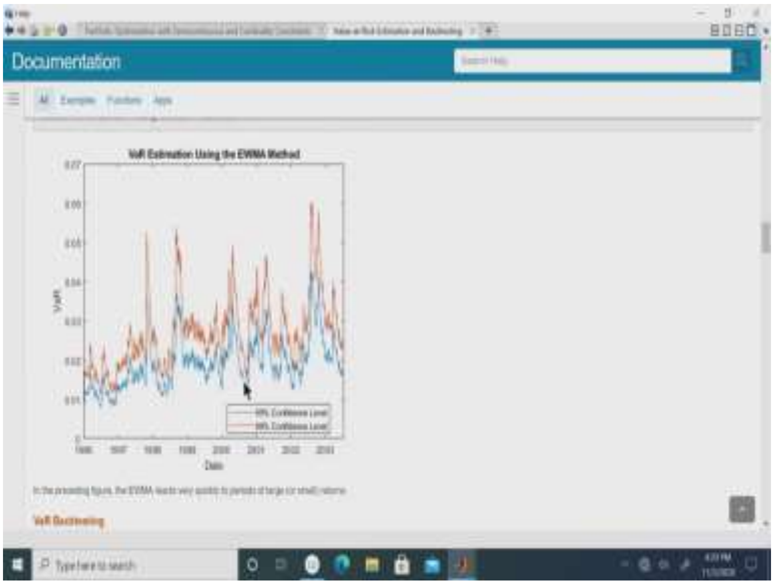

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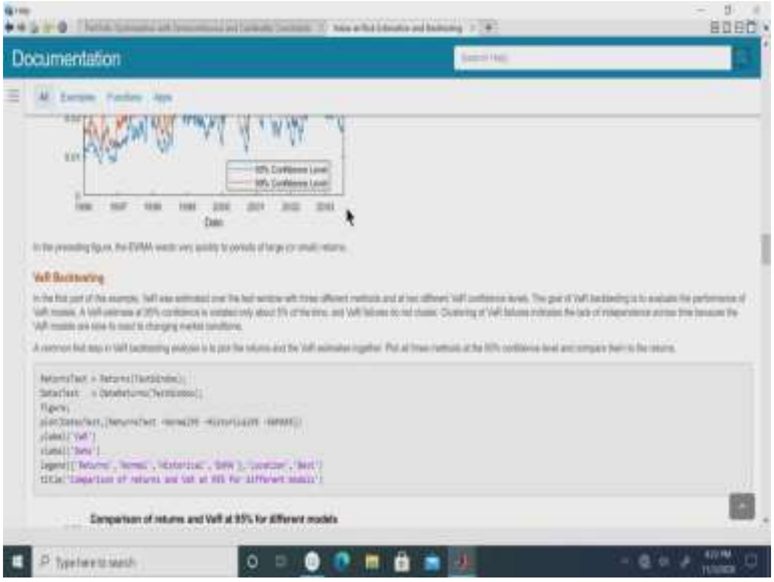

So, we look at the EWMA method, the value at risk using this method at 95 and 99 percent level is simply going to be the minus of Z score for both the cases into the sigma that you have estimated using the EWMA method and this gives you the value at risk using the EWMA method. So, if you observe very carefully here that, here we will add the 95 and again we have the plot of the estimation of the wave for using the 95 percent and 99 percent confidence level.

So, here you observe that EWMA given the more volatile nature of the graph as compared to the two preceding methods; you can conclude that the EWMA it reacts very quickly to periods of large or small returns, ok.

Now, once we have done these three approaches, we have calculated the VaR. Now, this VaR is actually going to be used in practice under the regulatory requirements in order to estimate what is going to be your likely amount of money that you are going to lose over and over a future time point say, typically one day.

So, you want to make sure that these three approaches that are being used to estimate the VaR are actually doing the work that they are supposed to do and this is what is done and this is accomplished to something which is known as the back testing.

So, even though we are using the term VaR back testing here; back testing is a generic term, where basically you look at the past data and you set up the model and you carry out the exercise, and you look at the past data and you can divide these into two parts.

So, suppose that you have 1000 data points and then you do the testing and setup by making use of 800 data points and you do the prediction. And then you compare how well your prediction is working as compared to the actual value that was realized over the subsequent period of time. So, this is a way of figuring out what is going to be how well the method is approaching.

So, suppose that I am considering a period of 2000 to 2020; I can make use of the period of 2000 to 2015 in order to make an estimation of the, estimation of VaR using the three approaches. And then with that method I will make an estimation of what happens being 2000. So, I will make use of the data between 2000 and 2015 to make an estimation of what happens between 15 and 20.

And I will compare to the actual value; because I already have the data of that value and see how do these two values compare with each other and that will give me an idea as to how much trustworthy is my approach the, approaches that I have done in order for practical implementation.

So, what you do here is the following. So, a very common first of all back testing, back testing analysis is to plot the returns and the VaR estimate together. So, what you do is that, you do the VaR estimate and you plot the VaR estimate and at the same time you plot the returns

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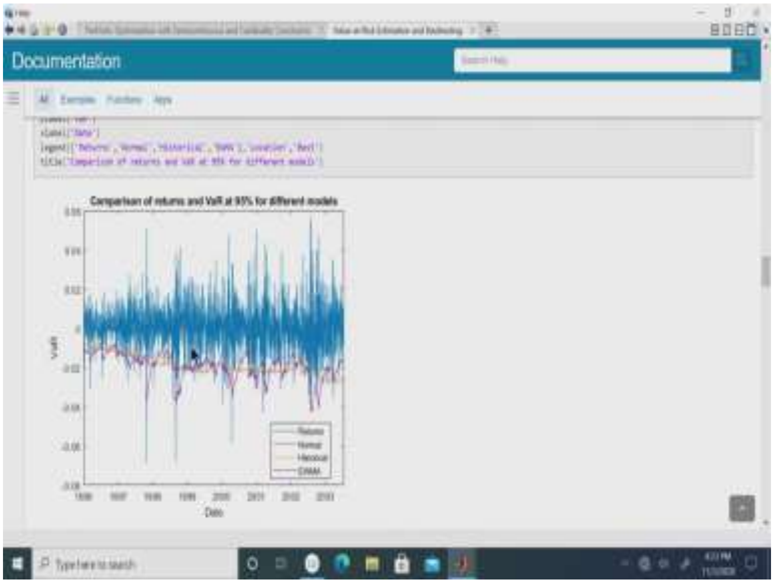

So, an example of this is that, this is a comparison of returns and the VaR at 95 percent using the various different models.

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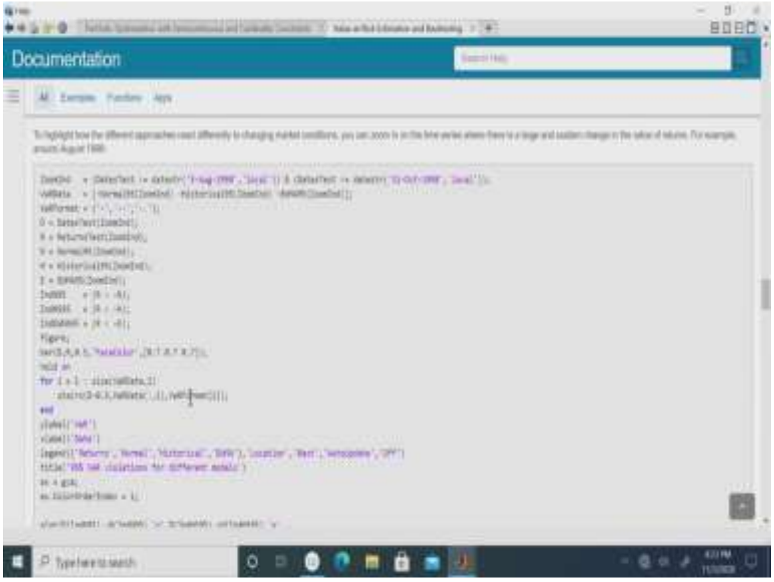

Now, if you now zoom in, now this is this looks you know a very very you know a spiky nature and very very volatile nature of graph, but you will get a better clarity if you actually zoom into this. So, since this is over a larger period of 1996 to beyond 2003.

So, what you can do is that, you can actually have a more localized view of what is going on here. So, accordingly we choose a localized period of 5th August 1998 to 31st October 1998 and you observe here carefully that this is from 5th August and this is up to 20th October.

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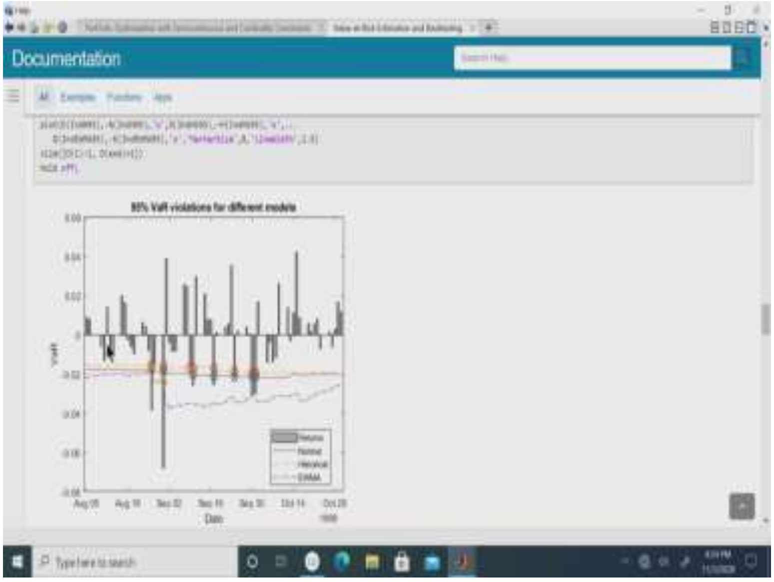

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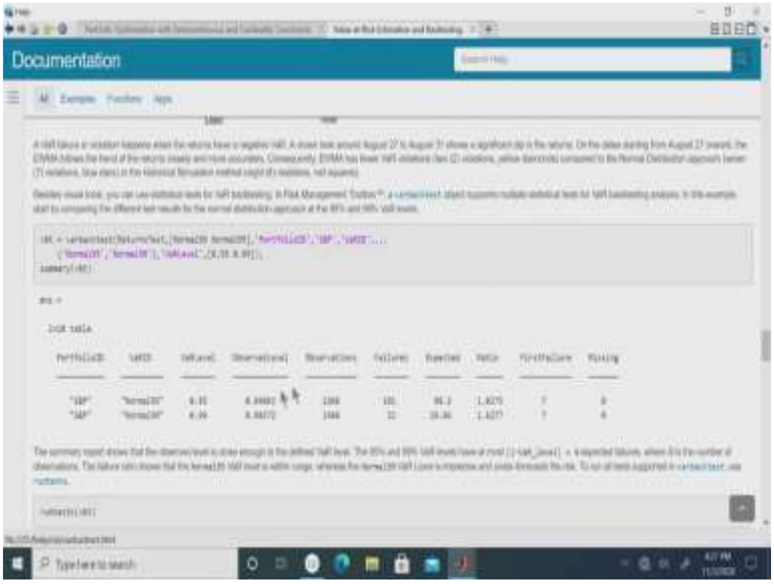

So, in this period what you have is that, these are the actual values; see the ones which are in bars or these grey coloured boxes, these are the actual values. And this graph which are there in golden dotted lines that is the historical data; then the one which is given as the orange line, these are the VaR graph that you have for using the normal distribution and finally, we have the purple dotted lines which are for the EWMA.

So, these three graphs are essentially the same graphs that we had seen earlier, when we are discussing each of the individual methods. Now, how do I decide, you know, so this is a manifestation of back testing; that you have used the past data in order to estimate the VaR and then you are actually looking at the returns return distribution.

Now, these are the values of VaR and remember that, VaR is intended to figure out what is going to be the losses and loss by definition is negative return. So, that means that, if I say that you have lost an amount of five; that means your return is actually minus 5. So, that is the reason why the bottom part of this graph. So, this is going, this is basically going to be the essentially a manifestation of the amount of money that you actually lost as a from here from your portfolio or your particular investment.

Now, let us look at them individually. Now, if you observe very carefully that, this lines that you had here this, this line which are in golden which are in orange and purple; these are the lines which give an estimation of the risk and value at the amount of money that is at risk or your likely level of losses that you are going to incur.

So, that means that, as long as these values of return that you have these are above this lines; that means that the your losses had, your losses are less than the estimated losses as estimated or as given by these three methods. So, as long as your this vertical bars, they are above this dotted lines or this particular line solid line; that then you are ok, but the moment it falls below those lines, that means you are in trouble.

So, now let us identify which are the troublesome points. So, if you observe here that, for example, EWMA, the purple line; if you observe very carefully here the purple line, there has been only two instances.

So, if you look at the purple line carefully which is at the bottom; the purple line only two instance, there have been only two instances where the purple line actually is at a higher level than the actual return, this is the one line here and this is another line here.

So, these are the two violations. So, these are given in this yellow and these are the two violations of VaR or the failure of EWMA in order to predict the correct VaR. So, and there are only two such instances. Now, as compared to the normal distribution which is the orange line; if you look carefully, there are seven instances where the normal distribution curve actually is at a higher level than the bottom tip of the bar.

And a historical simulation you will observe that, the historical simulation which is the yellow bar; obviously you are going to see that there are 8 violations. Now, given the fact that you see the consistently along the entire spectrum of the timeline; if you see that the historical line is above the normal line which is above the EWMA, so obviously you are going to expect that the maximum number of violations will happen in the top line which is the historical line.

And next we will have the normal distribution learning and finally, EWMA which is what has actually happened in reality. As you can observe that in case of historical violation, there are 8 violations; the for normal approach, there has been 7 violations, but for EWMA approach, there has been only 2 violations.

So, the remainder of this narrative actually, it gives you a detail of the violation. So, if you observe very carefully when this is actually carried out over a larger period of time. So, if you are calculating this value for the entire period, you will see that there are number of failures. So, at 95 percent level, there has been. So, if you are making use of 1966 observation; that means the entire timeline.

So, that means you observe that, in case of the 95 percent confidence level; there were 101 instances of failure and in case of 99 percent interval, there has been 32 instances of failure.

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So, you observe here that. So, these are several results that have been manifested in order to calculate or in order to estimate how much of under forecasting of risk was actually accomplished.

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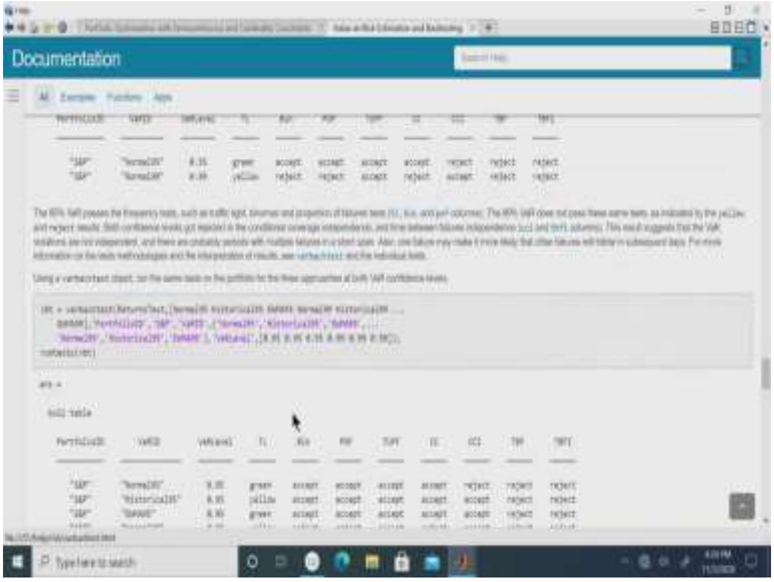

So, now here you observe carefully; so here we have these stabilated values of only the ones using the normal variant.

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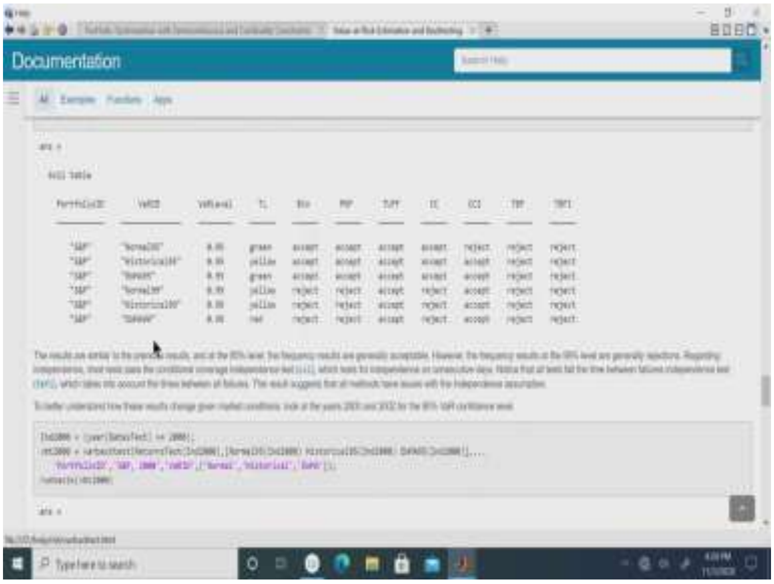

So, here this broader table which compares all three at 95 percent as well as 99 percent level will give you a much better idea. So, this is the table which will give you a broader elaboration of whether it was accepted or rejected; in case of each of the three approaches both at 95 percent confidence level as well as the 99 percent confidence level.

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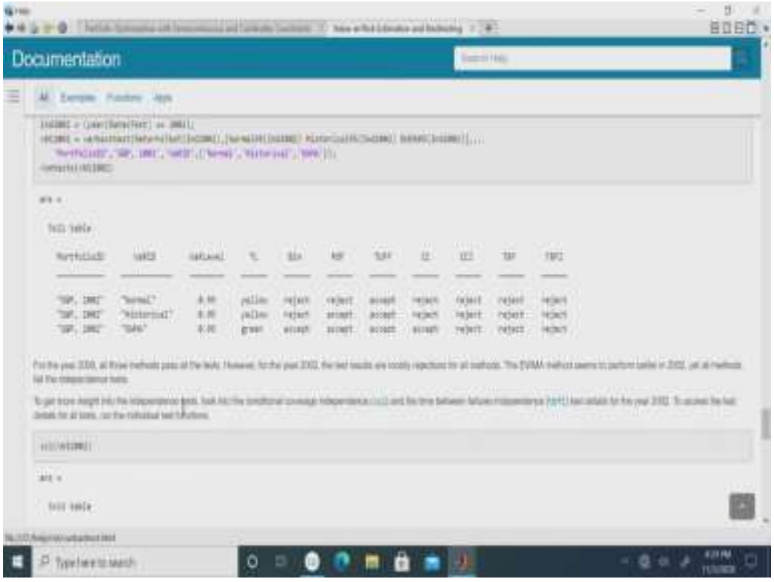

So, these are different variations of the approaches of how the testing can be done.

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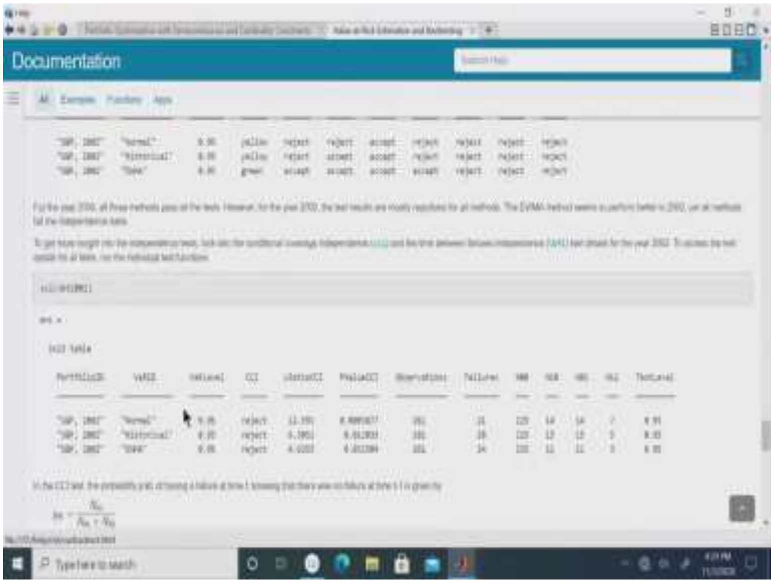

But at the end of the day, these just these reiterates in detail as to how important is it to carry out the back testing; a model has only the relevance in from practical applicability point of view, provided that it is able to actually have very very little violations or failures as was manifested in this broad main example that you consider, where you observe that the EWMA method has shown the least amount of violations or failures that were actually observed during this period of time.

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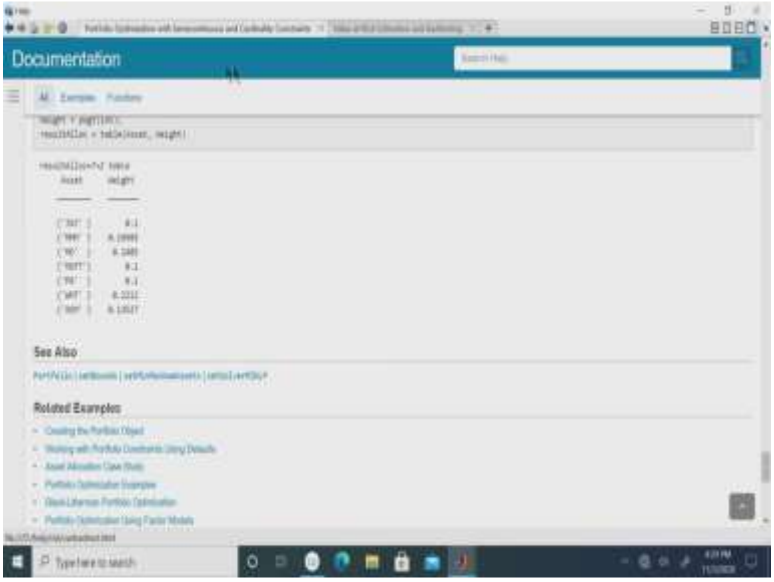

So, this concludes our discussion for today with these two particular examples; first on the constraint based portfolio optimization, and the other on value at risk estimation and back testing. So, this brings us to an end of this particular course; I hope all of you have enjoyed this course as much as I have. So, just to do a summary of this particular course of what we have done in today's lecture.

In today's lecture, we talked about two approaches; the first of the approach was actually motivated by the practical consideration that you must have constraints on the weights of the port of the assets in the portfolio and secondly, on the limitations that you have in terms of the number of assets in which you can make an investment.

So, we first considered the situation where we put a constraint on the minimum weight that we assigned to a particular portfolio a particular asset in a portfolio and that is necessary because; if you assign too low a weight, then from a practical point of view in terms of the cost that are involves, in particular the operational cost, it becomes invisible to have a very low weightage being assigned to any particular asset in the portfolio.

On the other hand, we have considered the situation where we put a numerical constraint on the number of assets that are going to be included in the portfolio. So, you put an upper bound on the number of assets; the upper bound is there, because you do not want to be investing in too many assets. Because investment to too many assets, increases the operational cost as compared to a limited number of assets.

And at the same time, we looked at how we are going to impose a lower bound on the number of assets; because if we have too few assets, that means that you are not accomplishing the basic principle of modern portfolio theory, namely diversification. In the second topic that we consider today, which was on value at risk; we looked at three methods, namely the normal distribution, historical simulation and the exponential weighted moving average that is EWMA.

So, in the normal distribution method and remember that all these three methods were used to estimate the value at risk. So, in the normal distribution method, which has a parametric method; what we did was, we made use of the assumption that the returns are normally distributed and estimated the standard deviation of that and use the standard deviation in order to estimate the value at risk of the portfolio or an individual asset.

In the historical simulation approach, which is a non-parametric approach; we just assume that the returns from the past are going to be rectificated in the future with equal probabilities and we will use the concept of percentiles or quantiles in order to determine what is going to be the value at risk.

And finally, when you talked about the EWMA method, which eventually turned out to be the most practical from the point of view of reliability and the EWMA method it assigns more weightage. So, this again, the EWMA method was again a parametric method and it was used to estimate the volatility.

So, the volatility in this case was estimated using a weighted formula where greater amount of weightage was assigned to more recent data points and lesser weightage was assigned to more past data points. And using that approach, we estimated the volatility and again once the volatility was determined; the value at risk was estimated by making use of the same formulation that we had done in case of the normal distribution approach.

And all these three methods were then tested using the back testing method; wherein we make use of the data to do the model and compare with the actual realized result to estimate the effectiveness and reliability of the method. And a testing done on a large sample set showed empirically that, the EWMA method was actually more effective as compared to the other method in its ability to make a correct prediction.

So, this brings us to the end of this particular weeks lectures as well as a conclusion to the entire course. It has been an interesting journey interacting with all of you. And I hope you all enjoyed the course as much as I did bringing this course to you. And I would like to now conclude just by wishing you all the good luck for your final examination.

Thank you for watching.