Mathematical Portfolio Theory
Module 07: Risk Management
Lecture 32: Value-at-Risk and its properties

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Hello viewers, welcome to this next lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. You recall that in the last class, we started off on a new topic, namely looking at risk measures beyond the Markowitz setup of a variance or equivalently standard deviation. And we identify two important risk measures namely value at risk and conditional value at risk. And in the previous class as a prelude to the discussion on value at risk and conditional value at risk; we introduced the concept of what is quantiles in terms of the cumulative distribution function and we looked at some of the examples. So, in today's class, we will be focused on starting our topic, discussion on the topic of Value at Risk or VaR and then we look at some of the Properties of VaR and we will conclude with looking at what is going to be the VaR in case of a, in case of a risky asset, namely a stock which follows the geometric Brownian motion.

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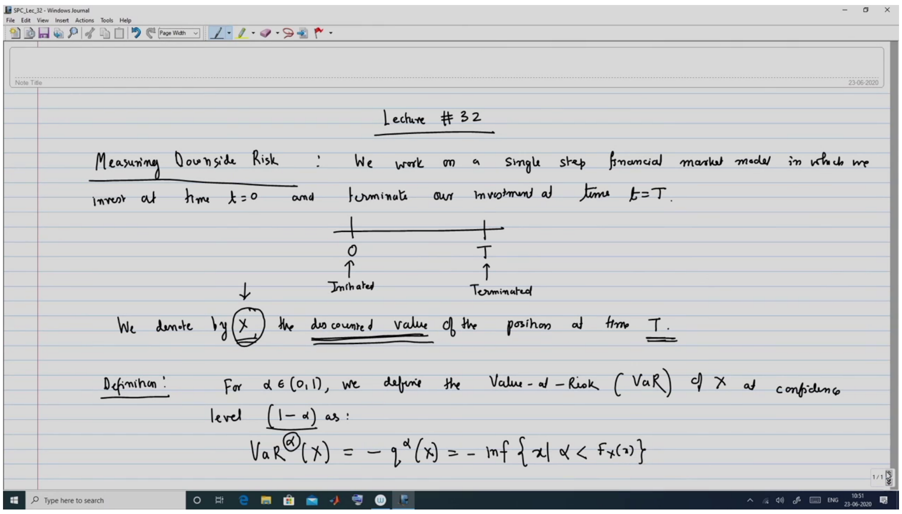

So, accordingly we start this lecture and this will be basically a part of the broader discussion on measuring downside risk. And you have already seen some of the downside risks when you are talking about the non mean variance framework.

And for the sake of brevity, we work on a single step financial market model, in which we invest at time t is equal to 0 and terminate our investment at time t is equal to capital T. So, that means that the investment horizon is going to be some time t equal to 0 when the investment is initiated and at time t equal to capital T when the investment is terminated.

So, we denote by X, the discounted value of the position at time capital T. So, basically if you look at the value of your position at time t and you take the discounting of that particular random variable that is the discounted value of the random variable at time t given by the value of the asset that will be denoted by the new random variable X. And for subsequent discussion we will be essentially when you talk about random variable X; we will mainly be focused on this particular random variable, alright.

So, let us start off with the definition of value at risk. So, for some alpha which lies strictly between 0 and 1; we define the value at risk which is abbreviated as VaR as I mentioned yesterday with the R in capital of X at confidence level 1 minus alpha as follows. So,

$$VaR^{\alpha}(X)$$

So, actually the confidence level is $(1-\alpha)$ and I indicate this with a superscript of alpha; this is defined as nothing, but minus the upper quantile of X. And this is minus and what was the definition of upper quantile? This was infimum of x, such that $\alpha \leq F_X(x)$.

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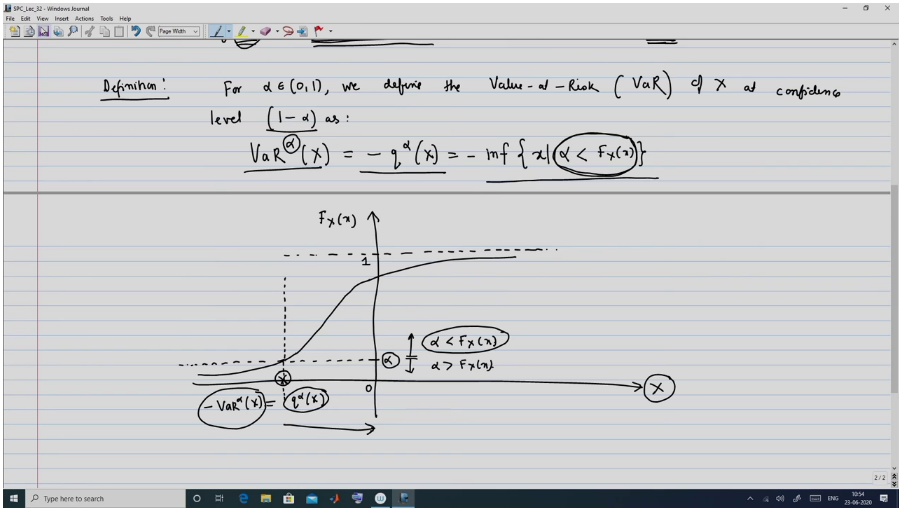

So, how does this look graphically? So, we take the random variable X on the x axis, and on the y axis we of course have $F_X(x)$. Then the cumulative distribution is something that obviously is going to look like this and eventually of course it as your as you as your X increases, this value approaches 1.

Now, what you do, is that we have fix some value of alpha out here, which obviously lies strictly between 0 and 1 and we look at this horizontal line corresponding to alpha and then we identify the corresponding value on the x axis.

So, now you observe that see this line is the line for alpha. So, the region that is satisfies the condition that alpha is strictly less than $F_X(x)$; that region is going to be nothing, but since your alpha is less than. So, it is basically this region that you have here, this region will be alpha is greater than $F_X(x)$; because alpha lies above the curve here and this region is the region where you have alpha is strictly less than $F_X(x)$.

So, accordingly, when I am going to calculate VaR; that is equivalently I have to calculate the minus infimum of all those x's for which alpha less than $F_X(x)$. So, I consider this region, where alpha less than $F_X(x)$ and consider all the X's, which are basically this part. So, it is essentially all the x's from this part onwards. And I look at the smallest value of x right and that smallest value of x and in this region of alpha being strictly less than$F_X(x)$, that smallest value of x will be given by this.

And what is this? This by definition is $q^{\alpha}(X)$ and from here you see that $q^{\alpha}(X)$ is nothing, but minus VaR of X. So, essentially what you need to do is that, you look at that cumulative distribution of x and then we identify what is this x for which what is $q^{\alpha}(X) = -VaR^{\alpha}(X)$, ok.

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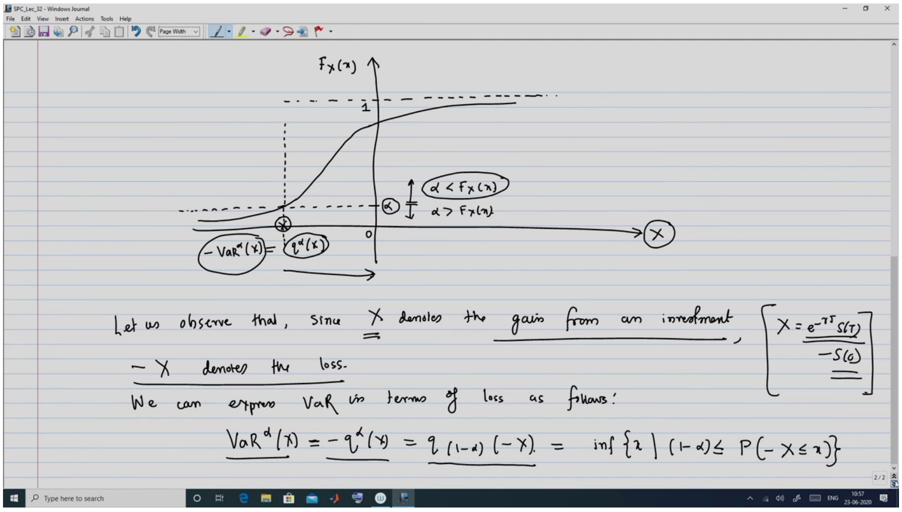

So, now let us observe, now let us observe this very carefully that, since what was X? You recall that your X was nothing, but the discounted value of the position at time t; so obviously X denotes the gain from an investment, right. So,

$$X = e^{-rT}S(T) - S(0)$$

So, you look at your original investment $S(0)$ and $e^{-rT}S_T$ is the present value of your final the wealth level. So, the difference between them is obviously going to be the gain.

So, accordingly X is here; it is it should be interpreted as the gain from a particular investment. And since X is the gain; so therefore minus X denotes the loss. So, once with this observation that minus X denotes the loss; we can now express VaR in terms of loss as follows. And what is this going to be? So, just, what is the definition of VaR?

$$VaR_\alpha(X) = -q^\alpha(X) = q_{(1-\alpha)}(-X).$$

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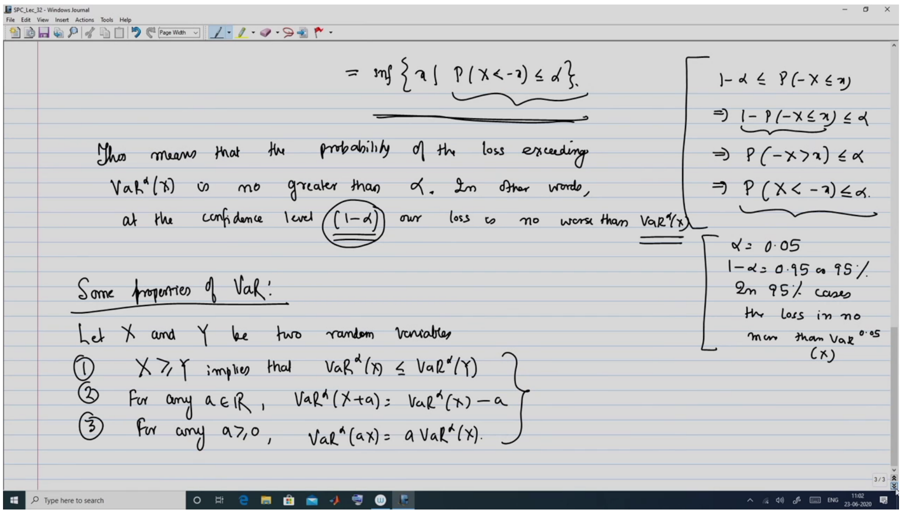

Some properties of VaR:
Let X and Y be two random variable

- $X \geq Y$ implies $VaR^\alpha(X) \leq VaR^\alpha(Y)$
- For any $a \in \mathbb{R}$, $VaR^\alpha(X+a) = VaR^\alpha(X) - a$
- For any $a \geq 0$, $VaR^\alpha(aX) = aVaR^\alpha(X)$

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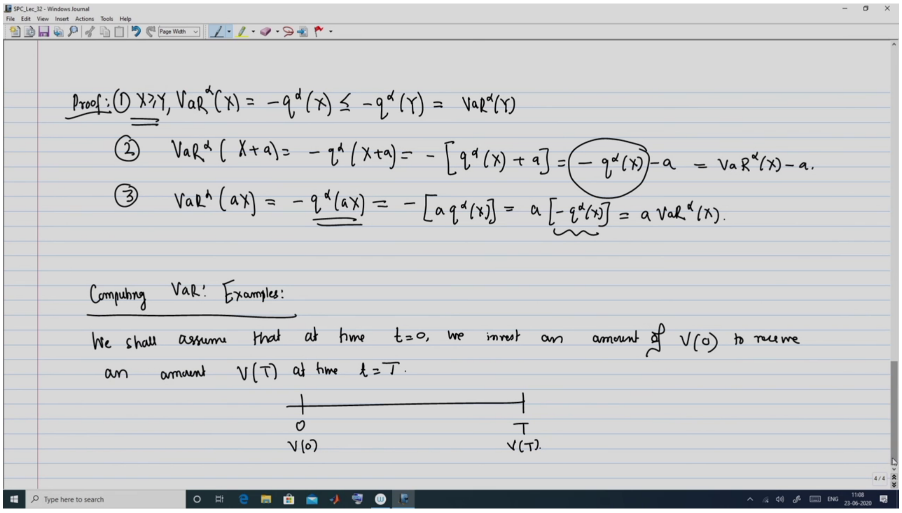

Outline of proofs are given above.
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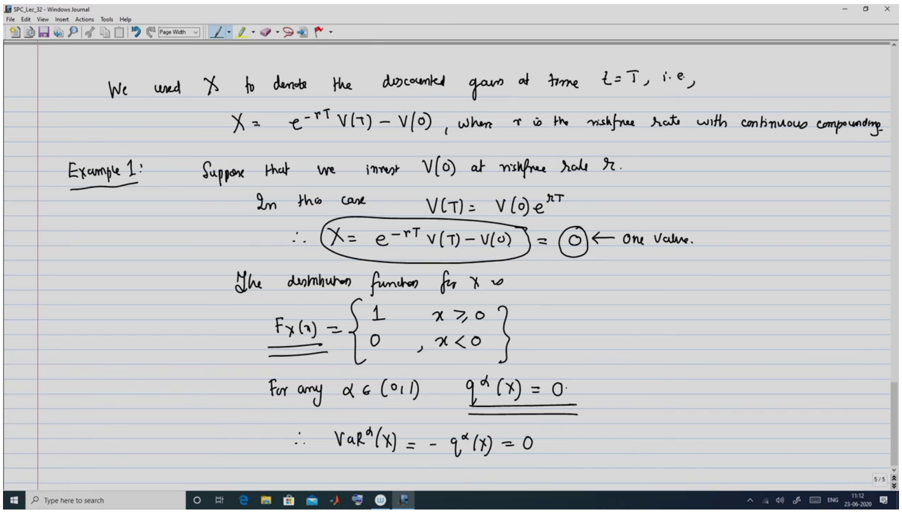

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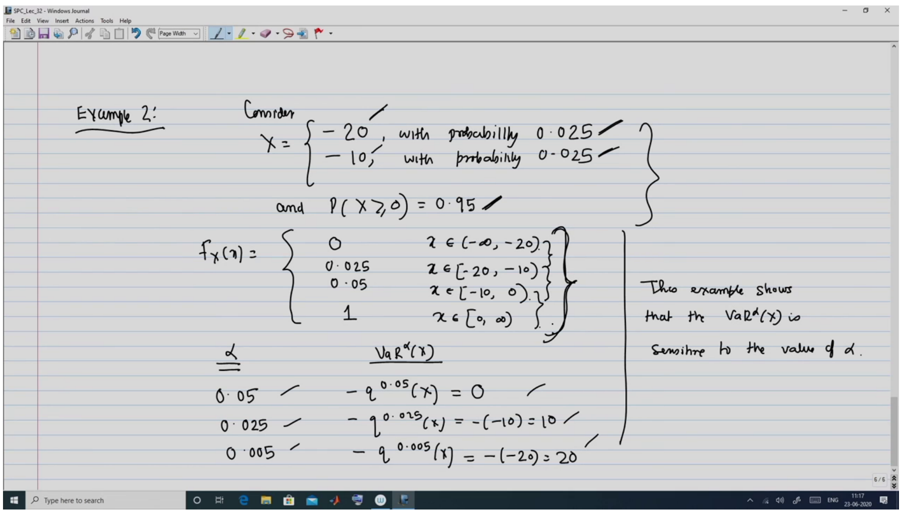

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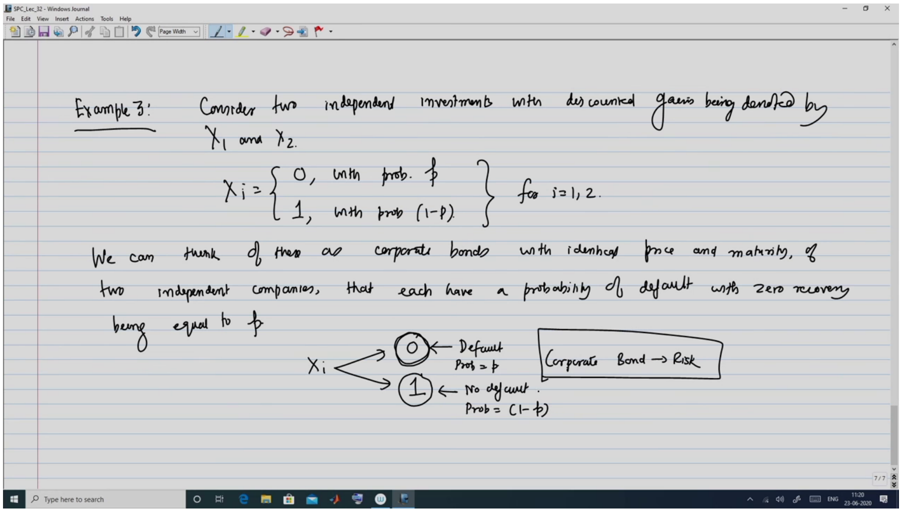

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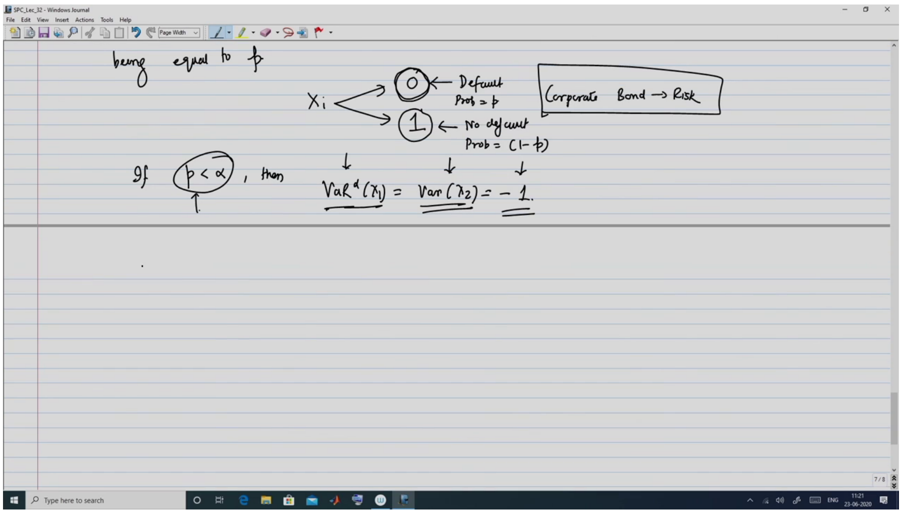

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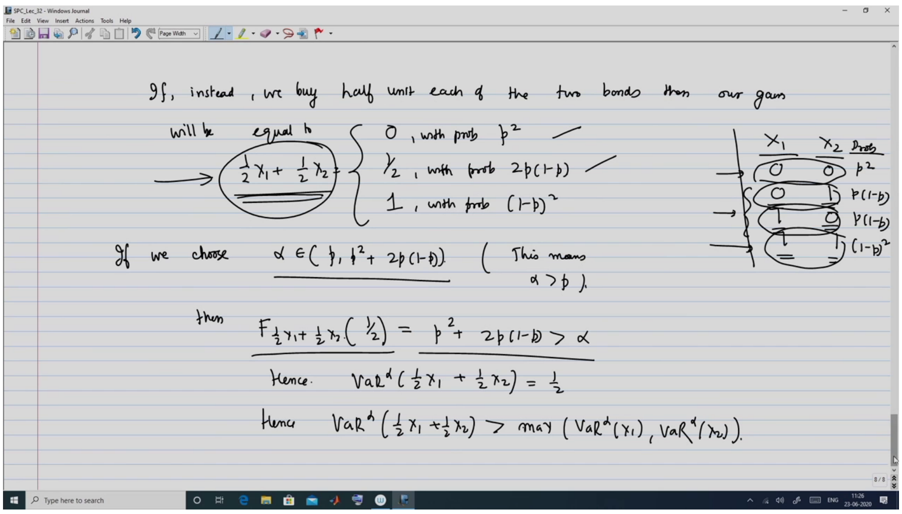

So, we consider an alternative strategy and say that, if instead we buy half unit each of the two bonds;

then our gain will be the random variable $\frac{1}{2}X_1 + \frac{1}{2}X_2$. Now, what are the values this random variable can take? So, what can happen is that, you could have. So, if I take X_1 and X_2 if; so either X_1 defaults, and X_2 defaults that is one possibility. If or X_1 may default, but X_2 does not default or you can have that X_1 does not default and X_2 defaults and you can have that neither of them default.

So, when both of them default, the probability is going and you remember that I had said that these are two independent companies. So, this means that, the probability of both the defaults, this probability is going to be simply p^2. The probability of a single default say, in this case as well as in this case and one no default; this probabilities are going to be $p(1-p)$ and $p(1-p)$. And the probability of no default in both the cases, this is going to be $(1-p)^2$.

So, this means that this random variable $\frac{1}{2}(X_1 + X_2)$, either can take the value half of the sum of these two which is 0 or it can take the value of half of 0 plus 1 or 1 plus 0. So, that is simply half or it takes half of 1 plus half of 1, which is equal to 1. So, for this value of 0, the probability is p square.

Now, for these two cases, we have the combined probability to be equal to twice p into 1 minus p and the third case, we have probability 1 minus p square. So, if we choose alpha to be in the interval say p plus p square plus twice p into 1 minus p; suppose we choose this alpha. So, this means that, obviously this means that alpha is a greater than p, right. So, it is consistent with this condition.

So, if I choose an alpha in this case from this interval, that will satisfy this condition and then I choose the same alpha in the second case. So, then what happens? Then the

$$F_{\frac{1}{2}(X_1+X_2)}(1/2) = p^2 + 2p(1-p) > \alpha$$

Hence, we can finally say

$$VaR^{\alpha}(\frac{1}{2}X_1 + \frac{1}{2}X_2) > max\big(VaR^{\alpha}(X_1), VaR^{\alpha}(X_2)\big)$$

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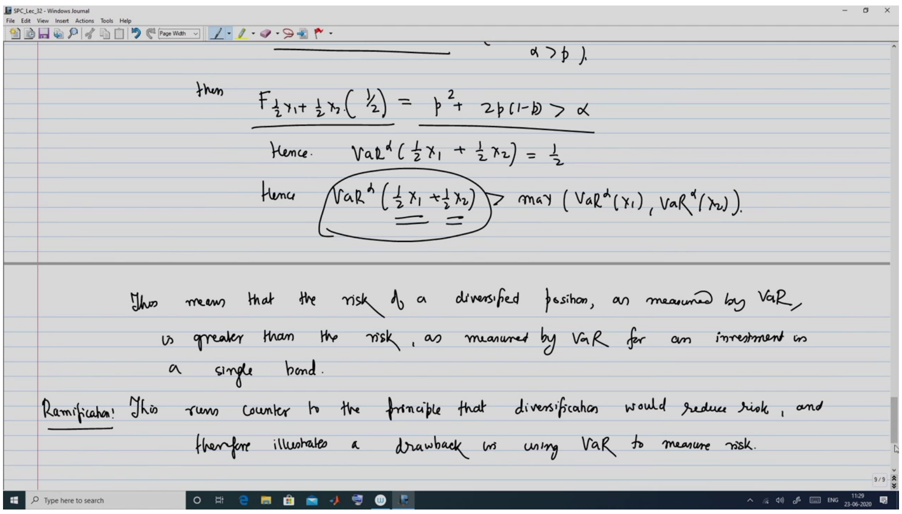

So, let us come to the main point in actually bringing this example into the picture and this means that, the risk of a diversified portfolio or a diversified position as measured by VaR. So, this is the VaR of the diversified portfolio; because you have invested in both the bonds is greater than the risk as measured by VaR for an investment in a single bond. So, the immediate ramification for this is the following that, this runs counter to the principle that diversification would reduce risk and therefore, illustrates a drawback in using VaR to measure risk. So, let me elucidate this in a little more detail, see when you started looking at VaR; VaR gave a very nice improvement over using standard deviation, because as a measure of risk because it looked at some sort of a worst case scenario in 95 percent of the time or rather or equivalently say in the 5 percent of the time. So, when I say that, we are looking at 95 percent VaR; so it basically means that, we are 5 percent certain that the losses will not exceed, we will exceed a certain amount that is the VaR and 95 percent chances are that you are safe and the losses will not exceed that amount. So, this point also. So, this is certainly an improvement over using just the standard deviation as a measure of risk. However, we must also keep in mind that, at the heart of portfolio theory in the paradigm of the Markowitz framework; the key aspect towards achieving a portfolio optimization is diversification. And through this example in which we consider two individual bonds and then a combination of those two bonds, which are identical; so that means that, your initial investment is identical in case of the individual bond or a half and half investment in those bonds. And it turns out that in this case, the value at risk that you have for alpha being greater than p; it turns out that as a result of diversification, the value at risk actually has increased as compared to a situation, where you would have just invested in either of the individual bonds. So, this essentially is a single counter example and there are many such counter examples that you can create which says that, the value at risk while bring you know a good alternative to using standard deviation or variance as a risk measure, suffers from this drawbacks that not in all situations can you achieve an improvement in terms of risk reduction just because you have carried out diversification. And it is this key weaknesses in case of value at risk that, motivates the usage of the introduction and usage of the conditional value at risk that we are going to introduce in the next class. So, let us continue our discussion keeping in mind this shortcoming; but nevertheless keeping in mind the usefulness of VaR as compared to just the standard deviation. Let us now continue our discussion and elaborate a little more on this by looking at a couple of important results; one in the discrete case and one in the continuous time case.

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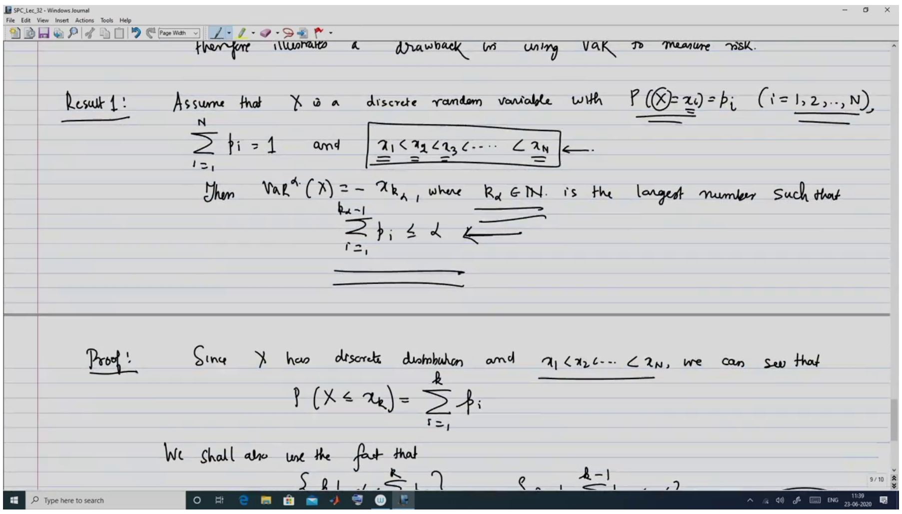

So, we first present this result one, it says the following that, assume that X is a discrete random variable. So, we have this random variables and suppose that we arrange this random variables in increasing order, the reason for which will become very clear. And we identify the corresponding probabilities of the random variable taking any of these increasing values or this ordered values x i to be equal to p i. Then what is the value at risk? Then value at risk of X this is going to be simply minus a x subscript k subscript alpha. And what is this x cross subscript k subscript alpha? So, where this k alpha is naturally going to be a natural number; because it is one of those 1 through N's is the largest number, such that the summation probability p i's i equal to 1 to k alpha minus 1, this summation is going to be less than or equal to alpha. And the reason for in this definition being introduced will be clear once you do the proof.

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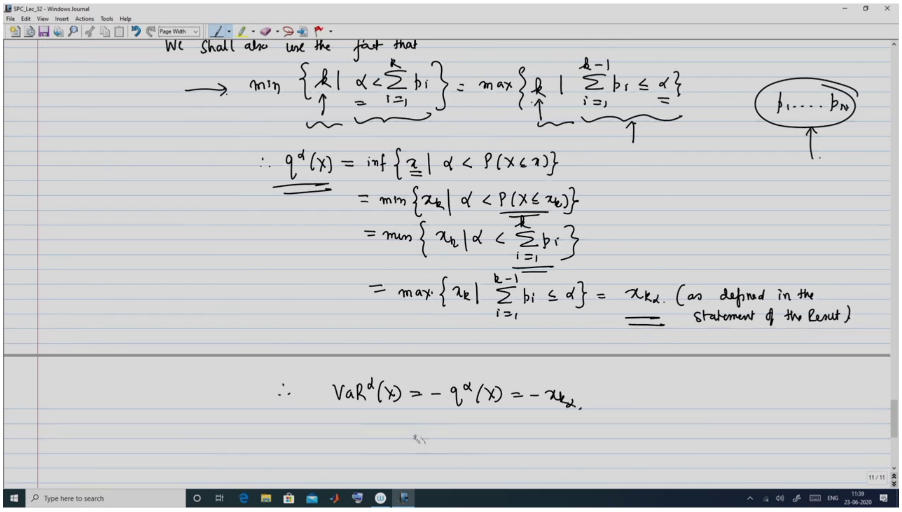

So, let us start the proof for this. Now, since X has a discrete distribution and your x 1 is less than x 2 all the way to less than x n; we can see that probability that X is less than or equal to x k and since these are ordered, so obviously probability that random variable X is less than equal to x k is going to be summation p i i is equal to 1 to k.

Now, we shall also use the fact that, minimum of all those case, such that alpha is less than summation p i i is equal to 1 to k; this will be simply the maximum of all those case such that summation p i i is equal to 1 to k minus 1 is less than or equal to alpha.

So, basically you can observe this very easily, if you actually order the, look at the ordered values of p 1 all the way to p of capital N and you basically set the alpha to be some sort of a bifurcation criteria. So, here of course, you know alpha is pre specified. So, accordingly you identify the two probabilities in between which this alpha lies. So, it is possible that alpha. So, accordingly you will essentially have that the sum of the probabilities in one case will be less than or equal to alpha here and sum of the probabilities in the other case is going to be greater than alpha. So, essentially this means that you look at it the cumulative distribution of the probabilities and then you essentially identify the point where alpha is going to lie. And so, essentially alpha is going to lie between sum of some case; at some point you are going to switch over alpha being less than a summation of all the probabilities to the situation, where the alpha is going to be greater than or equal to sum of all the probabilities and that switching point is the one and using that. So, essentially that means that, in one case the minimum the value of k is where alpha is less than summation of p i that is the same as the this and that is going to be the index for which the switching will happen and that is going to be exactly the same as the maximum of that value of k, such that alpha is going to be greater than or equal to summation of all the p i's. So, once we have identified this fact. So, therefore, we are now in a position to calculate the upper quantile which then can of course, be used to calculate what is the value at risk. So, the upper quantile by definition is the infimum of all those X, such that alpha is less than probability of X less than or equal to x and this is nothing but, it is the smallest. So, what are these X values? These are x k; so it is the smallest x k, such that alpha is less than probability of X less than or equal to x little k. And this is minimum of x k, such that alpha is less than summation p i i is equal to 1 to k. So, this comes from the definition of this probability and this minimum is going to be nothing, but maximum. So, making use of this results. So, these k's are obviously, this can be replaced by x k. So, this is going to be maximum of all those x k such that, summation of p i. So, this term here i equal to 1 to k minus 1 is less than or equal to alpha. And what is this? This is precisely what we had defined to be our k alpha. So, this is going to be nothing, but x of k alpha as defined in the statement of the result.

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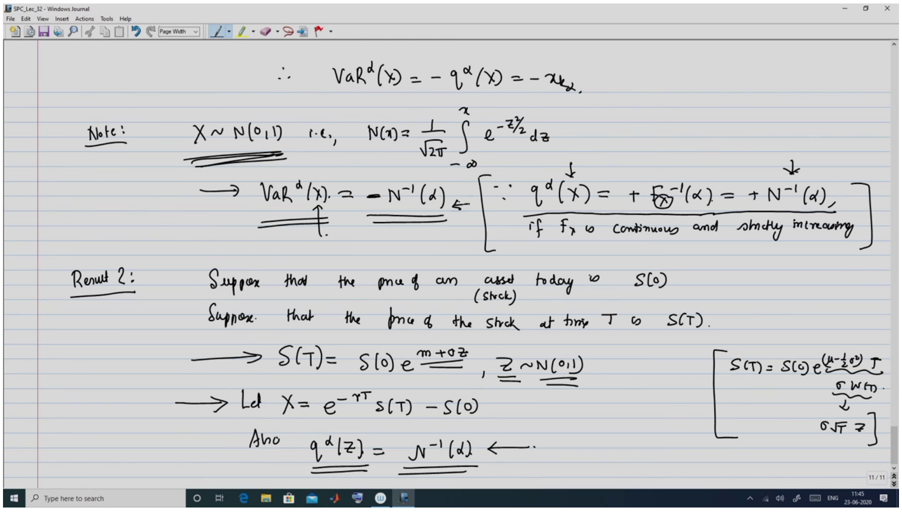

So, we are done. So, therefore, since the upper quantile is x k alpha. So, therefore, the value at risk is negative of upper quantile and this is minus x k alpha, alright. So, now, let us make a note that. So, let us now move to the continuous time setting. So, if X is a standard normal random variate; that is N of x the cumulative distribution of this is 1 over square root of 2 pi into integral minus infinity to x e raised to minus z square over 2 into d z. So, then your VaR alpha of X, if X is a standard normal random variate; this is simply going to be minus N inverse of alpha. And the reason is that, we make use of the result that the upper quantile q alpha of X is simply going to be minus N inverse of alpha is equal to minus F X inverse of alpha's, so you recall one of the results. And since the cumulative distribution is the cumulative normal distribution; so that is why I get minus N inverse of alpha. And this happened if F X is continuous and strictly increasing, which is the case in case of the normal distribution. So, this means that, if I take X to be standard normal random variate and you recall this result for the upper quantile being given by this. So, if your X is a standard normal random variate; so then the X is your N 0, 1. So, essentially then this will become simply minus N inverse of alpha. So, we found out what is the VaR in case, the random variable X is normally distributed and that is going to be minus N inverse of the pre specified value of alpha. And now, we are now in a position to do our second result and we have to recall the geometric Brownian motion model. So, suppose that the price of an asset today is S 0 and suppose. So, this is this asset is a stock, and suppose that the price of the stock at time capital T is S of capital T. Now, you will recall that S of capital T this is given by S naught S of 0 into sum e raised to m plus sigma z, with z following N 0 1 distribution. As you recall that one of the form which this was given was S 0 is equal to S T equal to S 0 into e raised to mu minus half sigma square into T plus sigma of W of T. So, this is another way of writing, where we take this term to be m and sigma of W T is going to be simply sigma of z; remember W T is N 0, 1 N 0, T, right. So, this actually can be should be written as, this is sigma of W T and this can be rewritten as a sigma into square root of T into sum N 0, 1 distribution, ok. So, what is going to be the X? So, accordingly the X which is the discounted gain is e raised to minus r T into S T minus S 0. And also from here you know that, q alpha of Z; because Z is N 0, 1. So, from here we get q alpha of Z is simply going to be equal to N inverse of alpha. So, just a slight correction here, this is actually plus. So, that is how I got the. So, the VaR was minus of this. So, please make this correction; this is actually plus of N inverse of alpha. So, I just reproduce this result here. So, we have three things now; we have S of T, a formula for that using geometric Brownian motion, accordingly we calculate the discounted gain. And since we have a Z sitting here which we will need; so accordingly we consider we, we just note that q alpha of Z is equal to N inverse of alpha, so from this result, ok.

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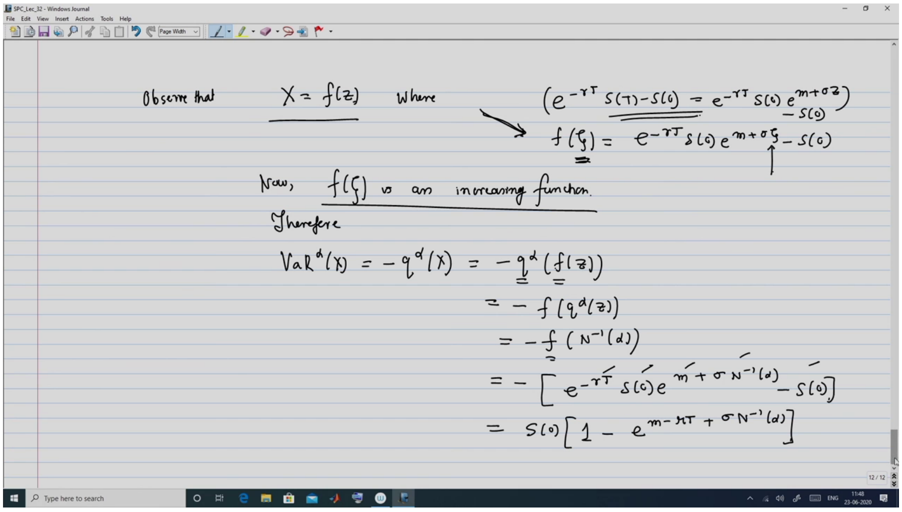
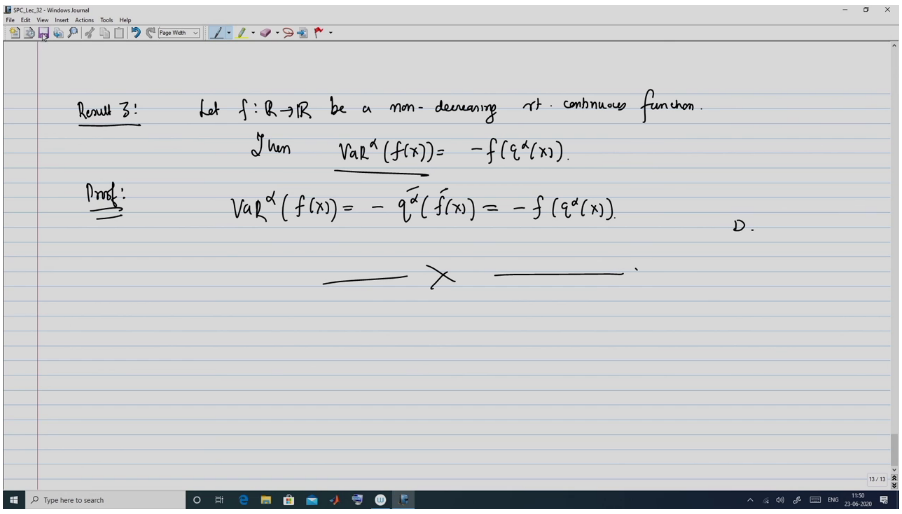


So, now observe that X is equal to f of z. So, I mean by X I mean this X, where this function f says zeta is nothing, but. So, e raised to minus r T. And what is S T? S T is S 0 into e raised to m plus sigma. So, let me actually write this in detail. So, this is going to be e raised to minus r T. So, we have e raised to minus r T s T minus S 0 this is e raised to minus r T into S 0 into e raised to m plus sigma z. So, therefore, I can define f of zeta which is a, this is going to be e raised to minus r T S 0 e raised to. So, there is a minus S 0 e raised to m plus sigma zeta minus S 0. So, if I put zeta is equal to z, then we get back or recover this relation, ok. Now, you observe carefully; now this f of zeta is an increasing function, right. So, therefore, what is we get? We get VaR alpha of X; what is this going to be? This is by definition minus q alpha of X the minus of upper quantile of X. What is X? X is f of z. So, this is minus q alpha f of z; but f is an increasing function. So, I can interchange this q the quantile and f. So, this is equal to minus f of q alpha of z. What is q alpha of z? Q alpha of z is N inverse minus alpha. So, N minus f inverse alpha, sorry this is a q alpha of z is N inverse of alpha, so just a correction. So, this is minus f of N inverse of alpha. And now we have to evaluate the function for zeta equal to N inverse of alpha. So, these becomes minus of e raised to minus r T S 0 e raised to m plus sigma N inverse of alpha minus S 0. So, I have just replace this zeta with N inverse of alpha and this can be written as S 0. (Refer Slide Time: 57:30)



So, I take the common factor of S 0 out and this becomes 1 minus e raised to. So, the S 0 is out and we combine m and r T. So, this becomes e raised to m minus r T plus we have this sigma N inverse of alpha. So, we just have a last brief result which says that, let f from R to R that is this is a real valued function be a non-decreasing right continuous function; then VaR alpha of f of X is going to be minus f of q alpha of X. And the proof for this is you know extremely straightforward. What is going to be start with the left hand side; so VaR of f X is going to be by definition minus q alpha of f X and we just interchange the f and q, so this becomes minus f of q alpha of X, alright. So, this brings us to the end of this discussion on value at risks. So, just to do a recap; we picked up from the results that we had done last time on the quantiles. And we now introduce the definition of value at risk of X to be negative of the quantile of X and we give a interpretation of the VaR in terms of looking at the loss and the percentage level with which you can predict this loss. We looked at three different examples the first was to recognize that the value at risk in case of a bond, since it is a risk free asset is going to be 0. And then we looked at another example in case of the value at risk, in case of a risky asset and finally, we looked at a very critical example of value at risk, which shows that diversification does not necessarily lead to reduction in the value at risk. And, we identified that this observation of the failure of diversification in achieving a lower measure of risk as given by the value of value at risk is something that is going to be used as a motivating reason for moving on to what is the conditional value at risk. And, then we looked at a few results; primarily the result pertaining to determining the value at risk in case of a random variable taking the discrete values and provided they are ordered in increasing fashion. And, then we looked at what is going to be the value at risk in case of a random variable forming following a standard normal random variate, which then was used to calculate the value at risk of its investment in a stock, where the behavior of the stock is modeled in the continuous time framework using the geometric Brownian motion. So, this brings us to an end to the discussion on value at risk. And in the next class, we will conclude this topic with the discussion on what is the conditional value at risk. Thank you for watching.