Mathematical Portfolio Theory
Module 06: Bond Portfolio Management
Lecture – 03 Convexity; Hedging and Immunization

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Module – 06 Bond Portfolio Management Lecture – 03 Convexity; Hedging and Immunization

Hello viewers, welcome to this next lecture on the NPTEL MOOC Course on Mathematical Portfolio Theory. So, far what we have been looking at, we have been looking at in the bond portfolio optimization in the framework of finding out matrices or measures, to ascertain what is the risk associated with the bond from the point of view of it is price movement resulting from change in the interest rate. And so far we have looked at, what is the pricing of the bond yield to maturity. And we have talked about duration, the properties of duration and what is going to be the duration of a portfolio and the concept of immunization or risk management in case of bond portfolio by matching of duration.

So, we continue our discussion in today's class by extending this to the notion of convexity. And then we will further discuss this concept of convexity in the context of a portfolio. And then we will see this in the framework of the percentage change of the bond prices and then we will look at a couple of example to illustrate this notion of immunization, which is the most basic notion in case of risk management for bond portfolios.

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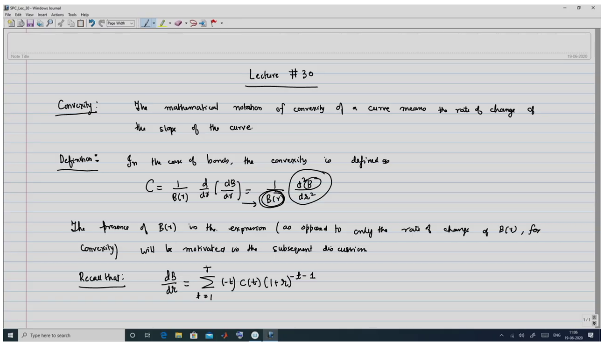


So, accordingly we now start this lecture, with the concept of convexity. So, recall that, the mathematical notation of convexity of a curve it means or is interpreted as the rate of change of the slope of the curve.

So, more formally in the context of our discussion, I will start off with the definition as follows, that in the case of bonds the convexity is defined as

$$C = \frac{1}{B(r)}\frac{d}{dr}(\frac{dB}{dr}) = \frac{1}{B(r)}\frac{d^2B}{dr^2}$$

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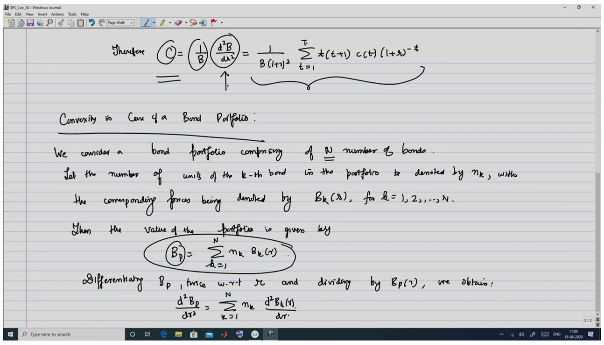


We have the expression of C. So, basically C is 1 over B into the second derivative and the second derivative here, is obtained by taking the derivative of this expression here, to eventually get this expression for convexity ah.

So, let us look at the convexity in case of a bond portfolio. So, we consider a bond portfolio comprising of capital N number of bonds. And let the number of unit, so the setup is the same as that of a duration in case of a bond portfolio. So accordingly, let the number of units of the k-th bond in the portfolio of this n number of bonds be denoted by $n_k$, with the corresponding prices being denoted by $B_k(r)$, and for k is equal to, remember there are n number of bonds, so $k = 1, 2, \ldots, N$

Then the value of the portfolio is given by $B_p$.

So now, what you do is, if you want to find out the convexity of this bond portfolio, so accordingly what we need to do is, then we differentiate $B_p$, twice remember this is the definition of convexity so we need the second derivative with respect to r and then divide this by $B_p(r)$.

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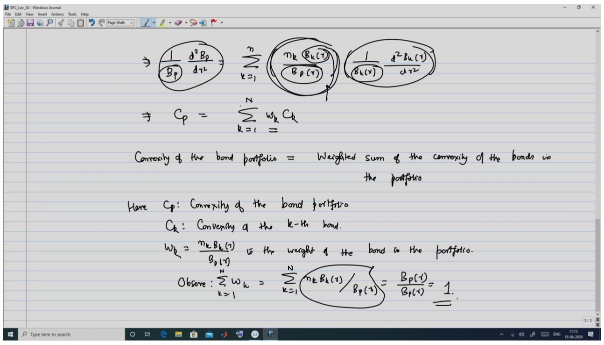


And this implies that if I want the definition of convexity, so I have to divide by $B_p$.

So, thus the convexity of the bond portfolio is going to be equal to the weighted sum of the convexity of the bonds in the portfolio. So, here $C_p = \sum_{k=1}^{N} w_k C_k$ is the convexity of the bond. So, $C_p$ is the convexity of the bond portfolio $C_k$ is the convexity of the k-th bond, $W_k = \frac{n_k B_k(r)}{B_p(r)}$.

And finally, you observe that summation $W_k$, k is equal to 1 to capital N. This is going to be 1.

So, the words defined in this way is again satisfies the condition, that the sum of the weights is equal to 1, ok.

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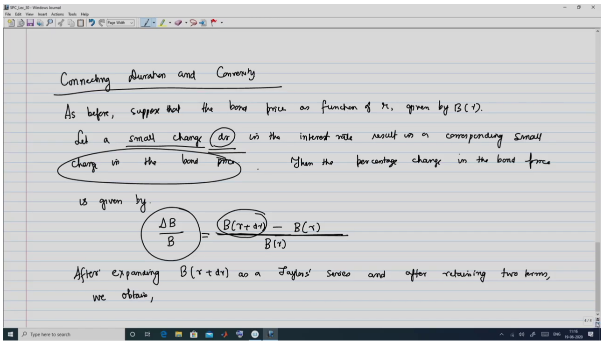


What you do next now is, we will now that you have defined durational convexity, the next thing we will do is now, that we will do a connection of duration and convexity ok. So, as before we consider the same setup and suppose, that the bond prices is a function of r, and is given by $B(r)$.

Now, let a small change $d_r$ in the interest rate result in a corresponding small change in the bond price. So, if a small interest rate change of $d_r$ results in the change in the bond price, then what we have? Then the percentage change in the bond price is given by the following.

$$\frac{\delta B}{B} = \frac{B(r+dr) - B(r)}{B(r)}$$

Now, let us focus on these particular term. So, and remember that we had said that $dr$ is a small change. So, accordingly we can make a Taylor series approximation, and accordingly we expand $B(r+dr)$, as a Taylor series.

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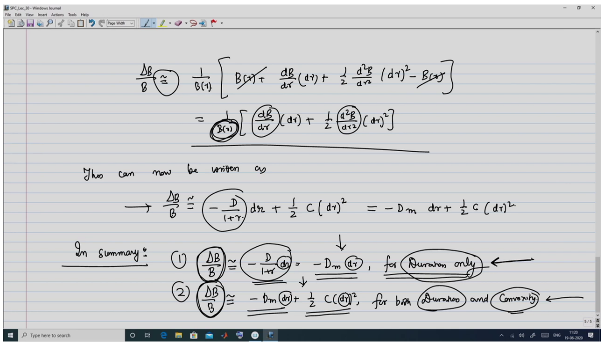

And after retaining the two terms of the trailer series, we obtain the above expression for

$$\frac{\delta B}{B} = \frac{B(r+dr) - B(r)}{B(r)}$$

We have,

$$\frac{\delta B}{B} = -\frac{D}{1+r}dr + \frac{1}{2}c(dr)^2 = -D_m dr + \frac{1}{2}C(dr)^2$$

So, this the first term where you only duration gives an estimate and the one which, second expression which involves the both duration and convexity; it gives us a better estimate as compared to the first case.

So, in some sort of this addition of the convexity term helps us obtain a correction or a better estimate as compared to the one that is given by, the just the duration only. And this is estimation for what? This is the estimation for the percentage change in the bond price as a result of the interest rate moving by some small amount of $dr$.

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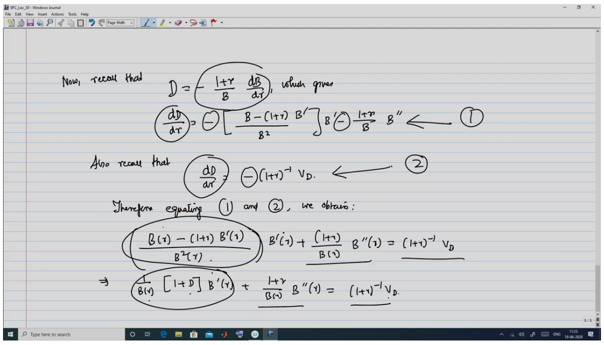


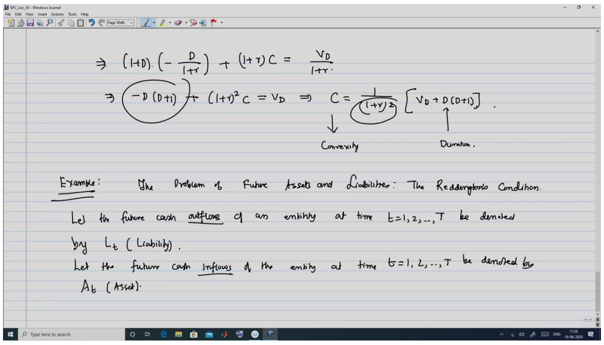

So, now; recall that

$$D = -\frac{1+r}{B}\frac{dB}{dr}$$

and

$$\frac{dD}{dr} = -(1+r)^{-1}V_D$$

Equating the above expression we get the required results. (Refer Slide Time: 21:14)

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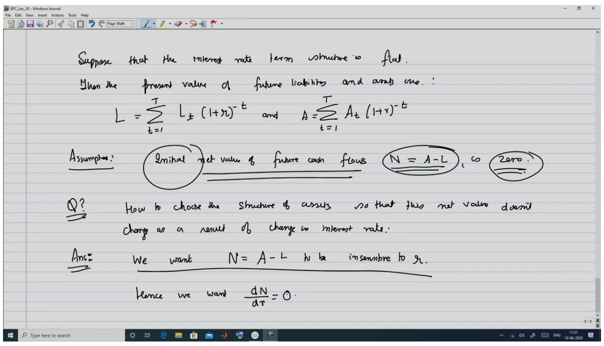


Now, suppose that so, as has been the case so far, we also assume in this case that the interest rate term structure is flat. So that means, the interest rate for all maturities is identical. Then the present value of future liabilities and assets are given by the following. So, if we have the payments liabilities of $L_t$ at time t. So, the present value of the t-th liability is

$$L = \sum_{t=1}^{T} L_t(1+r)^{-t}$$

and

$$A = \sum_{t=1}^{T} A_t(1+r)^{-t}$$

And the assumption is that, the initial net value of future cash flows. So, I will use $N = A - L$. So, this is the net value, and this is zero at the initial time.

And the question that you want to answer is that, how to choose the structure of assets, so that this net value does not change, as a result of change in interest rate.

So; that means, that this is unaffected by interest rate. And the broader answer to this is that, we want; so this is equivalent to saying that we want that this net value N given by A minus L to be insensitive to r. So, when I say that we want this to be insensitive to r, so hence; this means that we want that

$$\frac{dN}{dr} = 0$$

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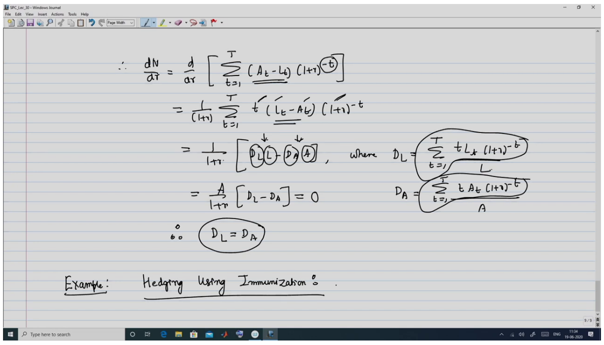

So, therefore, what is going to be my $\frac{dN}{dr}$? Equating it to zero gives

$$D_L = D_A$$

So that we are going to have a matching of our assets and liabilities, ok. So, next we look at another example, the last example; and this is on ‘Hedging using immunization’.

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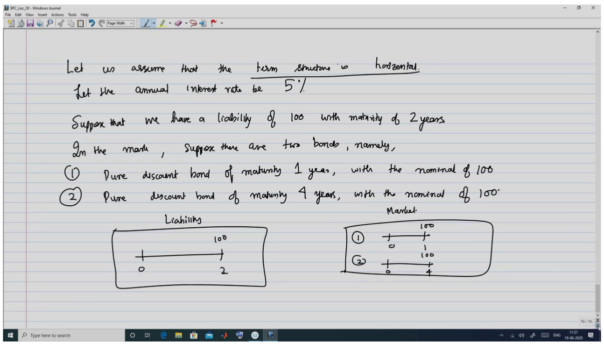

So here, let us have the problem set up. So, let us assume that the term structure is horizontal. And let the annual interest rate be 5 percent so; that means, there is a 5 percent interest rate for all maturities, because I have assumed that term structure is horizontal. Now, suppose that; we have a liability of 100 with maturity of 2 years. Now, in the market, suppose there are two bonds, namely, the first one is a pure discount bond of maturity 1 year, with the nominal of 100. And secondly, there is a pure discount bond of maturity 4 years with the nominal of 100. So, let us observe this graphically. So, what you have is that, on one side you have the liability ah; that means, from now to 2 years, at 2 years you have to pay an amount of 100. And accordingly, and then what you will have is that, in the market there are only two bonds. ah. The first bond is of maturity 1 year with a nominal of 100, and these are pure discount bond. And the second one has, has a maturity of 4 years with a pure discount bond. Now, what am I supposed to do here, I mean why am I calling it the hedging? The reason why I am calling it the hedging is that I have a liability of an amount of 100, and as I will shortly write that I start off with an amount of 90.70. Now, if it turns out that the market had a 2 year bond available at the interest rate of 5 percent, that all I would do is simply invest the money that I have now that is an amount of 90.70 for a period of 2 years. And at the end of 2 years I will exactly get an amount of 100 in order to meet my liability. However, in the market; unfortunately there is no 2 year bond and the only thing that I have in the market are 1 year bond and there is a 4 year bond, both are pure discount and both of them have a nominal of 100. So now, the problem is that if I invest a part of my money in the first bond for 1 year, then at the end of 1 year I have to reinvest that money. And then I do not know what is going to the prevailing interest at that point of time. And likewise, I have to invest a certain amount of money in the 4 year bond, which I have to liquidate or break at the end of 2 years. And again I do not know how much money I will receive, because the amount of money that I receive at the end of 2 years by breaking the bond is going to be dependent on what is going to be the prevailing interest rate at that point of time. So, the question that now faces me is that, since I do not have ah; I am sitting with an amount of 90.70 and since I do not have the choice of buying a 2 year bond, and then I have to decide that; I will have to go for a portfolio of bonds comprising of the first bond with a 1 year maturity and the second bond with a 4 year maturity.

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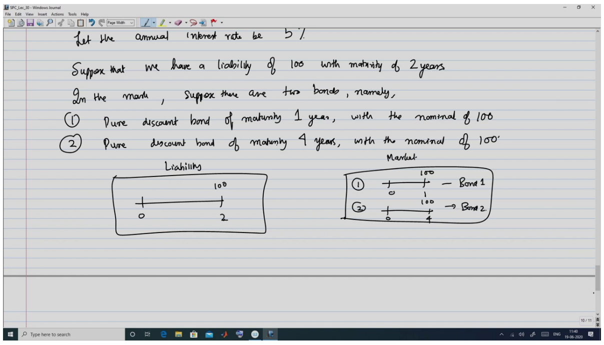


So, let me call this the first one to be bond 1 and the second one to bond 2. And the question is that, how much of the initial amount of money that I have; namely 90.70 as that amount of money, what proportion should I invest in the first bond with 1 year maturity and what proportion or the weight that I should assign in case of the second bond with the 4 year maturity. So, let me call this bond 1 as I have just now said, and this is going to be my bond 2, ok.

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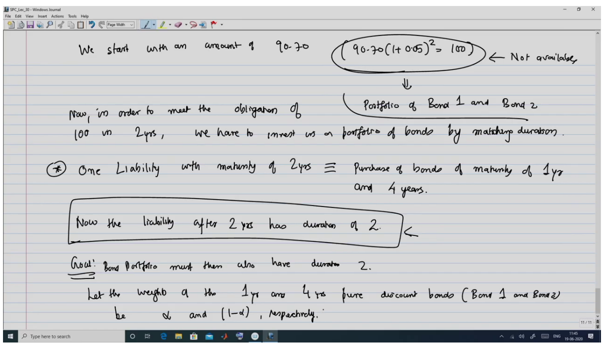


So now, as I as I mentioned while discussing this, the just now, we start with an amount of 90.70. So, if you have 90.70, if you had a 2 year bond; this into 1 plus 0.05 square, this would have gone to 100. But unfortunately, this kind of bond is not available. And so, I have to create a portfolio of bond 1, that is this bond and bond 2, that is this 4 year bond. Alright. So, now in order to meet the obligation of 100 in 2 years time, we have to invest in a portfolio of bonds by immunization, or what is known as matching duration. So, you have already seen that, you know the duration matching has to be done. So, this means that in order to put this in more specific term, what we say that is one liability with maturity of 2 years must be matched with, by purchasing of bonds of maturity of 1 year and 4 years. Now, the liability after 2 years has duration of 2. Now, why am I saying the liability after 2 years has a duration of 2. So, liability of this 100 that you have to pay after 2 years, this is like you having issued a bond of with the nominal value of 100 and the bond is a pure discount bond. So, you would recall that one of the properties of a pure discount bond in terms of duration is that, the in case of a pure discount bond it is duration is going to be exactly the same as the maturity of the bond. And that is the reason why that the liability of this this, 2 year liability of an amount of 100 to be paid at maturity. That liability is equivalent to a bond having a duration of 2. So accordingly, whenever I try to protect myself against the adverse movement of the interest rate. So in this case the bond portfolio that I am going to create, it has to be created in such a way that the duration of the bond portfolio has to exactly match this duration of 2 years. That means, I must choose my investment proportion or weights for bond 1 and bond 2 so that the resulting duration of the bond portfolio is going to be the same as 2. So, accordingly, so what is the goal? So, just to put it in more clear term, that the portfolio of bonds must then also have duration 2. So, let the weights of the 1 year and 4 years pure discount bonds, that is the one which are identified as bond 1 and bond 2. This be alpha and 1 minus alpha respectively.

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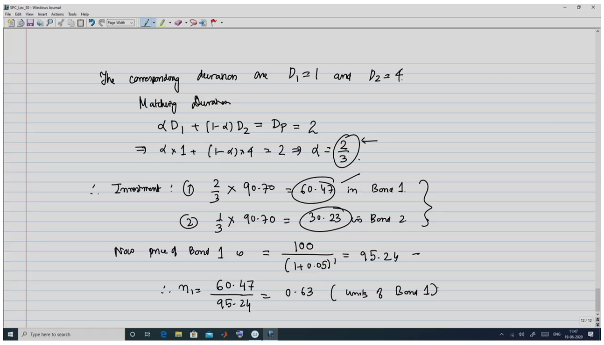


The corresponding duration, now remember that both this 1 and 4 year bond these are a pure discount bond. So, the corresponding duration are $D_1 = 1$ and $D_2 = 4$.

So, then we do this matching duration. And what is this going to be? It is going to be

$$\alpha D_1 + (1-\alpha)D_2 = D_p = 2$$

. Hence, we get

$$\alpha = \frac{2}{3}$$

So, therefore, what do you do? We invest. How much was the investment amount? My investment amount was 90.70 and I invest 2 by 3 of this amount and this is going to be 60.46, and this is my investment in bond 1. And similarly, you will have the remaining 1 by 3 you, of the total amount of 90.70. So, this is 30.23, this is invested in bond 2, alright.

So, based on the weight 60.70 is invested in bond 1 and 30.23 is invested in bond 2. Now, what is the price of the bond? The price of bond 1, how do we get the price of bond 1? It is basically the price of the bond 1 is equal to, it pays a nominal of 100. And since, it is a pure discount bond paying interest rate at 5 percent, so the; I have to divide this by 1 plus 0 point, so this is 1 plus 0.05 raised to 1, because it is a 1 year bond and this is 95.24.

So, therefore, $n_1$ is going to be equal to 60.47; that means, the number of bonds that you purchase is the amount invested in the bond which is 60.47 divided by the price of the bond which is 95.24 and this turns out to be 0.63 and this is the units of bond 1.

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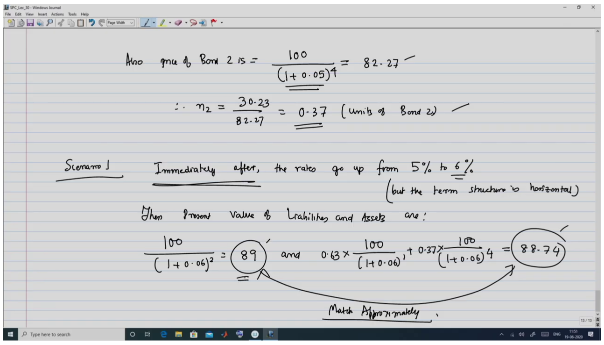


Now, also what is going to be the price of bond 2? Now, remember the bond 2 also has a nominal of 100 and this was a 4 year bond at a interest rate of 0.05 or that is 5 percent. So, this is going to be 100 divided by 1 plus 0.05 raised to 4, since it is a 4 year bond and this turns out to be 82.27.

So, therefore, what is $n_2$? The number of units of the second bond. This is going to be the amount of 30.23 invested in the second bond divided by the price of the second bond which is 82.27. And this is going to be 0.37 units of bond 2. And please do not assume that, you know the number of units must add up to 1, even though in this case, coincidentally $n_1 + n_2 = 1$, but it is; obviously, not necessarily the case.

Alright; so, what is now the purpose of my matching duration? The purpose of my matching duration is to protect myself against the interest rate movements. So, accordingly let us consider two scenarios of an imperial change in the interest rate movement. So, the first scenario is going to be the following.

So, immediately after you have made this purchases, say the rates go up from 5 percent to 6 percent, but of course, I still assume that the term structure is horizontal. So, this means that earlier we had a 5 percent interest rate for all maturities, now it is 6 percent for all maturities. Then, what is going to be; the present value of assets and liabilities are given by; So, what is this going to be? So, what is going to be your assets? So, actually I, let me do the liabilities first. So, the present value of liabilities and assets are.

So, what was your liability? Your liability was 100 and it is present value, since it is an immediate change in the interest rate, so it is present value is going to be 100 divided by 1 plus 0.06, because the interest rate now is 0.06 square and this is equal to 89. And in the second case, what is going to be the present value of bond 1? It is going to be 100 divided by 1 plus 0.06 raised to 1 and 100 divided by 1 plus 0.06 raised to 4 multiplied by the respective units which are 0.63 and 0.37. And we add this up and this turns out to be 88.74 and these two numbers, these match approximately. So, they should have similar.

So, earlier these values are 90.74, but now they are almost matching with a slight error resulting from, you know, the truncations resulting from the approximation used in determining the duration.

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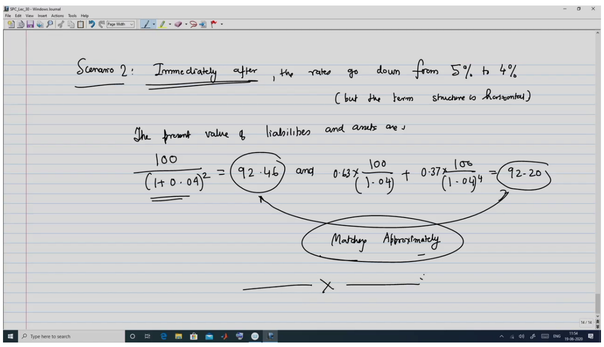


Now, so, this matches approximately, then you would consider what is the scenario 2? so, now, here we saw that there was an increase in interest rate. So, let us consider a scenario 2, where there is a decrease in the interest rate.

So, the assumption is that, immediately after the rates go down from 5 percent to 4 percent, but as before the term structure is still horizontal. So, now, instead of 5 percent being the interest rate for all maturities, it is going to be now 4 percent.

So, in this case also, the present value of liabilities and assets are the following; So, in this case your liability again you recall was 100. And then, it is present value is going to be 1 divided by now the interest rate is 0.04 which is equivalent to 4 percent.

So, the liability is going to be present value of the liabilities, 100 divided by 1 plus 0.04 square and this is going to be 92.46. And 100 divided by 1.04 is going to the present value pertaining to the first bond, and 100 divided by 1.04 raised to 4, this is going to be the present value of the second bond.

And then I multiply this by the respective number of units which are 0.63 and 0.37 and I add them up, and this adds up to 92.20 and again you see that there.

So, earlier again this, both these numbers this number here; and this number here, they were 90.70, but now again they are more or less matching approximately.

So, this brings us to the end of this lecture and this topic on bond portfolio optimization. So, just to do a recap of whatever we have done today, we continued our discussion on by extending the notion of duration and we have extended it to a second order matrix for determining or which is linked to the risk associated with the bond in particular the liquidity risk. And we defined what is known as the convexity.

And then we looked at, what is the, what is going to be the convexity in case of a portfolio. And you looked at approximations to the percentage change in the bond price, as a result of interest rate movement in terms of about duration and convexity. And then we looked at two examples, illustrating the notion of immunization or matching of durations. So, this as I said concludes this module on bond portfolio optimization. And from the next week, we will start off on the new topic, that is on risk management using the notion of value at risk and conditional value at risk.

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Thank you for watching.