Mathematical Portfolio Theory
Module 06: Bond Portfolio Management
Lecture – 02 Duration: Immunization

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Hello viewers, welcome to this next lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. You would recall that in the previous class we had talked about what is bond and we looked at what is the price of the bond, interest rates and so on and we have defined what is going to be the duration of the bond. And we had identified that the duration of the bond is synonymous with the risk that is associated with the bond and the risk associated with the bond here means in the context of how the price of the bond changes depending on how the interest rate actually moves during the lifetime of the bond. And, this usually is a risk that will arise when the owner of the bond decides to liquidate the bond prior to the maturity of the bond. So, in today's class we will extend this notion and we will talk a little bit about what is certain properties of the duration of a bond and a portfolio of bonds. And, we will talk about the concept of risk mitigation of a portfolio of bonds by using the notion of immunization.

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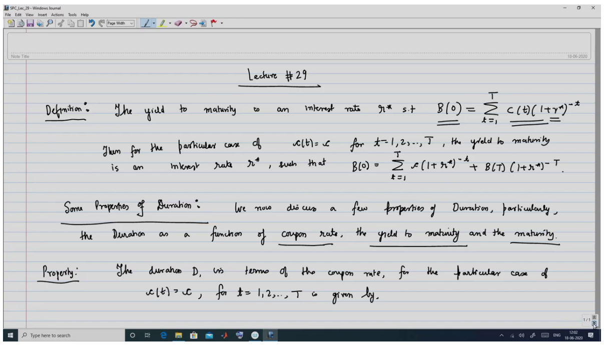

So, accordingly we start this lecture with the definition of something called yield to maturity. So, the yield to maturity is an interest rate which are denote by r^* such that it is given by the expression

$$B(0) = \sum_{t=1}^{T} c(t)(1+r^*)^{-t}$$

So, this is some sort of a uniform interest rate that is applicable so that the present value of the coupon becomes equal to the price of the bond. Then for the particular case of all the coupons being identical to c for t is equal to 1 2 all the way to capital T. The yield to maturity is an interest rate r^* such that

$$B(0) = \sum_{t=1}^{T} c(t)(1+r^*)^{-t} + B(T)(1+r^*)^{-T}$$

We next talk about some properties of duration. So, we now discuss a few properties of duration particularly the duration as a function of 3 things namely coupon rate, the yield to maturity and the maturity itself. So, start off with the first property which says that the duration D in terms of the coupon rate for the particular case of $c(t) = c$.

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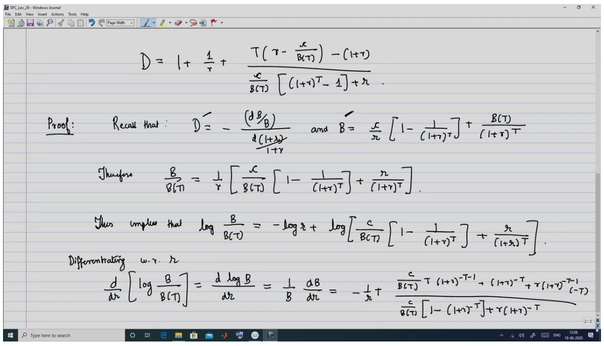

For t is equal to 1, 2 all the way to capital T is given by D,

$$D = 1 + \frac{1}{r} + \frac{T(r - c/B(T)) - (1+r)}{c/B(T)[(1+r)^T - 1] + r}$$

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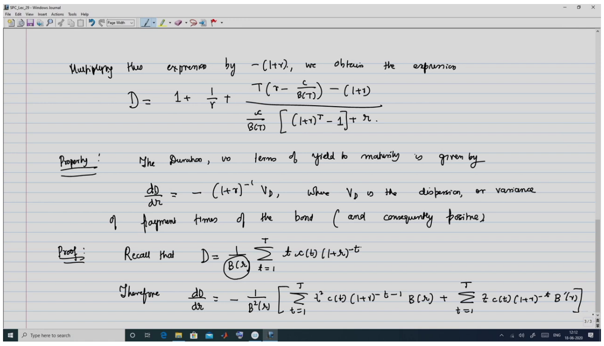

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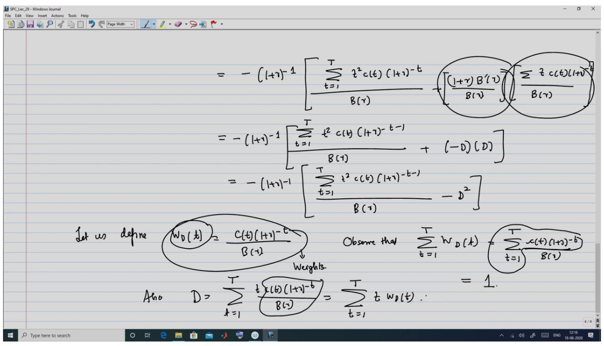

Finally, we get

$$D = \sum_{t=1}^{T} t w_D(t)$$

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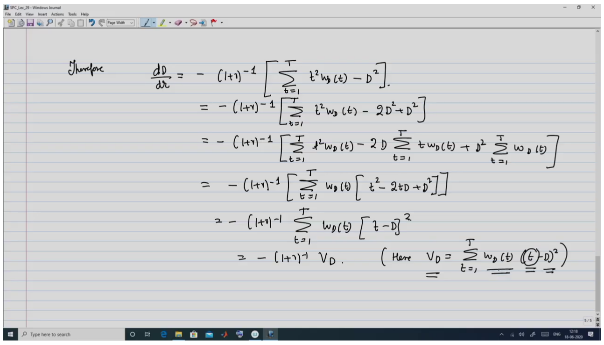

So, therefore, $\frac{dD}{dr}$ what is this going to be? This is going to be

$$-(1+r)^{-1}V_D$$

V D was defined as the dispersion or variance of the payment times of the bond. So, this is going to be the variance of the payment time of the bonds here t is like the random variable.

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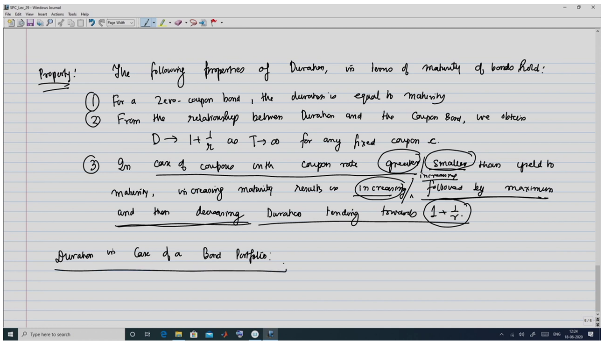

So, we come to the last of the three properties and this property is that the following properties of duration in terms of maturity of bonds hold. So, I will enumerate the properties. So, for a zero coupon bond, the duration is equal to maturity.

The second property is that from the relationship between duration and the coupon bond we obtain that the duration tends to $(1+\frac{1}{r})$ in case of a perpetual bond that is at T tends to infinity for any fixed coupon c. And, thirdly in case of coupons with coupon rate greater than or smaller than yield to maturity increasing maturity results in increasing followed by maximum and then decreasing duration tending towards $(1+\frac{1}{r})$.

So, this means that in case of coupons with when the coupon rate is greater than yield to maturity it results in increasing duration tending towards $(1+\frac{1}{r})$ and when the coupon rate is smaller than the yield to maturity it follows that initially it is an increasing duration followed by maximum and then decreasing duration alright.

So, for greater than the yield to maturity we will have a increasing duration and when it is less than the yield to maturity we will have a piecewise kind of a behavior where there is an increasing followed by the maximum and then there is a decreasing behavior of the duration in both case eventually tending to $(1+\frac{1}{r})$ ok.

So, now we come to a very important topic and that is on looking at what is going to be the duration of a portfolio of bonds, remember that our whole concept of bond portfolio optimization is driven by the idea that bonds are also susceptible to risk and accordingly we need to make an appropriate choice of the weights of the bonds in a portfolio driven by what is the risk associated with the portfolio of the bond and that risk association is reflected through the duration of the portfolio of bonds.

So, naturally the next thing that we need to look at is we are going to look at the duration in case of a portfolio of bonds comprising of say N number of bonds. So, accordingly we start the topic of duration in case of a bond portfolio.

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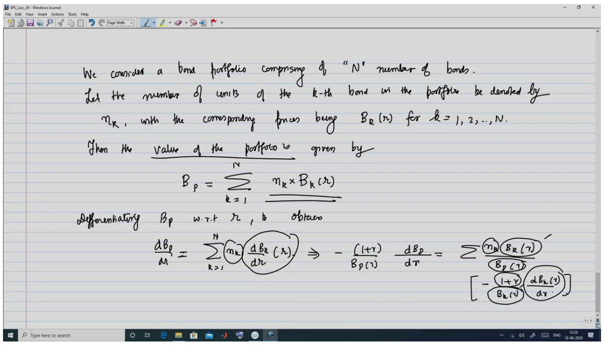

So, we consider a bond portfolio comprising of say capital N number of bonds and we let the number of units. So, I am proceeding in exactly the same way that I had done in case of stock. So, that the number of units of the k-th bond out of this capital N bond in the portfolio be denoted by say n subscript k with the corresponding prices being $B_k(r)$ for k is equal to 1 2 all the way to capital N.

Then, the value of the portfolio is given by the following:

$$B_p = \sum_{k=1}^{N} n_k B_k(r)$$

So, now what we do is that we differentiate B_p with respect to r to obtain duration $\frac{dB_p}{dr}$.

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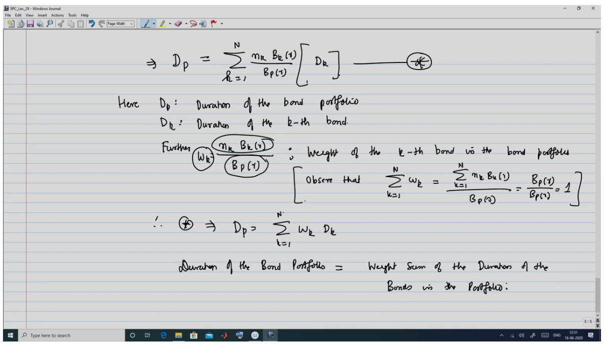

So, this will give me duration of portfolio, D_p

So, here what do we have here your D_p is the duration of the bond portfolio your D_k is the duration of the k-th bond. So, W_k is; obviously, going to be by definition, the weight of the k-th bond in the bond portfolio and you observe that

$$D_p = \sum_{k=1}^{N} w_k D_k$$

So, now that we have defined what is going to be the duration of bond portfolio it is now imperative that you move on to how to use this concept of the duration of a bond in order to take care of the risk aspect of the bond that is for better risk management of my bond portfolio and this concept of the management of a bond portfolio as far as the risk of it is concerned that arises from the movement of the underlying interest rate is what is known as the process of immunization.

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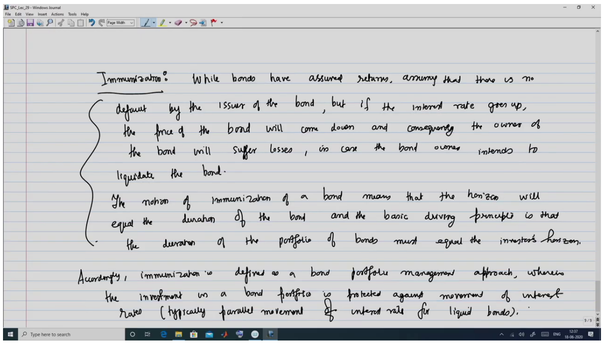

So, accordingly we start off with our discussion on immunization. So, in order to motivate this let me first observe that while bonds have assured returns assuming that there is no default by the issuer of the bond, but if the interest rates goes up, the price of the bond will come down and consequently the owner of the bond will suffer losses, in case the bond owner intends to liquidate the bond. The notion of immunization of a bond means that the horizon will equal the duration of the bond and the basic driving principle is that the duration of the portfolio of bonds must equal the investors horizon.

So, armed with this motivation here we accordingly can say, that immunization is defined as a bond portfolio management approach, wherein the investment in a bond portfolio is protected against movement of interest rates typically parallel movements of interest rate for liquid bonds.

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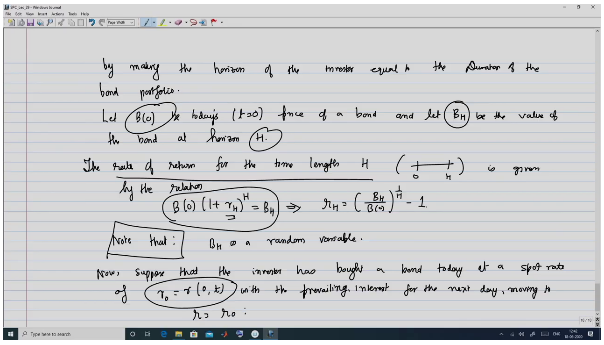

And this is done by making the horizon of the investor equal to the duration of the bond portfolio. So, this means that if you are considering the portfolio of bonds. So, obviously; that means, that it is going to be constituted by purchasing several number of bonds now each of those bonds will have a duration.

So, you have to decide on the appropriate weight that you have to assign to each of those bonds. So, that the weighted sum of the duration of the bonds the and the appropriate choice here means the appropriate choice of the corresponding weights and the consequent weighted sum of the durations is going to be the duration of the portfolio of bonds.

And in order to carry out the process of immunization you will need that the duration of this bond portfolio given by the weighted sum of the duration of the individual bonds must be such that the duration is equal to the horizon of the investors point of view and accordingly this becomes like an optimization problem while you decide what is going to be the weights such that D_p becomes equal to the horizon H in case of each of the individual investors.

So, in order to be little more specific so, let say that $B(0)$ be today's; that means, at time $t = 0$ price of a bond and let B_H be the value of the bond at horizon H. The rate of return for the time length H; that means, from this during the interval 0 to capital of H is given by:

$$B(0)(1+r_H)^H = B_H$$

.

So, accordingly I have this relation and where my r_H is the rate of return for the time length H and this gives me that

$$r_H = (\frac{B_H}{B(0)})^{1/H} - 1$$

Now suppose that the investor has bought a bond today at a spot rate of $r_0 = r(0,t)$ with the prevailing. So, at the time of purchase of the bond the interest rate was r naught given by this with the prevailing interest rate and consequent to that the prevailing interest rate for the next day having moved to some r which is different from r naught.

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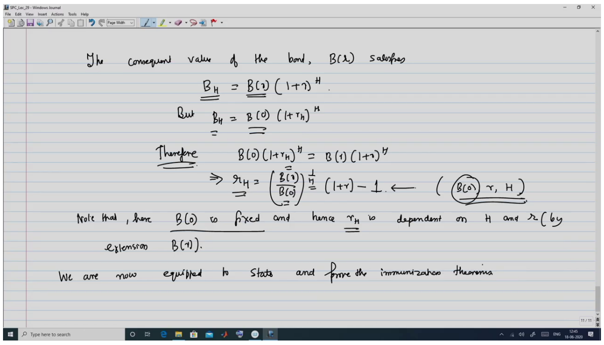

Now, we can easily the above results.

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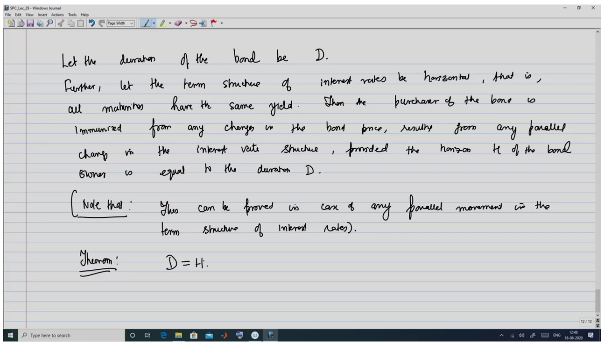

So, let the duration of the bond be D and this can be extended to a bond portfolio. Further let the term structure of interest rates be horizontal that is a straight line. So, that is all maturities have the same yield.

So, we are just starting off with a very simple case, then the purchaser of the bond is immunized from any changes in the bond price resulting from any parallel change in the interest rate structure, provided; that means, subject to the condition that the horizon H of the bond owner is equal to the duration of capital D.

So, now note that this can be proved in case of any parallel movement in the term structure of interest rates. So, we now state the following theorem in a very the statement of theorem is very brief that D is equal to H.

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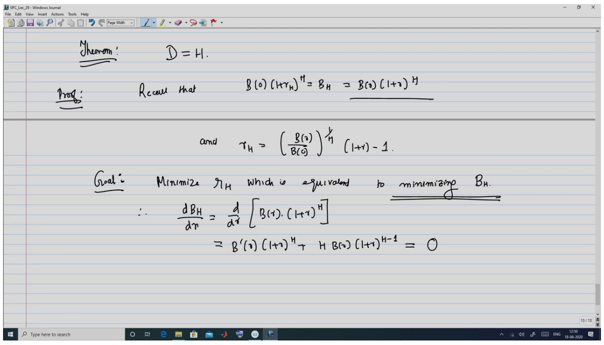

So, let us start off with the proof of this theorem now recall that what we had

$$B_H = B(0)(1+r_H)^H = B(r)(1+r)^H$$

and the other relation that we had was r of capital H was B r over B 0 raised to 1 over H multiplied by 1 plus r minus 1.

$$r_H = (\frac{B(r)}{B(0)})^{1/H}(1+r)-1$$

So, you recall these two relations and what is going to be our goal, the goal is to minimize r_H which is equivalent to minimizing B_H. Therefore, we have the expression of $\frac{dB_H}{dr}$

Now, since I am trying to minimize B_H. So, this means that I have to take this equal to 0 ok.

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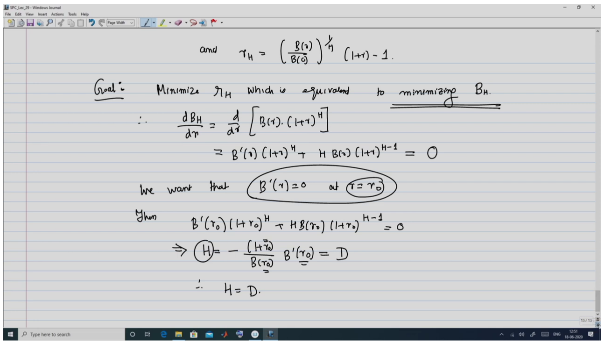

So, what do we want? So, we want that at the initial time point that

$$B'(r) = 0,\ r = r_0$$

After solving, we finally get

$$H = D$$

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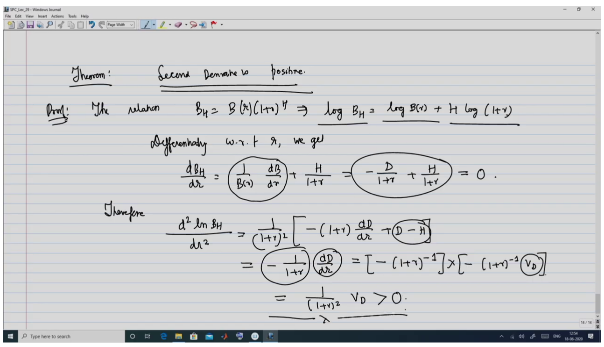

So, now we have to prove another theorem and then this theorem is that the second derivative is positive remember that we are trying to minimize B_H. It is shown above.

So, this brings us to the end of this lecture. So, just to do a brief recap we essentially did two things in this lecture the first is that we looked at the properties of duration and it is behavior vis a vis the coupon rate the yield to maturity and the maturity itself and the second component that we did was that we looked at the notion of immunization and we proved two theorems involving the first and the second derivative. And identify the key component of the process of immunization is that the duration of the bond or the portfolio of bonds must be equal to the horizon of the investor that is the period for which the investor is looking to hold on to their position. So, in the next class we will look at an extension of this case to get a slightly better results in terms of assessing the risk of a bond and that is what is known as the convexity of a bond. (Refer Slide Time: 51:35)

Thank you for watching.