Mathematical Portfolio Theory
Module 06: Bond Portfolio Management
Lecture 01: Interest rates and bonds Duration

Professor Siddhartha Pratim Chakrabarty[1]

[1]*Department of Mathematics, Indian Institute of Technology Guwahati, India*

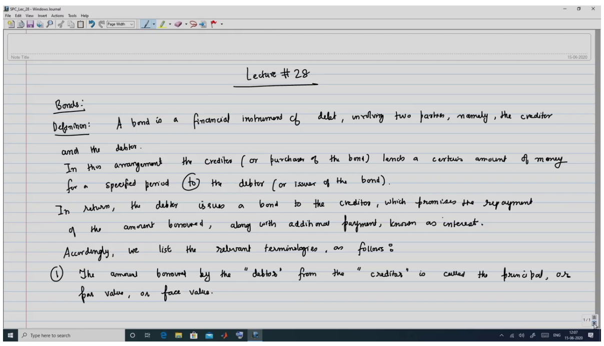
Hello viewers. Welcome to this next lecture on this NPTEL, MOOC course and Mathematical Portfolio Theory. So, far we have been talking mostly about equity based portfolio optimization, that means a portfolio optimization involving risky assets and we just only consider a scenario where you could include only one bond to a portfolio of several risky assets or stocks. So, now we are going to focus on a new topic with an emphasis on a portfolio that comprises of several bonds an exclusively of the bonds and the risk associated with the bond. And remember that we had talked a little bit about bonds earlier, and what is the source of this risk, and we had discussed several possibilities possible sources of risk in case of a bond from the point of view of a bond holder. And in this part of this course we will mostly emphasize on the topic of the sensitivity of the bond prices vis-a-vis the interest rate.

So, remember what happens is that suppose you are investing in a bond for a certain period of time and after a point of time you decide that you want to sell off your bond to somebody else.

Now, what might happen is that you based on the interest rate that was initially applicable to the bond you have a certain expectation or estimation of what the price of the bond that you will get in the intermediate time points. However, in the intermediate time point the interest rate might have change and this will have an immediate ramification on the price of your bond. So, what might happened is that if the interest rate goes up then the bond price is goes down and you will lose money as a bondholder. But in case the interest rate goes down then the price of the bond goes up and then you will get a gain as the bond holder. And this is something that you are going to discuss in detail in today's class.

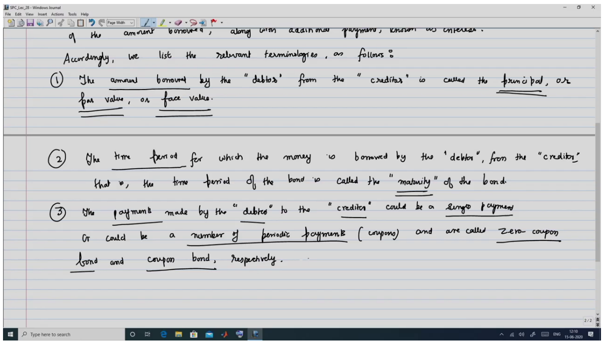
And we will begin with a prelude to a background to bonds in general including the pricing of the bond, and then we will move on to the concept of duration which is a measure of risk for a bond. And the eventual goal for this part of the course is to look at two important measures of risk, namely, duration and convexity, and how it is applicable in case of an optimization of a bond portfolio the term for which is known as immunization.

(Refer Slide Time: 02:53)



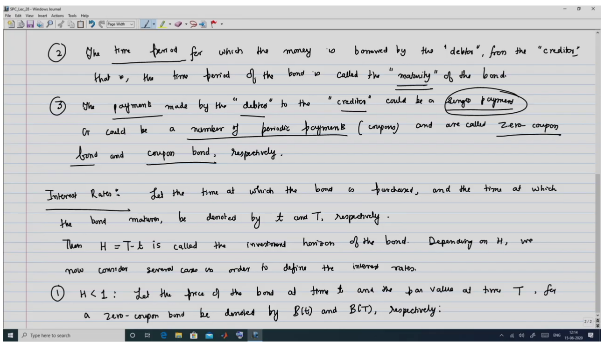
So, accordingly we start this lecture. So, first of all let us talk about bonds and we start off with the definition of a bond. So, a bond is a financial instrument of debt involving two parties namely, the creditor and the debtor. So, in this arrangement the creditor or who extends the credit or the purchaser of the bond lends a certain amount of money for a specified period to the debtor or issuer of the bond. And in return the debtor issues a bond to the creditor which promises the repayment of the amount borrowed from the creditor along with additional payment known as interest. So, accordingly we list the relevant terminologies as follows. So, the first one is the following that the amount borrowed by the debtor from the creditor is called the principal or par value or face value.

(Refer Slide Time: 06:49)



The second terminology is the regarding the time. So, the time period for which the money is borrowed by the debtor from the creditor that is the time period of the bond is called the maturity of the bond. Thirdly, the payments made by the debtor to the creditor could be a single payment or could be a number of periodic payments or coupons. And these are called a zero-coupon bond and coupon bond respectively. So, the 3 point that we are talking about is first of all we talk about the amount borrowed and this is what is known as the principal or par value or face value and the time period for which the money is borrowed this is known as the maturity of the bond. And the payment made by the debtor to the creditor if those payments are a single payment then the bond is called a zero-coupon bond. And if there are number of periodic payments then the bond is known as a coupon bond.

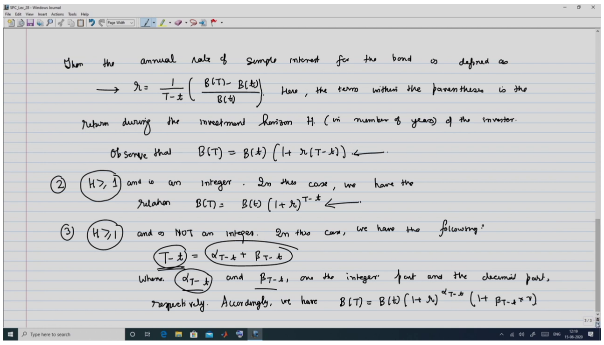
(Refer Slide Time: 09:30)



So, we next come to the interest rates an introduce a few notations. So, let the time at which the bond is purchased and the time at which the bond matures be denoted by small t and capital T, respectively. Then, $T-t$ which I will denote by H this is called the holding period or the investment horizon of the bond.

Now, depending on H, we now consider several cases in order to define the interest rates. So, the first one is when H is less than 1, so we enumerate all this. So, the first one is H less than 1. So, let the price of the bond at time t and the par value at time T. So, the single payment here this is sometimes referred to as the par value. So, this par value at time T for a zero-coupon bond be denoted by $B(t)$ and $B(T)$, respectively.

(Refer Slide Time: 12:26)



Then the annual rate of simple interest for the bond is defined as

$$r = \frac{1}{T-t}(\frac{B(T)-B(t)}{B(t)})$$

Let us now look at the second case if your H is greater than or equal to 1 and is an integer. So, we looked at the case H less than 1, and now we look at two sub cases of H greater than or equal to 1 and this is an integer. So, in this case we have the relation

$$B(T) = B(t)(1+r)^{(T-t)}$$

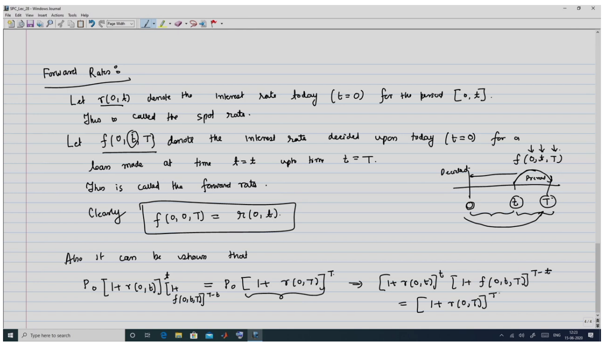
And for the third scenario of H being greater than or equal to 1, and is not an integer; in this case we have the following:

$$T - t = \alpha_{T-t} + \beta_{T-t}$$

So, basically, I take capital T minus small t and decompose it to an integer and a decimal part. So, α_{T-t} is the integer part and β_{T-t} is the decimal part, respectively. So, accordingly we have the following:

$$B(T) = B(t)(1+r)^{\alpha_{T-t}}(1+\beta_{T-t}r)$$

(Refer Slide Time: 16:45)



So, next let us come to what are known as the forward rates. So, let $r(o,t)$ denote the interest rate today and I will denote today as t is equal to 0 and this is the interest rate for the period starting today and going up to time t. This $r(o,t)$ is called the spot rate, ok.

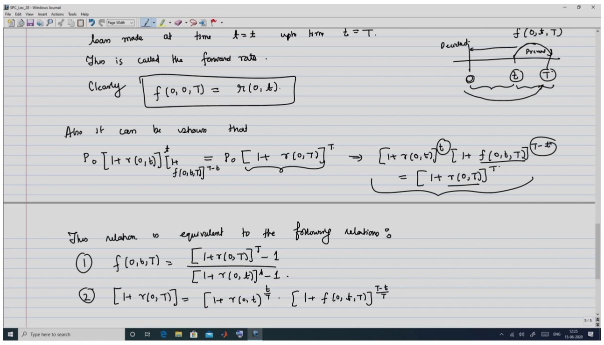
Let us the introduce another notation. So, let $f(0,t,T)$, denote the interest rate decided upon today that is at t equal to 0 for a loan made at time t is equal to little t up to time t equal to capital T. And this is that means, $f(0,t,T)$ is called the forward rate.

So, just to put things in perspective, so we have this time today at 0 and at time 0 we decide on the rate for a loan that is going to be applicable from time small t to time capital T. So, the decision; so, the interest rate is going to be applicable from 0 to capital T as given by small t to capital T, but the rate is fixed at time t equal to 0.

So, the interest rate will prevail in this period, but this will be decided at time t equal to 0. So, clearly if I take small t is equal to 0, so that means,

$$f(0,0,T) = r(0,t)$$

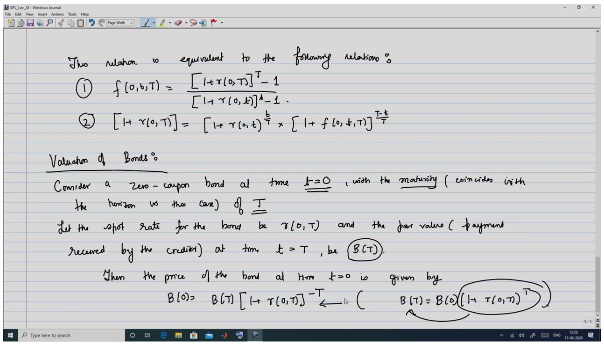
(Refer Slide Time: 20:47)



So, this relation, I mean this relation is equivalent to the following relations. So, the first one is I can solve this for f of 0, little t, capital T and this turns out to be equal to 1 plus r of 0 capital T raised to capital T minus 1 divided by 1 plus r 0, little t raised to little t minus 1.

And the second relation will be for r 0, T. So, this you can solve to be 1 plus r 0, T. So, we take the 1 over capital T-th root on both sides, so this becomes 1 plus r 0, t, this term here raise to t over capital T into 1 plus f of 0, little t, capital T; again this is going to be capital T minus small t over capital T.

(Refer Slide Time: 22:03)



Now, let us come to the concept of valuation of bonds. So, for this we first consider a zero-coupon bond at time t is equal to 0, with the maturity. And in this case of course, the maturity also coincides with the horizon in this case of capital T. So, the maturity is capital T and this is the same as the horizon.

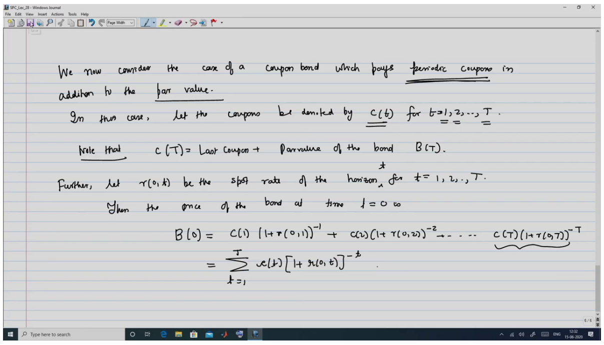
Now, let the spot rate for the bond, and remember the bond is from t equal to 0 to capital T. So, the spot rate for the bond be $r(0,T)$. And the par value, and remember what is the par; value par value was basically the payment received at the end by the creditor. So, this par value at time t is equal to capital T be denoted by B of capital T. So, as we had done before.

Then, if you receive an amount of $B(T)$, then the price of the bond or the amount that the creditor should lend at the initial time $t = 0$ is going to be given by

$$B(T) = B(0)(1+r(0,T))$$

So, suppose that we initially start off with an amount of B 0 and invest for the time capital T. So, this will go by a factor of 1 plus r 0, capital T raised to T and this is going to be B of T. So, getting this term on the other side gives us this relation, ok.

(Refer Slide Time: 24:40)



Now, that we are done with the zero-coupon bond, so we now consider the case of a coupon bond which pays or has payouts of periodic coupons in addition to the par value. So, in the zero-coupon bond you only had the par value, but now in case of coupon bonds in addition to the par value we will have periodic coupons.

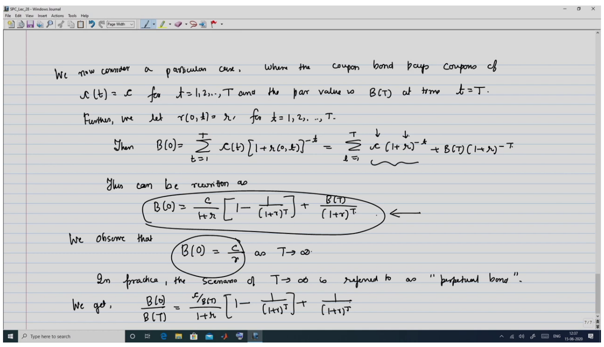
So, accordingly we need to have a notation for this coupon. So, in this case we let the coupons be denoted by c of t for t is equal to 1, 2 all the way to capital T. So, these are the bunch of periodic coupons being paid at time t equal to 1, 2 all the way to capital T.

Note that. So, you have to be very careful that when you talk about the coupons, we have to be careful about the last coupon. So, note that c of capital T is not just the last periodic coupon, but it is the last coupon plus the par value of the bond B of capital T. Further, let r 0, t be the spot rate of the horizon for t is equal to; so, for the horizon a little t for little t equal to 1, 2 all the way to capital T.

Then, the price of the bond at time t is equal to 0 is given by the following:

$$B(0) = \sum_{t=1}^{T} r(t)[1 + r(0,t)]^{-t}$$

(Refer Slide Time: 27:45)



Now, $B(0)$ can be written as

$$B(0) = \frac{1}{1+r}[1 - \frac{1}{(1+r)^t}] + \frac{B(T)}{(1+r)^T}$$

So, in this expression if you let T tends to infinity we will have

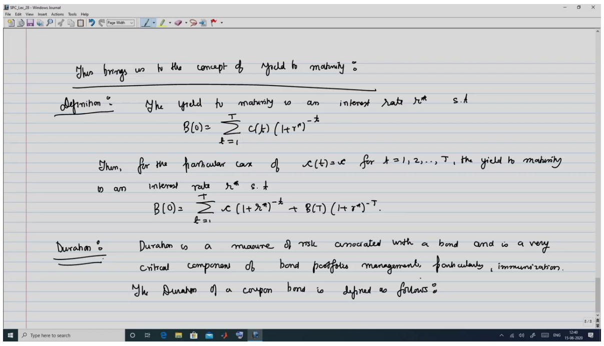
$$B(0) = \frac{c}{r}$$

So, in practice you might wonder what is this infinite maturity. So, in practice the scenario of this T tending to infinity is referred to as what is known as a perpetual bond, ok.

So, coming back to this relation. So, we also we finally, get that

$$\frac{B(0)}{B(T)} = \frac{c/B(T)}{1+r}[1 - \frac{1}{(1+r)^T}] + \frac{1}{(1+r)^T}$$

(Refer Slide Time: 31:47)

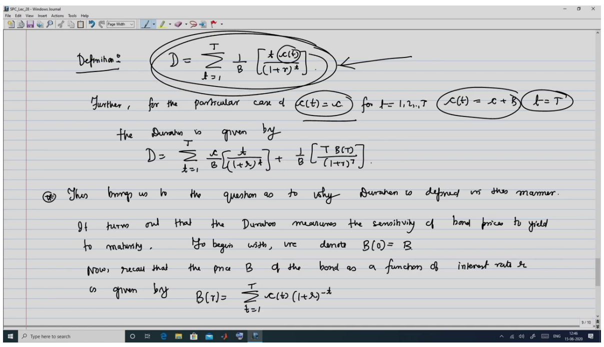


So, this brings us to the concept of yield to maturity. So, we start off with the definition and the yield to maturity, so we bring into the observation that the yield to maturity is an interest rate r star such that B 0 is going to be summation c of t into 1 plus r star raised to minus t, where t goes from 1 to capital T.

Then, for the particular case of c of t is equal to little c, for t is equal to 1, 2 all the way to capital T. The yield to maturity is an interest rate r star such that B of 0 is equal to summation c into 1 plus r star raised to minus t, t is equal to 1 to capital T plus B of capital T into 1 plus r star raised to minus t.

So, we next come to the concept of what is known as duration and this is the point where we start about the bond portfolio management concepts. So, duration is a measure of risks. So, it is a measure of risk just like we had in case of stock it is a measure of risk that is associated with a bond and is a very critical component of bond portfolio management particularly, what is known as immunization. And this is a concept that we will deal off deal with the later stage. So, the duration of a coupon bond is defined as follows.

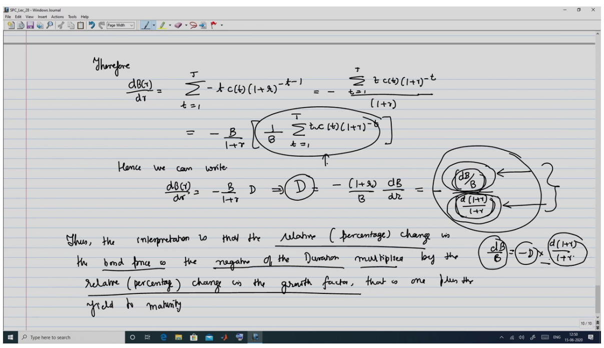
(Refer Slide Time: 35:05)



Duration is defined as:

$$D = \sum_{t=1}^{T} \frac{1}{B}[\frac{tc(t)}{(1+r)^t}]$$

(Refer Slide Time: 38:55)

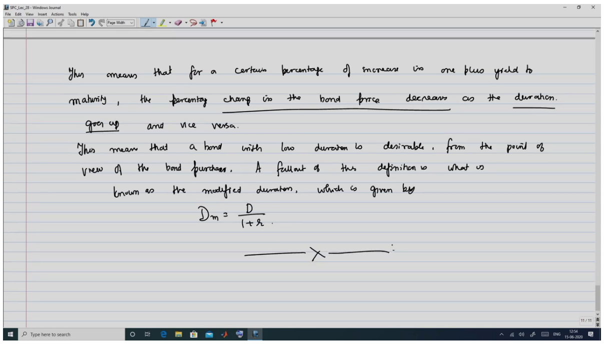


$$\frac{dB(r)}{dr} = -\frac{B}{1+r} * D$$

, Hence,

$$\frac{dB}{B} = -D * \frac{d(1+r)}{1+r}$$

(Refer Slide Time: 43:37)



So, this means that for a certain percentage of increase in 1 plus yield to maturity, the corresponding percentage change in the bond price. So that means, for a change in this as given by this term, the corresponding change in the bond price decreases as the duration goes up. So, as the duration goes up this term to minus D will become smaller. So, accordingly this will decrease. So, this bond change in bond price decreases as the duration goes up and vice versa. So, further this in turn means that a bond with low duration is desirable from the point of view of the bond purchaser. So, a fallout of this definition is what is known as the modified duration which is given by

$$D_m = \frac{D}{1+r}$$

So, this brings us to the end of today's lecture. Today's lecture we started off a new topic that is on bond portfolio optimization, and we focus primarily on the background to this namely, we looked at what is the interest rate, what is yield to maturity and then what is how do you determine the price of a bond, in case of bonds which have zero-coupons that means, there is only final payment and in case of coupon bonds.

And then, we introduce the definition of a duration and looked at a justification of why the duration is defined in that particular manner, and why is this an indicator of the risk associated from the point of view of a purchaser of a bond. So, in the next class, we will continue our discussion on this by looking at several properties of duration and then we will also talk about the duration in case of a portfolio of bonds. Remember that this is a course on portfolio theory. So, we will look at the duration in the paradigm of a portfolio comprising of several bonds.

(Refer Slide Time: 47:13)

Thank you for watching.