Mathematical Portfolio Theory

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Module 05: Optimal Portfolio and Consumption
Lecture 06: Duality/Martingale Approach in Discrete and Continuous Time

Hello viewers, welcome to this lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. You would recall that, in the previous class we had worked on extending the Hamilton Jacobi Bellman Framework, and you look at some examples. And then we are started of talking about the single period optimization using the martingale or the duality approach. And in today's class, we will continue our discussion on this by looking at a couple of examples. And then we will talk about the multi-period setup in case of the martingale approach for discrete time. And then, we will talk about, what we are going to talk about is going to be the continuous time setup. And in the continuous time setup, we will look at the martingale approach, in order to decide on what is going to be our optimal portfolio. So, accordingly we start this lecture.

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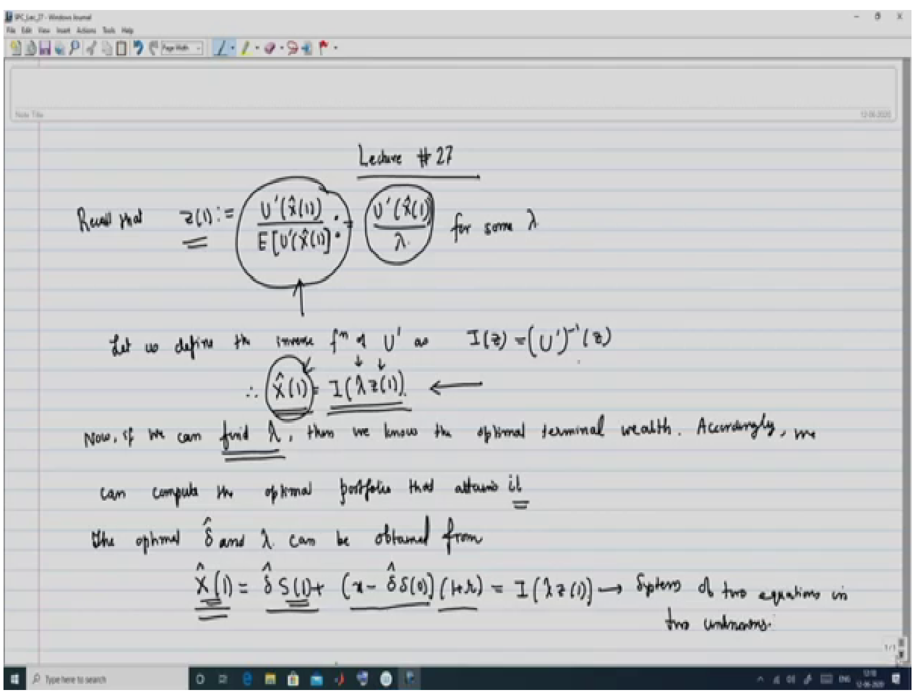

By recalling that $Z(1)$ was defined as

$$Z(1) := \frac{U'(\hat{X}(1))}{E[U'(\hat{X}(1))]} = \frac{U'(\hat{X}(1))}{\lambda},$$

for some λ. So, we begin with defining the inverse function of U' as

$$I(Z) = (U')^{-1}(Z)$$

and therefore, we have

$$\hat{X}(1) = I(\lambda Z(1)).$$

Now, if we can find this parameter lambda, then we know. So, here if we find the lambda; we already know what is Z 1, so we can calculate I of lambda Z 1. So, the determination of X hat of 1 then reduces to finding of lambda. So accordingly, then we know the optimal terminal wealth. And thus we can compute the optimal portfolio that attains it, and by it I mean that attains $\hat{X}(1)$. So, in conclusion; the optimal delta hat, which is the optimal holding in the stock and lambda which in turn will give you your $\hat{X}(1)$, can be obtained from

$$\hat{X}(1) = \hat{\delta}S(1) + (x - \hat{\delta}S(0))(1 + r) = I(\lambda Z(1)).$$

So remember that $\hat{X}(1)$ is going to be delta hat of the stock price at time 1 plus x minus delta hat S naught that is the investment in the bond grown by a factor of 1 plus r. And this is from this relation; this $\hat{X}(1)$, which has this expression must be equal to I or the inverse function lambda into Z 1. And, this is nothing but system remember that Z 1 has two values and S 1 has two values and accordingly x 1 has two values. So, this gives is a system of two equations in two unknowns.

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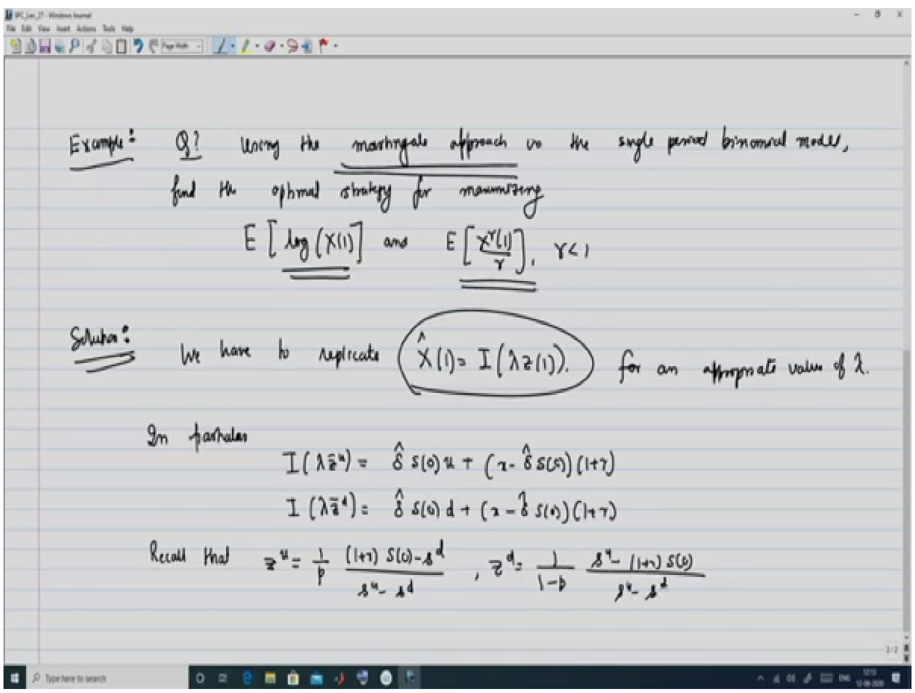

So, let us now look at an example. So, the question is that, using the martingale approach in the single period binomial model, find the optimal strategy for maximising

$$E[\log(X(1))],$$

that is for log utility function. So, what I am asking is, use the martingale approach to find the optimal strategy with log utility and power utility for a single period model, alright, so, for gamma less than 1. So, the solution is as follows, we have to replicate, and by replication we mean that we have to get lambda, so that

$$\hat{X}(1) = I(\lambda Z(1)).$$

That means, you have to figure out our delta hat and lambda in such a way that this condition is satisfied. And this must be done for an appropriate value of lambda, alright. So; in particular, we have

$$I(\lambda \bar{Z}^u = \hat{\delta}S(0)u + (x - \hat{\delta}S(0))(1 + r),$$

and

$$I(\lambda\bar{Z}^d = \hat{\delta}S(0)d + (x - \hat{\delta}S(0))(1+r).$$

So now; recall that, we had the expression for Z^u and Z^d determined in the previous class. So,

$$Z^u = \frac{1}{p}\frac{(1+r)S(0) - s^d}{s^u - s^d}$$

and

$$Z^d = \frac{1}{1-p}\frac{s^u - (1+r)S(0)}{s^u - s^d}.$$

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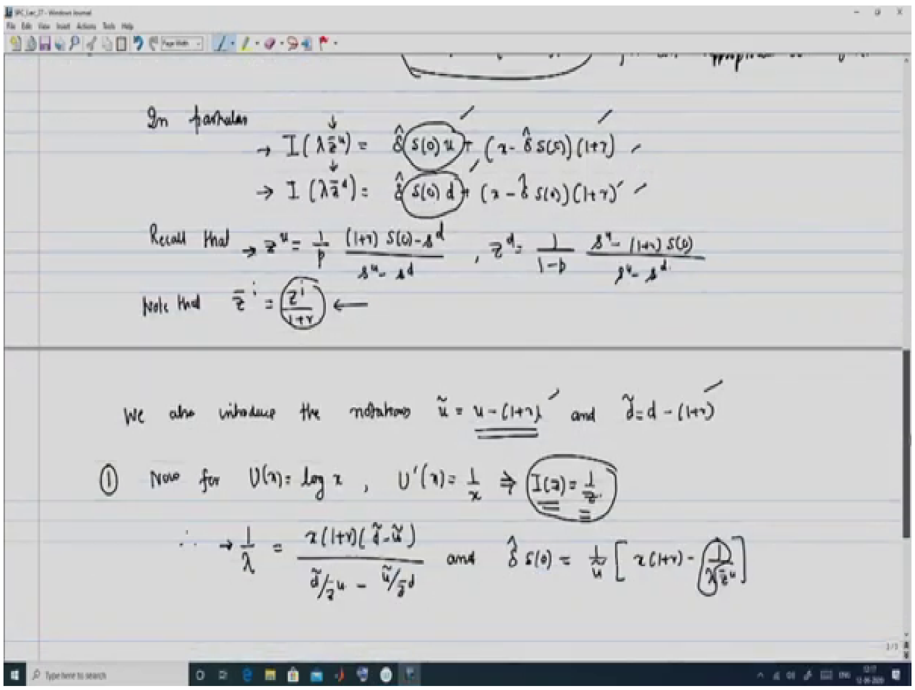

So now here, we have used the notation $\bar{Z}^u$ and $\bar{Z}^d$. So; accordingly, note that this notation $\bar{Z}^i$ where i could be u or d, this is

$$\bar{Z}^i = \frac{Z^i}{1+r}.$$

So, basically this is the discounted value of z. We also introduce the notations. So, apart from this notation; we also introduce a couple of more notations, which will be u tilde is u minus 1 plus r. So; that means, this and this combine will give you $\hat{\delta}S(0)[u-(1+r)]$, so that $[u-(1+r)]$, I will denote as $\tilde{u}$ and accordingly, for the second one, we will have $[d-(1+r)]$, which I will denote as d tilde. So; now, for the utility function being the log utility, as is the case with the first part of the problem, what is $U'(x)$?

$$U'(x) = \frac{1}{x}$$

and this implies that the inverse is going to be $\frac{1}{Z}$. So, inverse function

$$I(Z) = \frac{1}{Z}.$$

So accordingly, this term is going to be 1 over lambda Z bar of U, where Z bar of U can be obtained from here. And accordingly, here also I will get 1 over Z lambda Z bar of d, where you will get Z bar of d by making use of this relation. So therefore, we will get that, we get these two equations this and this here, and you can use this two equation to solve for lambda, to get

$$\frac{1}{\lambda} = \frac{x(1+r)(\tilde{d}-\tilde{u})}{\tilde{d}/\bar{Z}^u - \tilde{u}/\bar{Z}^d}.$$

And accordingly, you will get

$$\hat{\delta}S(0) = \frac{1}{\tilde{u}}\left[x(1+r) - \frac{1}{\lambda \bar{Z}^u.}\right]$$

So, this was the first problem.

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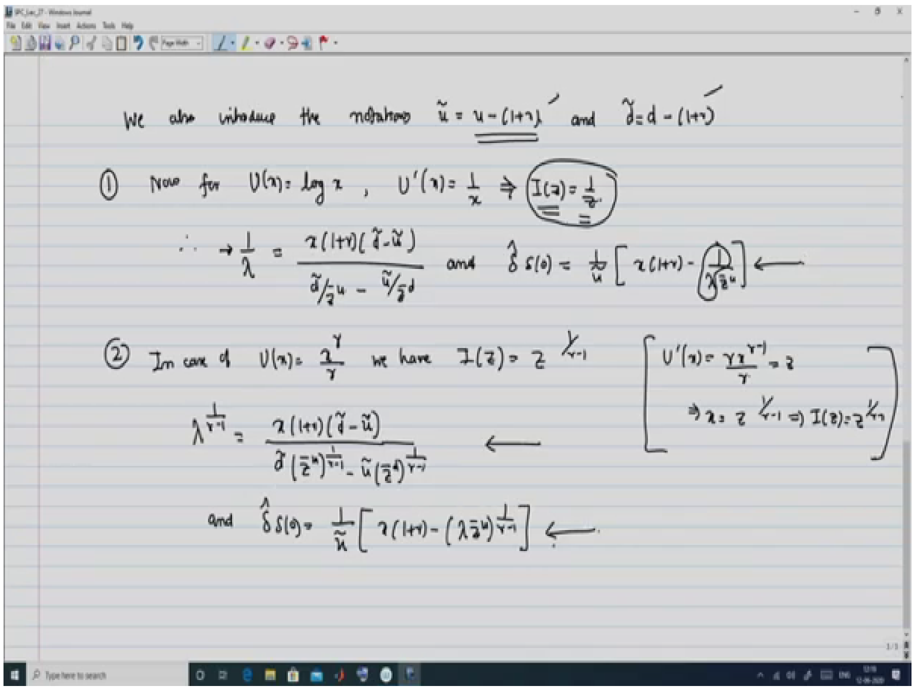

Now, for the second problem, in case of the power utility, that is x raised to gamma over gamma; we have, what is going to be I of z? So, this is going to be Z raised to 1 over gamma minus 1. So, for here you notice that U prime of x is going to be gamma x raised to gamma minus 1 into gamma and I call this some y so; that means, x is going to be y raised to 1 over gamma minus 1. So actually, let me call this as Z and this is nothing but I of Z is going to be equal to Z raised to 1 over gamma minus 1. So, for this we obtain that

$$\lambda^{\frac{1}{\gamma-1}} = \frac{x(1+r)(\tilde{d}-\tilde{u})}{\tilde{d}(\bar{Z}^u)^{\frac{1}{\gamma-1}} - \tilde{u}(\bar{Z}^d)^{\frac{1}{\gamma-1}}}.$$

And you get

$$\hat{\delta}S(0) = \frac{1}{\tilde{u}}[x(1+r) - (\lambda \bar{Z}^u)^{\frac{1}{\gamma-1}}].$$

So, this is going to be your lambda and this is going to be your consequent portfolio delta hat, given by this relation. So, in both the cases we have obtained the delta hat in terms of lambda that was obtained in the previous step.

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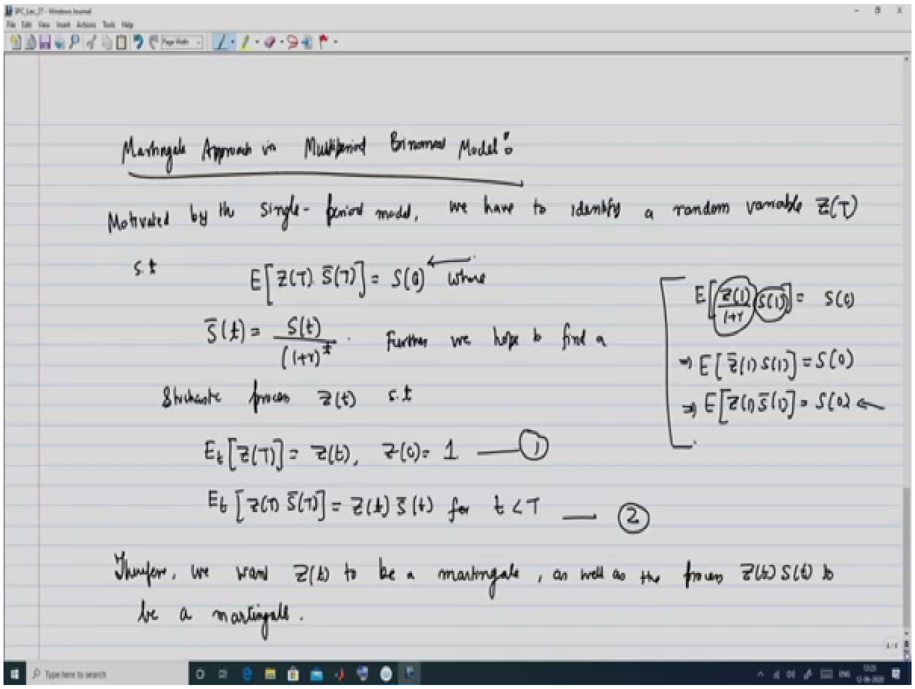
We next now, move on to; from the single period model to the martingale approach in multi period binomial model. So, let us start off with the observation, that motivated by the single period model, we have



to identify a random variable $Z(T)$ such that the following condition holds. So, I am just going to extend the condition that; so, earlier what we had? We had

$$E\left[\frac{Z(1)}{1+r}S(1)\right] = S(0).$$

Now this can be written in the form

$$E[\bar{Z}(1)S(1)] = S(0).$$

So, motivated by this; we now, take expected value of Z of T and now this instead of dividing Z 1 by 1 plus r, I can divide this by I can divide S 1 by 1 plus r. So, in which case this form is going to be

$$E[Z(1)\bar{S}(1)] = S(0).$$

Further, we hope to find a stochastic process, Z of t such that,

$$E_t[Z(T)] = Z(t), \;\; Z(0) = 1.$$

And the other relation is

$$E_t[Z(T)\bar{S}(T)] = Z(t)\bar{S}(t) \;\text{ for }\; t < T.$$

So, let me call this equation 2. So, therefore, we want Z of t to be a martingale, as well as the process Z t S t to be a martingale. Remember, we had introduced the definition of martingale in the previous class.

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So, in particular,

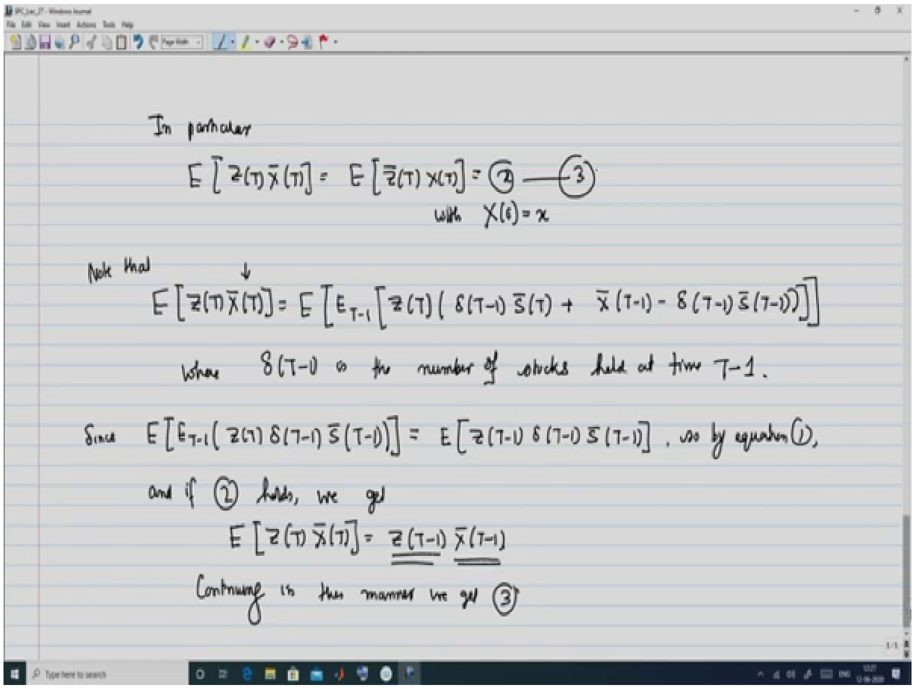
$$E[Z(T)\bar{X}(T)] = E[\bar{Z}(T)X(T)] = x.$$

Let me call this equation 3, with of course, this little x is nothing, but the initial wealth level X of 0. Now, we note that

$$E[Z(T)\bar{X}(T)] = E[E_{T-1}[Z(T)(\delta(T-1)\bar{S}(T) + \bar{X}(T-1) - \delta(T-1)\bar{S}(T-1))]]$$

So, I am basically writing the discounted wealth process. So, this is going to be delta of T minus 1, S bar of T plus X bar of T minus 1 minus delta into T minus 1 into S bar of T minus 1. Where, the newly introduced notation of delta of capital T minus 1 is, the number of stocks held at time capital T minus 1. Now, since

$$E[E_{T-1}[Z(T)(\delta(T-1)\bar{S}(T) = E[Z(T-1)(\delta(T-1)\bar{S}(T-1)].$$



So, by equation 1, and if 2 holds, we get

$$E[Z(T)\bar{X}(T)] = Z(T-1)\bar{X}(T-1).$$

And continuing; in this manner we get relation 3. So, eventually we will go to Z of 0 which is 1 and X bar of 0 which is going to be equal to just little x. So, that is how we end up getting this relation.

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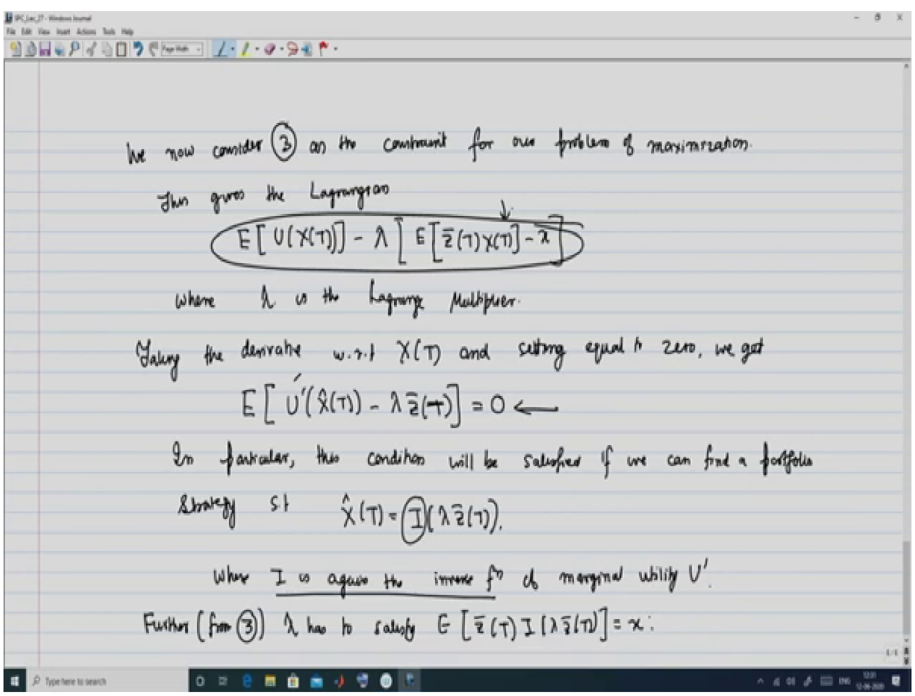

We now, consider equation 3. So, that is this equation here. This as a constraint for our problem of maximization of the expected terminal wealth. So, this gives the Lagrangian

$$E[U(X(T))] - \lambda[E[\bar{Z}(T)X(T)] - x],$$

where λ is the Lagrange multiplier, are not to be confused with the λ that we had done earlier. Now, taking now you want to basically solve the optimisation problem using this Lagrangian. So, taking the derivative, with respect to X of T and setting equal to 0, we get

$$E[U'(\hat{X}(T)) - \lambda\bar{Z}(T)] = 0.$$

So, in particular, this condition will be satisfied if we can find a portfolio strategy such that; so, from this relation we get the condition that

$$\hat{X}(T) = I(\lambda\bar{Z}(T)),$$

where I of course, is the inverse of U'. So, I is against the inverse function of U prime, and remember that U prime is nothing, but the marginal utility. Further, from 3 lambda has to satisfy, which relation? That is

$$E[\bar{Z}(T)I(\lambda\bar{Z}(T))] = x.$$

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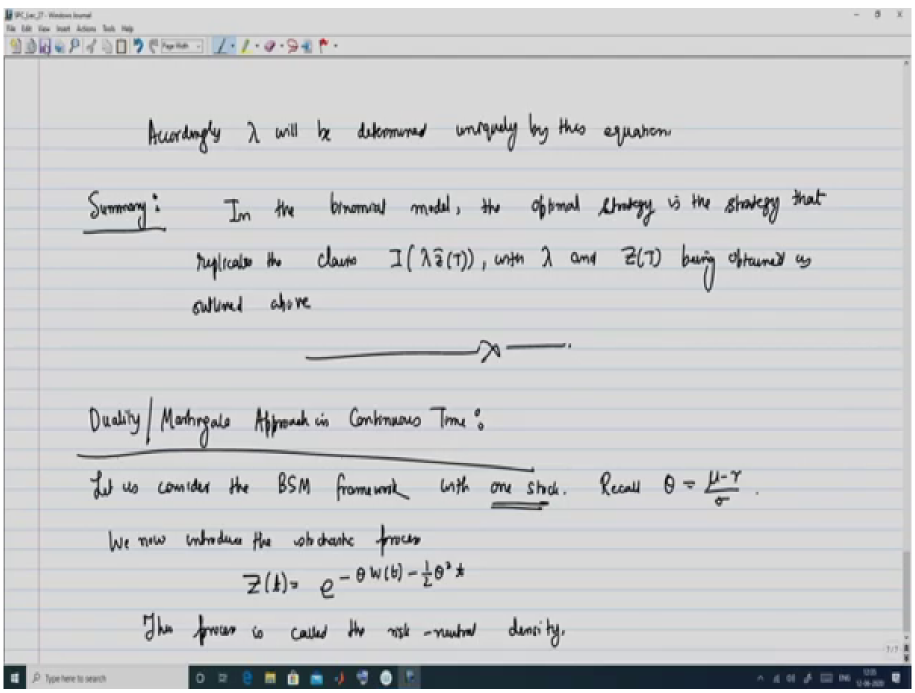
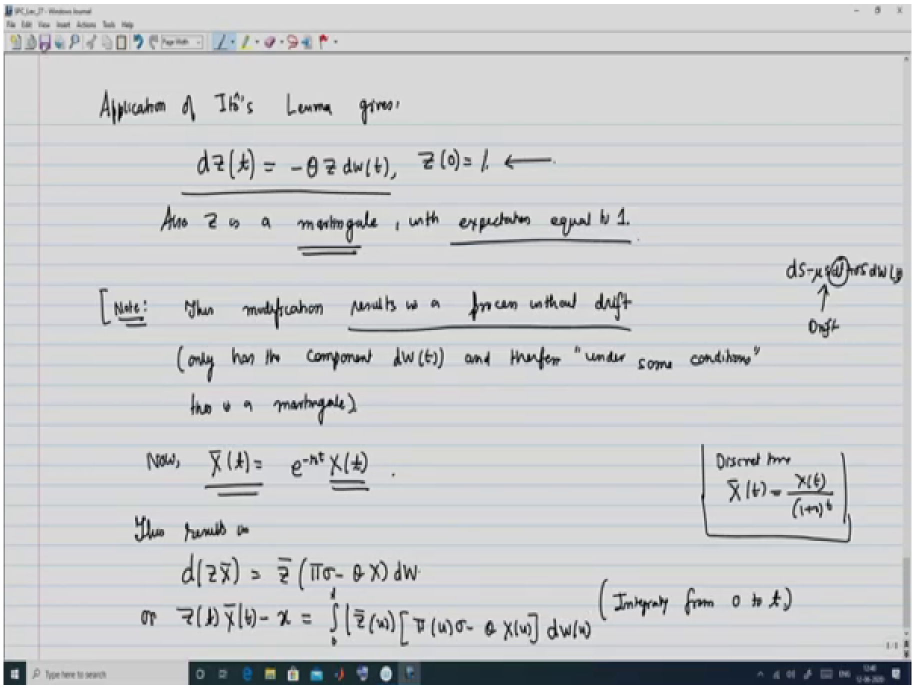


Accordingly, lambda will be determined uniquely by this equation; that means, this equation. So, in summary, we can state the following that in the binomial model, the optimal strategy is the strategy that replicates the claim $I(\lambda\bar{Z}(T))$, with lambda and Z of capital T being obtained as outlined above. So, this brings us to the last topic of the duality or the martingale approach in continuous time. Let us consider, the black shows martin framework; and by this mean, that is a continuous time model 1 the portfolio comprises of a stock and a bond. So accordingly, this black shows framework comprises of 1 stock and 1 bond. Also, recall from our discussion on the H J B equation, that is

$$\theta = \frac{\mu - r}{\sigma}.$$

So, I am just recalling the notation, since I am going to use this now. So, we now, introduce the stochastic process, which I will define as Z of t. So, this is basically the same Z as before, but in the continuous time setting, is given by

$$Z(t) = e^{-\theta W(t) - \frac{1}{2}\theta^2 t}.$$



Remember, the asset price is follows the geometric Brownian motion minus half theta square of little t, and this process is called the risk neutral density. So, this is some sort of the equivalent of the Z that we had in the discrete time case.

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So, you can show that, using Itos lemma. So, Itos lemma is nothing, but the Taylor series expansion in the world of stochastic calculus. So, an application of Itos lemma this gives d z. So, just like you obtain d Z in the deterministic case, using ah Taylor series. So, in the stochastic case, we have Itos lemma which gives

$$dZ(t) = -\theta Z dW(t), \quad Z(0) = 1.$$

So, you can see from here that Z of 0 will be e raised to w of 0, will render this term 0 and t equal to 0 will render this term 0. So, Z of 0 is going to be simply e raised to 0 which is equal to 1. Also, a Z is a martingale, and I will just make a brief note, without actually proving why this is a martingale; with expectation equal to 1. So, regarding the expectation being equal to 1, you can actually calculate the expected value of this and use the probability density function for the winner process w t, which is the normal distribution. And the other claim that, I have made here, is that Z is a martingale and equal to 1. And for this we just make a note, thus this modification that is using the Itos lemma results in a process without drift. So, without drift basically means that, as you remember that we had d s is equal to mu s d t plus sigma is d w of t and this term was known as the drift term. So, drift term is the term which has d t; so, since these does not have a d t term, so it; that means, you have obtained a process without the drift, that is; only has the component d W of t and therefore, under some conditions. So, because of the absence of this drift term; so, under some conditions, this is a martingale. Now, X bar of t, what is this? So, X bar is basically the bar will denote the discounted process. Now, X bar of t means, that is a discounted process of X of t and now, since this is a continuous time setup, so the discounting factor is going to be

$$\bar{X}(t) = e^{-rt}X(t).$$

At the discrete time, this was

$$\bar{X}(t) = \frac{X(t)}{(1+r)^t}.$$

So, the equivalent in the continuous time setting is going to be this form. So, this results in the relation

$$d(Z\bar{X}) = \bar{Z}(\pi\sigma - \theta X)dW,$$

where X is basically the wealth process. Or if you integrate this, what will you get? If you integrate from 0 to little t, what is this going to give you? This is going to give you

$$Z(t)\bar{X}(t) - x = \int_0^t \bar{Z}(u)[\pi(u)\sigma - \theta X(u)]dW(u).$$

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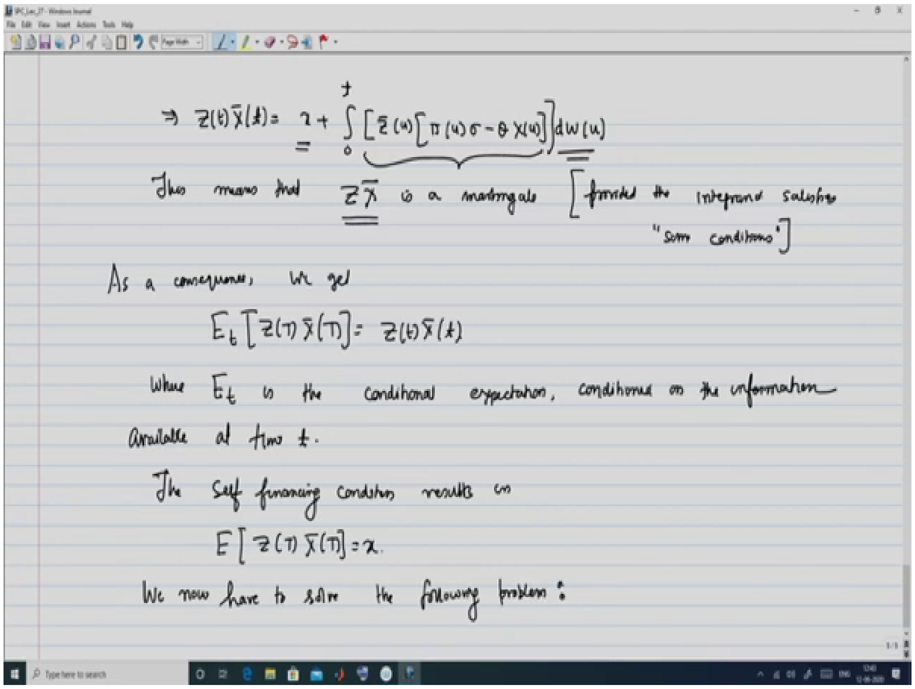

Or equivalently, we can just write this as

$$Z(t)\bar{X}(t) = x + \int_0^t \bar{Z}(u)[\pi(u)\sigma - \theta X(u)]dW(u).$$

Now, you observe that here, we only have the d w term and there is no d t term. So; this means, that $Z\bar{X}$ is a martingale. Again, provided the integrant, that is this term here, this satisfies some conditions. Now, since $Z\bar{X}$ is a martingale. So, as a consequence, we get that

$$E_t[Z(T)\bar{X}(T)] = Z(t)\bar{X}(t).$$

Where, this newly introduced expectation E subscript t is the conditional expectation, conditioned on the information available at time t. So, accordingly the self-financing condition results in, so; that means, at t time little t equal to 0, this results in

$$E[Z(T)\bar{X}(T)] = x.$$

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So accordingly, we now have to solve the following problem

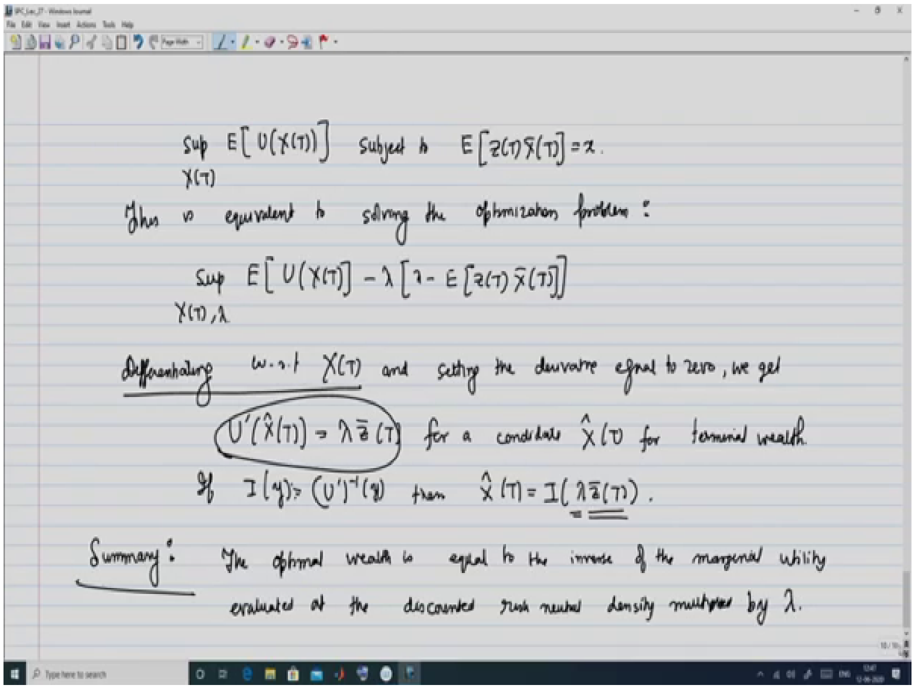
$$\sup_{X(T)} E[U(X(T))]$$

subject to

$$E[Z(T)\bar{X}(T)] = x.$$

And, this is equivalent to solving the optimization problem of supremum. So, I am defining the Lagrangian

$$\sup_{X(T),\lambda} E[U(X(T))] - \lambda[x - E[Z(T)\bar{X}(T)]].$$



So, differentiating with respect to X of T and setting the derivative equal to 0, we get

$$U'(\hat{X}(T)) = \lambda \bar{Z}(T)$$

for the candidate $\hat{X}(T)$ for terminal wealth. So, if

$$I(y) = (U')^{-1}(y),$$

then from this relation we obtain that

$$\hat{X}(T) = I(\lambda \bar{Z}(T)).$$

So, let me just summarize the continuous time discourse. So; here, the optimal wealth is equal to the inverse of the marginal utility evaluated at the discounted risk neutral density. Remember, $Z(T)$ is the risk neutral density, so $\bar{Z}(T)$ is the discounted risk neutral density multiplied by λ.

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And the condition is that, we have to choose lambda such that, what was the other condition? The other condition was, if we took the derivative with respect to lambda, we recovered this condition. So accordingly, we need to choose lambda such that, expected value of $\bar{Z}(T)$, so we have the $\bar{Z}(T)$. So, I can switch the $\bar{Z}(T)X(T)$, so this expression is going to be $\bar{Z}(T)X(T)$ and I can replace this $X(T)$ from here to obtain

$$E[\bar{Z}(T)I(\lambda \bar{Z}(T))] = x.$$

So; this means, that you figure out what is the lambda making use of this relation. So, this means, that you make use of this relation to figure out what is λ and then substitute the lambda to obtain what is $\hat{X}(T)$. So, this brings us to the end of this lecture. So, just to do a brief recap, what we have done in today's lecture is, we looked at an example of the single period optimization using the martingale approach and gave the setup for the multi period optimization. And then we moved on to the continuous time framework, and gave the approach using martingales in order to find what is going to be the optimal portfolio. So, this brings us to the end of this module on optimal portfolio and consumption. And from the next week, we will start off with a new topic namely on bond portfolio optimization.

Thank you for watching.

Differentiating w.r.t $X(T)$ and setting the derivative equal to zero, we get

$$U'(\hat{X}(T)) = \lambda \bar{Z}(T)$$ for a candidate $\hat{X}(T)$ for terminal wealth.

If $I(y) = (U')^{-1}(y)$ then $\hat{X}(T) = I(\lambda \bar{Z}(T))$

Summary: The optimal wealth is equal to the inverse of the marginal utility evaluated at the discounted risk neutral density multiplied by λ.

We have to choose λ such that

$$E\left[\bar{Z}(T)\, I(\lambda \bar{Z}(T))\right] = x$$