Mathematical Portfolio Theory

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Module 05: Optimal Portfolio and Consumption
Lecture 05: Hamilton-Jacobi-Bellman PDE; Duality/ Martingale Approach

Hello viewers. Welcome to this lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. You would recall that in the last class, we started talking about the optimal portfolio in the case of a continuous time setup and we looked at the wealth equation; wherein, we consider a portfolio comprising of a stock with the stock following the geometric Brownian motion and a bond. And based on that wealth equation, we talked about how we can optimize the expected utility of the final wealth level and this resulted in us getting the equivalent of the dynamic programming in case of the continuous time setup, namely, the Hamilton, Jacobi, Bellman equation. So, in today's class, we will continue our discussion on the HJB equation and we begin with a couple of examples and then, what we will do is that we will move on to the inclusion of the case, where we also in addition to the wealth process must also consider what is the consumption process and then, we will look at a description of that.

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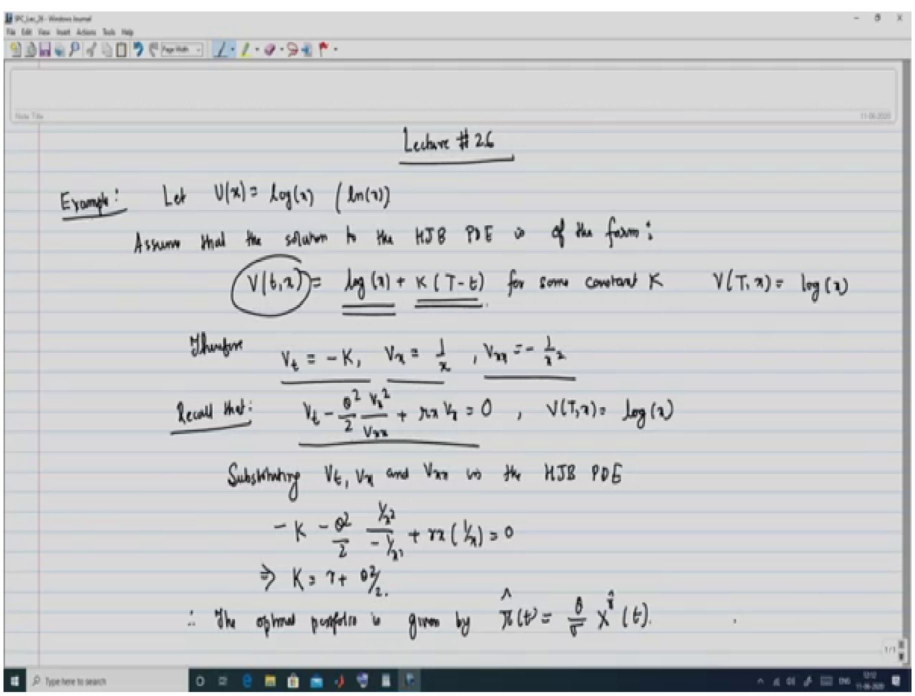

So, accordingly, we begin this lecture with this example. So, here let us consider that the utility function is log x that means, natural log of x and we assume that the solution to the HJB equation is of the form V t, x. Remember the HJB equation involved the value function

$$V(t,x) = \log x + K(T-t)$$

for some constant K. So, actually the constant is a K and here of course, we need the condition that

$$V(T, x) = \log x$$

. Because if you put small t is equal to capital T, this term becomes 0 and you are only left with log of x. So, the assumption of this form for the value function, therefore, we obtain that

$$V_t = -K, \;\; V_x = \frac{1}{x}, \;\; V_{xx} = -\frac{1}{x^2}.$$

So, now, recall that, what was the HJB equation? The HJB equation was

$$V_t - \frac{\theta^2}{2}\frac{V_x^2}{V_{xx}} + rxV_x = 0, \;\; V(T, x) = \log x.$$

So, now, what you do is that we can substitute these three evaluated expressions in the HJB equation. So, substituting V t, V x and V xx in the HJB PDE, we get

$$-K - \frac{\theta^2}{2}\frac{\frac{1}{x^2}}{-\frac{1}{x^2}} + rx\left(\frac{1}{x}\right) = 0$$

$$\implies K = r + \frac{\theta^2}{2}.$$

So, therefore, the optimal portfolio is given by pi of t and we indicate the optimal portfolio

$$\hat{\pi}(t) = \frac{\theta}{\sigma}X^{\hat{\pi}}(t).$$

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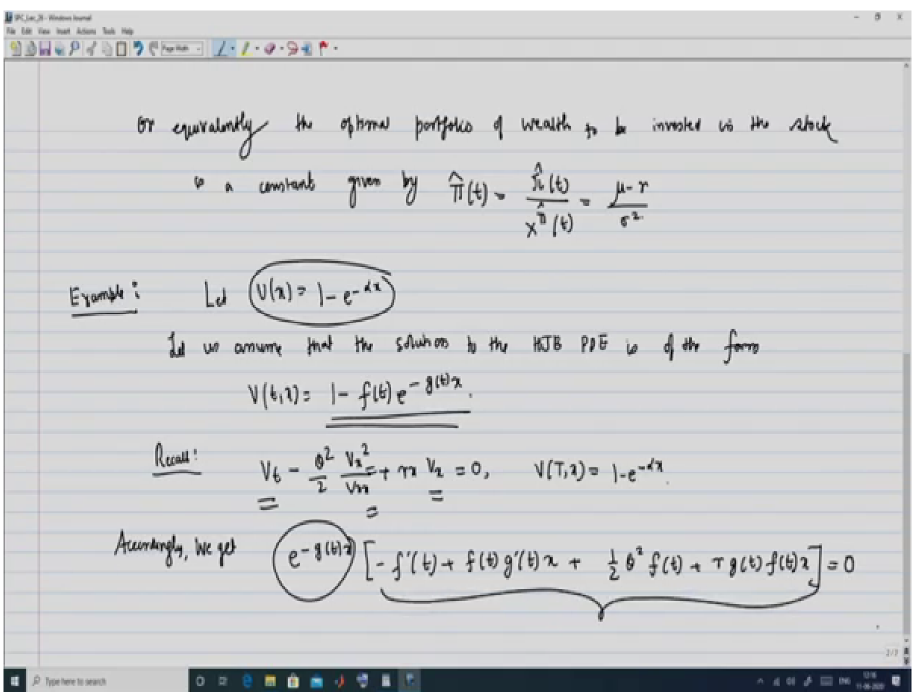

Or equivalently, the optimal portfolio of wealth to be invested in the stock is a constant given by $\hat{\pi}(t)$. So,

$$\hat{\Pi}(t) = \frac{\hat{\pi}(t)}{X^{\hat{\pi}}(t)} = \frac{\mu - r}{\sigma^2}.$$

Let us now consider another example. So, let the utility function

$$U(x) = 1 - e^{-\alpha x}$$

and let us assume based on the form of the utility function, the solution to the HJB PDE is of the form

$$V(t,x) = 1 - f(t)e^{-g(t)x}.$$

So, again, we recall the HJB equation, what was this? This was

$$V_t - \frac{\theta^2}{2}\frac{V_x^2}{V_{xx}} + rxV_x = 0, \;\; V(T,x) = 1 - e^{-\alpha x}.$$

So, we can now evaluate these values. So, after evaluating $V_t, \; V_x$ and V_{xx} we get accordingly, we will get

$$e^{-g(t)x}\left[-f'(t) + f(t)g'(t)x + \frac{1}{2}\theta^2 f(t) + rg(t)f(t)x\right] = 0.$$

Now, what you are going to do? So obviously, this term is not going to be equal to 0. So, accordingly, I have this entire expression being equal to 0. So, what I can do now is we will now look at the term which comprises of f prime. So, I have an f prime here and f here. So, what we are going to do is that we are going to segregate these terms in which only involve f or f prime, that is the function f and then, those which involve both f and g or f for g prime. So, these two are going to be collected together. So, one way of achieving that this is going to be equal to 0 is that this plus this are going to be 0 and this expression plus this expression is going to be 0.

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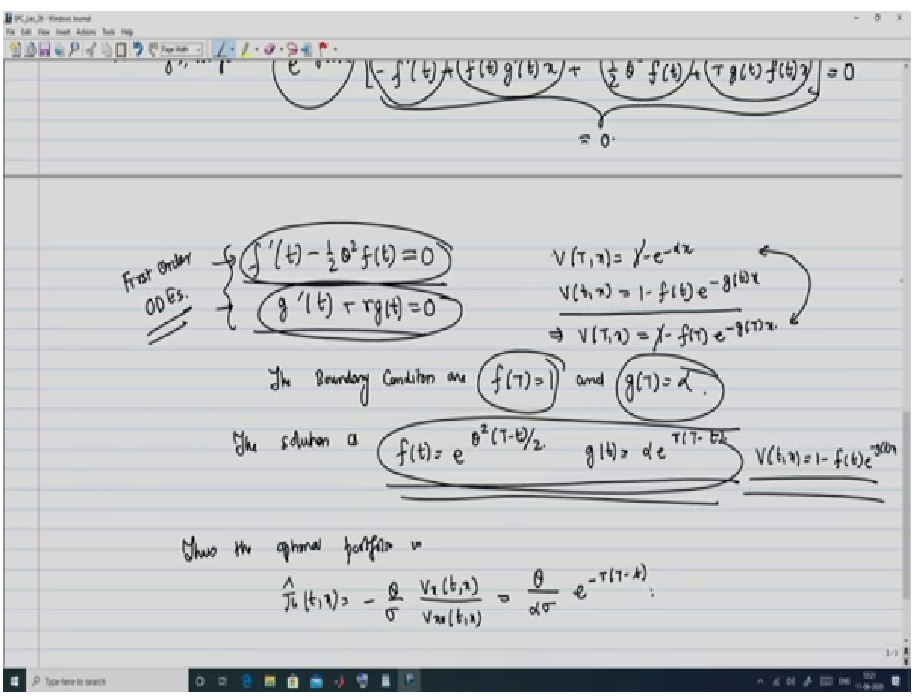

So, accordingly, we get

$$f'(t) - \frac{1}{2}\theta^2 f(t) = 0,$$

$$g'(t) + rg(t) = 0.$$

f prime of t minus half theta square f of t is going to be equal to 0 and for this along with this expression, we will get a common factor of f t which obviously, cannot be 0 because in that case V of t x is just going to be a constant. So, accordingly, we will have that our g prime of t from here plus r g of t is going to be equal to 0 from this term. Now, you observe that both of them, these are first order ODEs or Ordinary Differential Equation. So, now, what we do is we need in order to solve this first order ODEs, we need some condition. So, that can be obtained from the fact that V of t x is equal to 1 minus e raised to minus alpha x and we had assumed our V little t of x is 1 minus f of t into e raised to minus g t of x. So, this is going to give us V capital T, x is going to be 1 minus f of capital T e raised to minus g of capital T into x. So, what I do is that, we equate both these expression and accordingly, what we get from both these expression is that we have the boundary condition or the final condition r. From here, you get; so, the 1s will cancel out. So, this

$$f(T) = 1 \text{ and } g(T) = \alpha.$$

So, then we can solve this first order ODEs subject to this condition for f of t. So, the solution is

$$f(t) = e^{\theta^2(T-t)/2}, \quad g(t) = \alpha e^{r(T-t)}, \quad V(t,x) = 1 - f(t)e^{-g(t)x}.$$

So then, using that we can obtain the optimal portfolio is

$$\hat{\pi}(t,x) = -\frac{\theta}{\sigma}\frac{V_x(t,x)}{V_{xx}(t,x)} = \frac{\theta}{\alpha\sigma}e^{-r(T-t)}.$$

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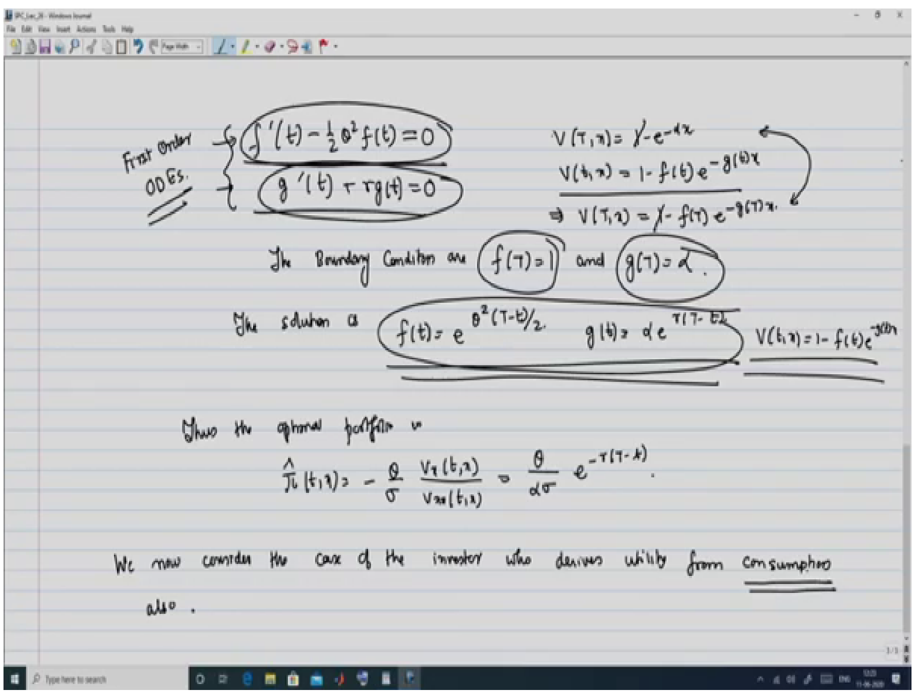

So, we now consider the case of the investor, who derives utility from consumption also. So, this means that in addition to the wealth process, now the consumption process also has to be taken into consideration.

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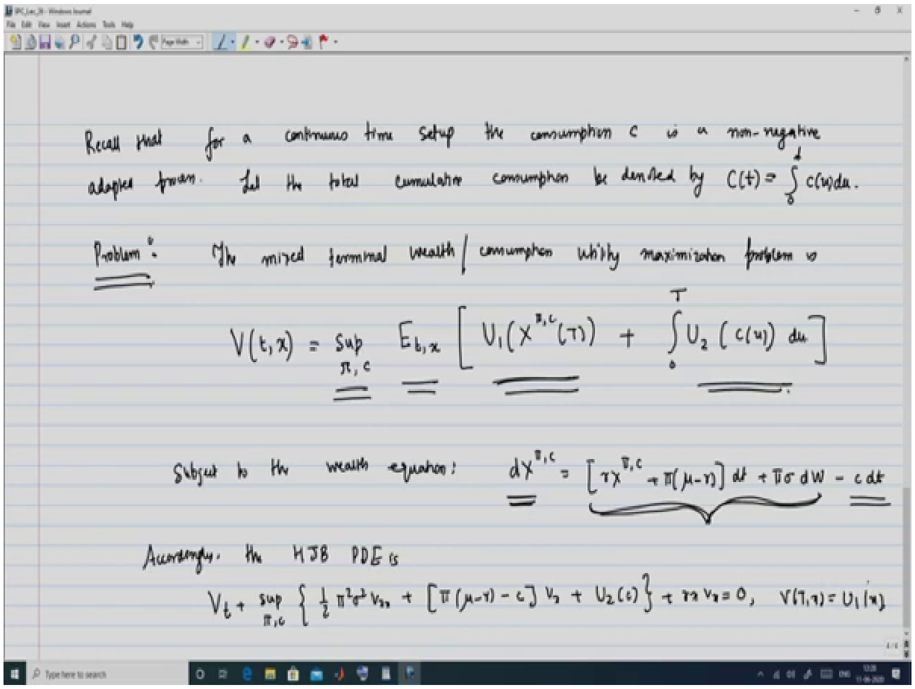
So, recall that for a continuous time setup, the consumption c is a non-negative adapted process. So, accordingly, let the total cumulative consumption be denoted by

$$C(t) = \int_0^t c(u)du.$$

So, now we are in a position to state the problem. So, the mixed that means, it is a combination of terminal wealth, consumption utility maximization problem is given by. So, we will look at what is going to be the



utility of the final wealth that is X of T and the utility of a consumption and the integral of that from 0 to T of du and this utility of the wealth, this is going to depend on pi and c. And since, this is a random variable, so obviously, we need to calculate its conditional expectation E subscript t comma x and you take the supremum of this over the portfolio process pi and the consumption process c and this results in the value function which depends on time t and the wealth level at time t being equal to little x. So,

$$V(t,x) = \sup_{\pi,c} E_{t,x}\left[U_1(X^{\pi,c}(T)) + \int_0^T U_2(c(u))du\right].$$

Now, this is subject to the wealth equation

$$dX^{\pi,c} = [rX^{\pi,c} + \pi(\mu - r)]dt + \pi\sigma dW - cdt.$$

So, this is the expression for the wealth process without the consumption and along with it, now we include a term for the consumption. So, you observe that this was the change in the wealth level without the consumption was given by this factor. And then, the change of the wealth level is also going to be affected by the consumption c over the interval of dt which has to be subtracted. So, accordingly, you can show that for the case where you consider both wealth and consumption, the HJB PDE is

$$V_t + \sup_{\pi,c}\left\{\frac{1}{2}\pi^2\sigma^2V_{xx} + [\pi(\mu - r) - c]V_x + U_2(c) + rxV_x = 0,\ \ V(T,x) = U_1(x).\right\}$$

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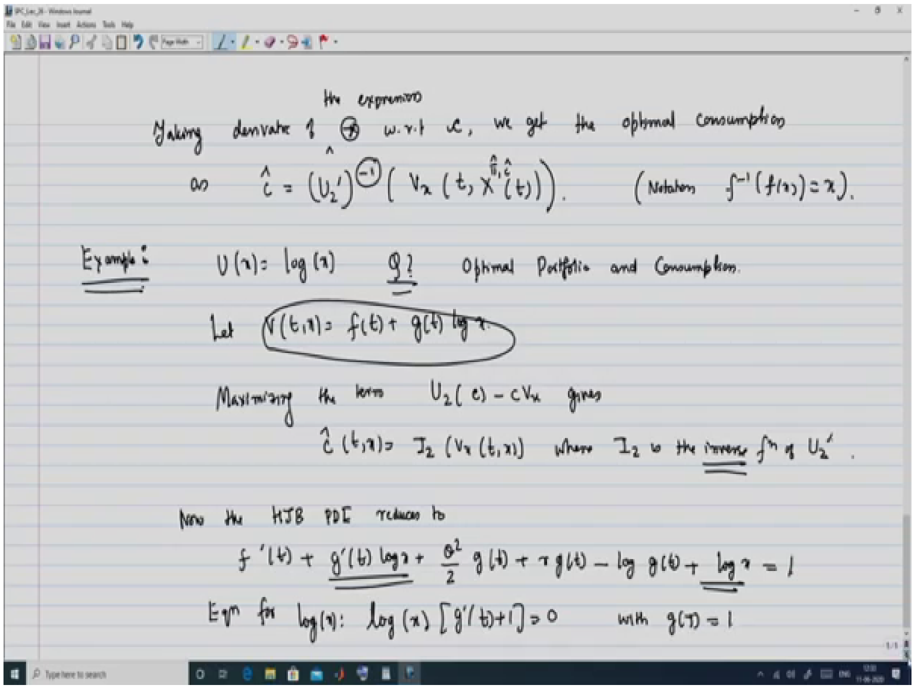
So, now we need to take the supremum of this expression here. So, taking, so let me identify this expression as star. So, taking derivative of the expression star with respect to c, we get the following. So, there are only two terms comprising of c, this term and this term. So, we accordingly, we take the derivative to in order to obtain the optimal consumption as

$$\hat{c} = (U_2')^{-1}(V_x(t, X^{\hat{\pi},\hat{c}}(t))).$$

So, here the notation, this inverse notation is nothing but the notation f inverse of f x is going to be equal to x. So, now, it is time to consider an example to accommodate this consumption. So, again, we use the utility



function to be the log utility and the question that we want to address is what is going to be the optimal portfolio and of course, what is the optimal consumption. So, here let

$$V(t, x) = f(t) + g(t) \log x.$$

So, maximizing the term $U_2(c) - cV_x$ gives that

$$\hat{c}(t, x) = I_2(V_x(t, x))$$

where I_2 is the inverse function of U_2'. Now, the HJB PDE with this assumption for V T, x, this reduces to f prime of t plus g prime of t log x plus theta square over 2 into g of t plus r g t minus log of g t plus log of x is going to be equal to 1. So, the equation for g t will then be what? It is going to be log of x. So, actually is the equation, I should say for log of x. That means, this term and this term. So, this is going to be

$$\log x[g'(t) + 1] = 0, \text{ with } g(T) = 1.$$

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So, hence, the solution is

$$g(t) = T - t + 1.$$

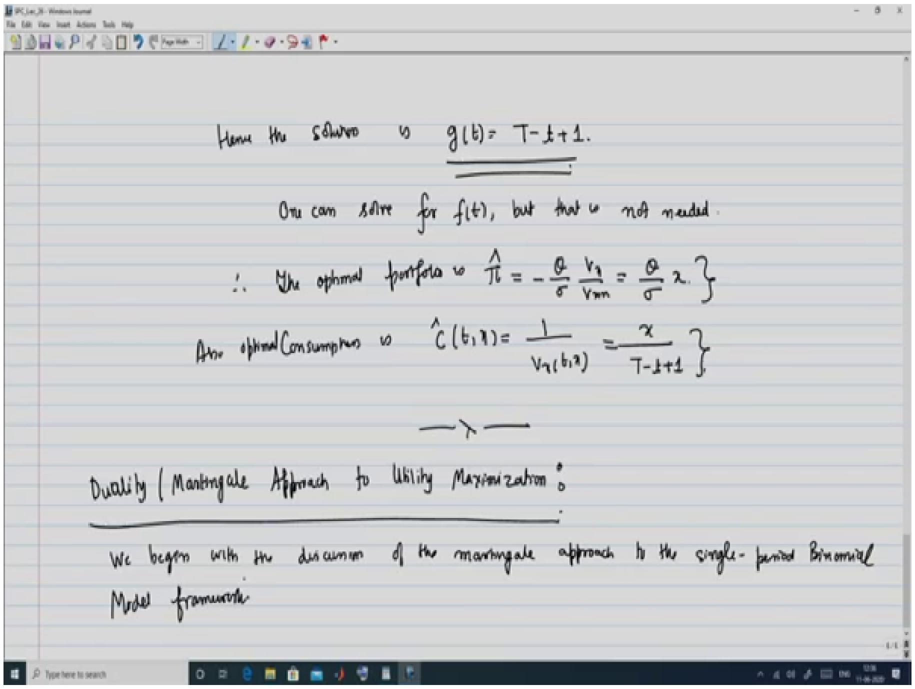
So, once you have got the value of g of t, you can replace it in this expression and then, you can one can solve for f of t, but that is not needed. I mean of course, you have to do this if you are trying to calculate V. But we are trying to calculate, what is the optimal portfolio consumption. So, evaluation of f t is not required. So, therefore, the optimal portfolio is

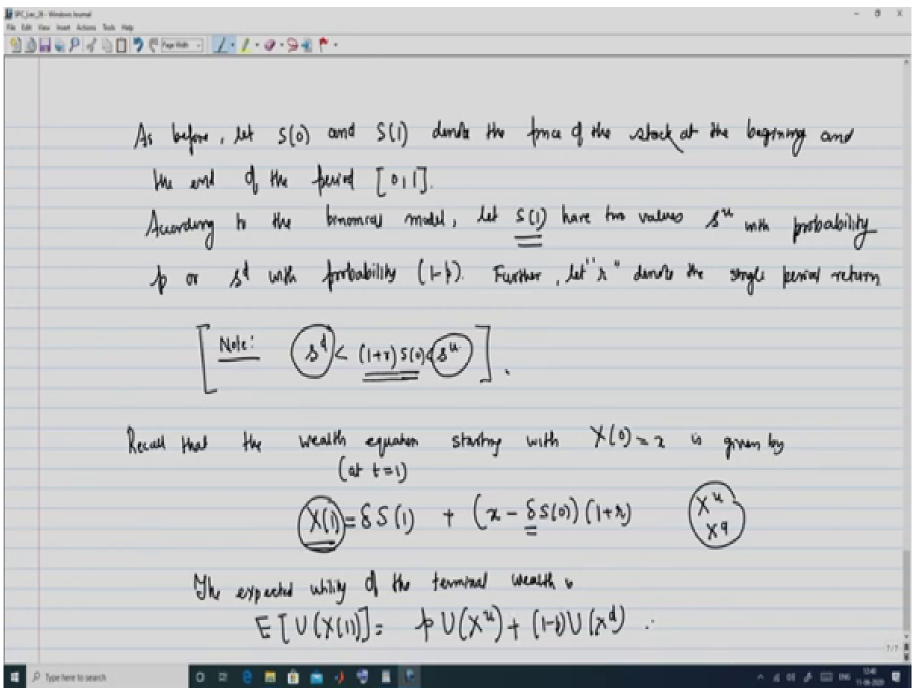
$$\hat{\pi} = -\frac{\theta}{\sigma}\frac{V_x}{V_{xx}} = \frac{\theta}{\sigma}x$$

and the optimal consumption is

$$\hat{c}(t, x) = \frac{1}{V_x(t, x)} = \frac{x}{T - t + 1}.$$

So, thus, you have obtained the optimal portfolio as well as consumption in case of the log utility. So, we now come to what is known as the duality or martingale approach to utility maximization. So, we begin





this part of the class, with the discussion of the martingale approach to the single-period binomial model framework.

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So, as before let $S(0)$. So, we are considering a single period model. So, accordingly, we say that let $S(0)$ and $S(1)$ denote the price of the stock at the beginning and of the period $[0, 1]$. So, according to the binomial model, let $S(1)$ have two values. So, we know that in the binomial model $S(1)$ will have two values. So, let those values be $s(u)$ with probability p or $s(d)$ with probability $1 - p$. Further, let r denote the single period return. So, just to note that, s d is less than s u and this must align it in this must sandwich a $(1 + r)S(0)$. So, this means that if we invest in amount of $S(0)$ in a risk free asset, it will go up to the value of $S(0)(1 + r)$ at time 1 and this must lie between the lower value of the stock s d at time 1 and the higher value of the stock s u at time 1. Now, recall that the wealth equation starting with the wealth level $S(0) = x$ is given by. So, this wealth equation at $t = 1$, this is given by $S(1)$ and how do we get this. So, you start off with an amount of x and you buy delta naught stocks, whose total cost is $\delta S(0)$ and you invest then in a risk free asset and this delta stock at time 1, it gets the value $S(1)\delta$. So, that is the value of the delta number of stocks at time 1 and this is equal to $X(1)$. Now, this is like the terminal wealth. So, what we need to look at is that; so, since x of 1 is the wealth at the terminal point t equal to 1. So, accordingly, the expected utility of the terminal wealth is what is this? It is going to the expected utility of the terminal wealth, that is $X(1)$; this is going to be the utility of; so, $X(1)$ can take two values; namely, X^u and X^d. So, the corresponding utilities are going to be $U(X^u)$ and $U(X^d)$ and the respective probabilities are p and $1 - p$. So, the expected utility is going to be sum of these two.

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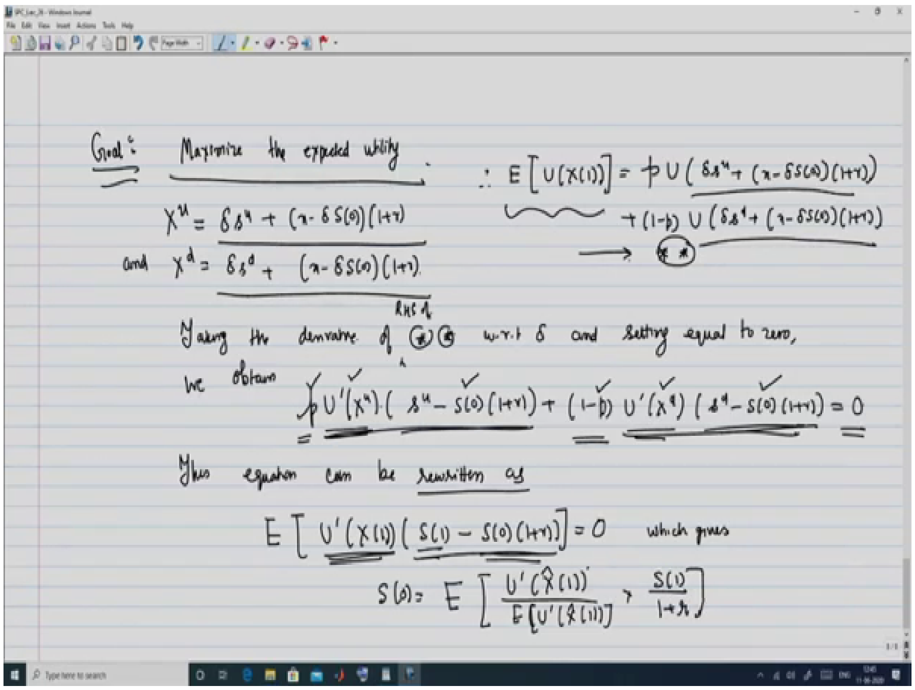

And what is the goal? So, the goal is to maximize the expected utility. Now, let us be more specific about what my X^u and X^d are going to be. So, what is going to be X^u? X^u is the wealth resulting from the stock going up to s^u. So,

$$X^u = \delta s^u + (x - \delta S(0))(1 + r)$$

and in an analogous manner

$$X^d = \delta s^d + (x - \delta S(0))(1 + r).$$

So, therefore, what is going to be the expected utility here? So, this is going to be; so, the expected utility at the terminal wealth now, in more precise term, this is going to pX^u. So, I will replace this form of X^u. So,

$$E[U(X(1))] = pU(\delta s^u + (x - \delta S(0))(1 + r)) + (1 - p)U(\delta s^d + (x - \delta S(0))(1 + r)).$$

and my goal is essentially to maximize this utility. So, accordingly, taking the derivative of let me call this double star with respect to δ and setting equal to 0, we obtain that

$$pU'(X^u)(s^u - S(0)(1 + r)) + (1 - p)U'(X^d)(s^d - S(0)(1 + r)) = 0.$$

So, this expression or rather this equation can be rewritten as

$$E[U'(X(1))(S(1) - S(0)(1 + r))] = 0$$

So, you see the random variable here is U prime of X u into this expression and U prime of X d into this expression. So, X 1, U prime of X 1 will be U prime of X u with probability p and U prime of X d with probability 1 minus p and S 1 minus S 0 into 1 plus r is going to be this expression with probability p and this expression of with probability 1 minus p. So, if I consider this entire thing as a random variable, these two random variables. So, with respective probabilities p and 1 minus p. So, this can be written in the compact form as the expected value of U prime of X 1 into S 1 minus S 0 into 1 plus r and this is going to be equal to 0 and then, which gives that. So, I can solve this for $S(0)$ and this is going to be

$$S(0) = E\left[\frac{U'(\hat{X}(1))}{E[U'(\hat{X}(1))]} \times \frac{S(1)}{1+r}.\right]$$

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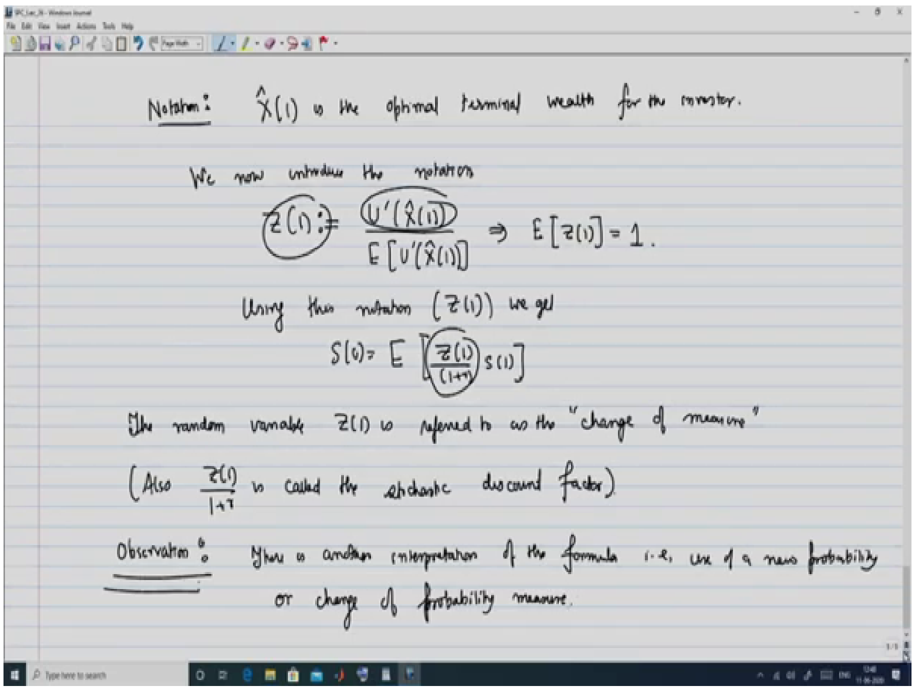

So, here, the notation that you have used here X hat of 1, so this notation X hat of 1 is the optimal terminal wealth for the investor. So, we now introduce the notation and we call this expression here, let us call this since this depends on time 1, let us call this some $Z(1)$ and this is defined as

$$Z(1) := \frac{U'(\hat{X}(1))}{E[U'(\hat{X}(1))]}$$

$$\implies E[Z(1)] = 1.$$

So, using the this notation that is Z 1 we get. So, this will result in this value of S 0 being written as

$$S(0) = E\left[\frac{Z(1)}{1+r} S(1)\right].$$

Now, see Z of 1 is U prime of X hat of 1. So obviously, this is a random variable. So, this random variable Z 1 is referred to as the change of measure and you will soon be clear why we are calling this as change of measure. Also, this factor Z 1 over 1 plus r is called the stochastic discount factor. Now, the I want to make an important observation. So, if you look at option pricing, you will see how important this observation is and that there is another interpretation of the formula that is use of a new probability or change of probability measure.

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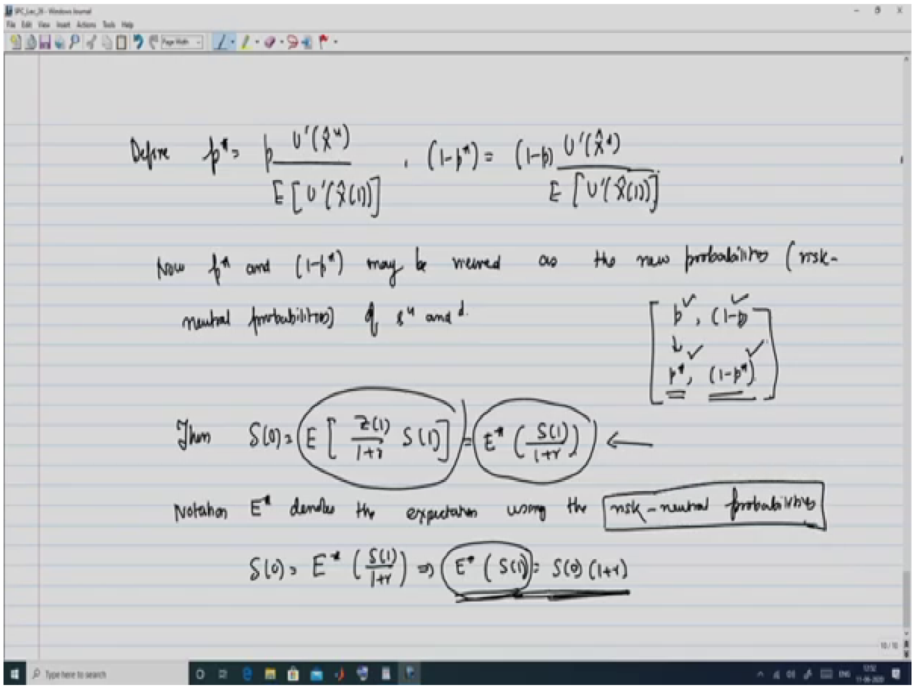


So, accordingly, we will define

$$p^* = p\frac{U'(\hat{X}^u)}{E[U'(\hat{X}(1))]},$$

$$1 - p^* = (1-p)\frac{U'(\hat{X}^d)}{E[U'(\hat{X}(1))]}.$$

Now, here p star and 1 minus p star may be viewed as the new probabilities or what is known as risk neutral probabilities of s u and s d. So, remember that earlier, we had the probability p and 1 minus p and this will now be changed to p star and 1 minus p star. So, this and that is the reason why. So, since this is a change of the probability measure. So, that is the reason why we had referred to Z 1 as the as change of measure. So, accordingly, we can write that then

$$S(0) = E\left[\frac{Z(1)}{1+r}S(1)\right].$$

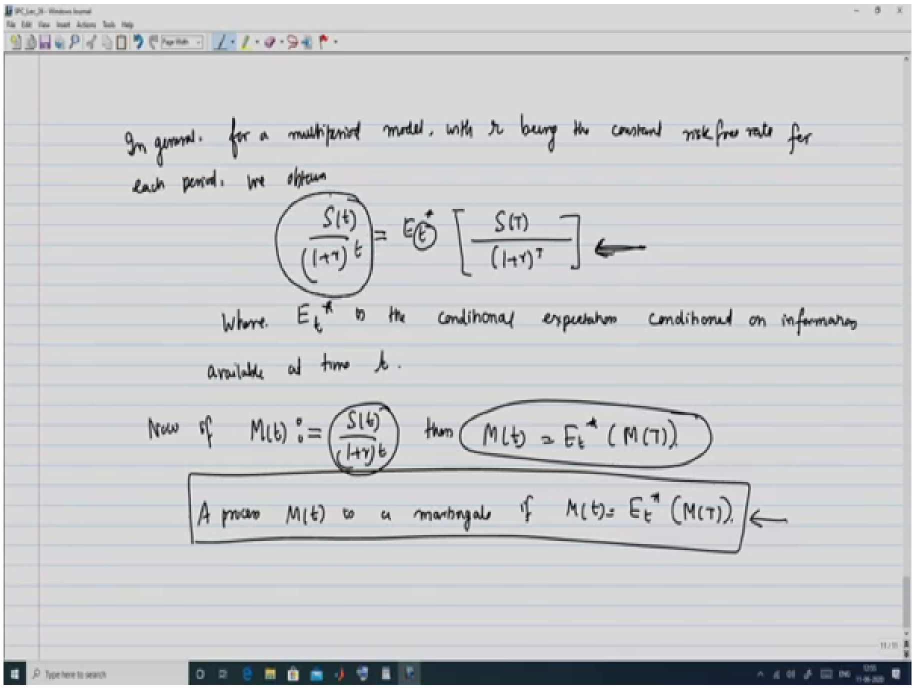
We had got and this can then be; so, this is one expression. And then, we reconcile it with a previous expression and it can be shown that this is nothing but

$$E^*\left[\frac{S(1)}{1+r}\right].$$

And here, we note that the notation, E star denotes the expectation using the risk neutral probabilities. So, this expectation here makes use the probability p and 1 minus p and this, expectation here uses the probability p star and 1 minus p star. So, the reason why you know these are called risk neutral probabilities is that you observe that here, we have

$$S(0) = E^*\left[\frac{S(1)}{1+r}\right] \implies E^*[S(1)] = S(0)(1+r).$$

So, if we had invested an amount of S naught directly in a risk free asset, this would grow to S naught, S 0 into 1 plus r and this is going to be your same as the expected value of the stock price at 1 which is a random variable and since, the expected stock price at time t equal to 1 under p star and 1 minus p stars become the



same as that you obtain from a risk free asset. So, the probability which causes this to be equal namely p star and 1 minus p star, accordingly is called as the risk neutral probability.

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So, in general for a multi period model with r being the constant risk free rate for each period, we obtain that

$$\frac{S(t)}{(1+r)^t} = E_t^*\left[\frac{S(T)}{(1+r)^T}\right].$$

So, here this notation each star with subscript E t is the conditional expectation conditioned on information available at time little t. Now, if we introduce the notation $M(t) := \frac{S(t)}{(1+r)^t}$. Then, from here we get, $M(t) = E_t^*(M(T))$ and a process $M(t)$ is a martingale, if $M(t) = E_t^*(M(T))$. So, a martingale process is defined to be a process M t which satisfies this property and in particular, if I choose my M t to be this expression, then the above expression here reduces to this definition of martingale and hence, the discounted stock price process is a martingale.

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Now, we observe that $Z(1)$ can take two values. So, remember that this depends on whether the stock goes up or down. So, accordingly, depending on s^u and s^d, we take it can take two values Z^u and Z^d with probability p and $1-p$ respectively. Now, since $E[Z(1)] = 1$, therefore,

$$pZ^u + (1-p)Z^d = 1.$$

Also, if you observe carefully,

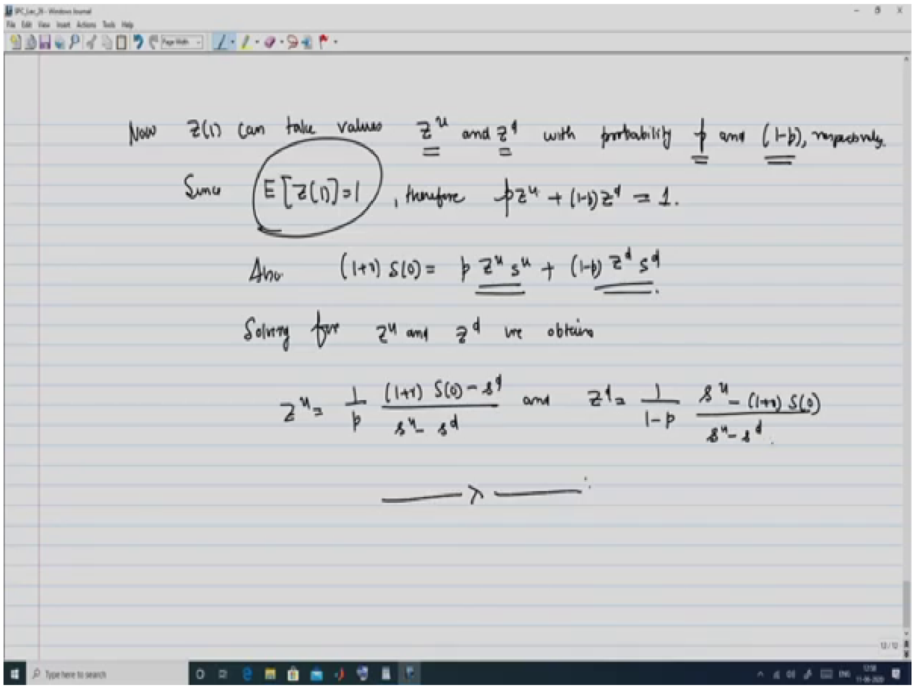
$$(1+r)S(0) = pZ^u s^u + (1-p)Z^d s^d.$$

So, S 1 can take two values. So, this becomes $(1+r)S(0)$. So, this is going to be $Z^u s^u$ and the other value would be; so, these are going to be the values of $Z(1)S(1)$. So, these are going to be $Z^u s^u$ or $Z^d s^d$ with respective probabilities p and $1-p$. So, then, what we get is we now solve solving for Z^u and Z^d, we obtain that

$$Z^u = \frac{1}{p}\frac{(1+r)S(0) - s^d}{s^u - s^d}$$

and

$$Z^d = \frac{1}{1-p}\frac{s^u - (1+r)S(0)}{s^u - s^d}.$$



So, this brings us to the end of this lecture. Just to do a recap of what we have done so far. So, we picked up from our formulation for the Hamilton, Jacobi, Bellman equation and then, we looked at two examples to determine what is going to be our optimal portfolios and then, we moved on to the situation, where we includ e in addition to the wealth process, we take into account what is the consumption process. And then, we stated the Hamilton Jacobi Bellman equation, when both these processes are involved and we looked at one example and then, we moved on to the case of the martingale approach for portfolio optimization. And in todays class, we set up what is going to be the background for that and we also identified a very important concept that is used in finance in general that is the notion of risk neutral probabilities. So, in the next class, we will continue our discussion and we will look at how we can now make use of this martingale approach in order to determine, what is going to be an optimal portfolio.

Thank you for watching.