Mathematical Portfolio Theory

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Module 05: Optimal Portfolio and Consumption
Lecture 02: Optimal portfolio for single-period discrete time model

Hello, viewers. Welcome to this lecture on the NPTEL-MOOC course on Mathematical Portfolio Theory. In the previous lecture, we started off with a new topic on optimal portfolio and consumption. And we mentioned that we will look at the optimal portfolio as well as the consumption in both discrete and continuous time model. And, we started off by talking about a market model and consumption as well as utility functions. And the approach that we will take for discrete time portfolio optimization is going to be what is known as dynamic programming due to Richard Bellman. So, we continue our discussion in the discrete time framework for this problem of portfolio optimization and begin todays lecture.

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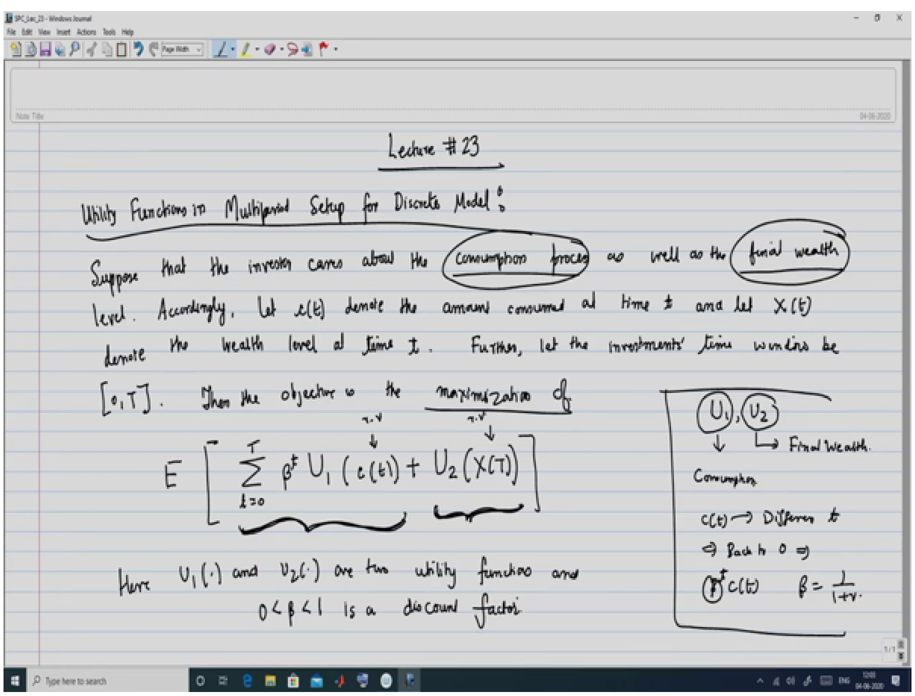

So, we pick up from where we less left last time that is utility functions, in multi period setup for discrete model. So, we begin with the supposition that the investor cares about the consumption process as well as the final wealth level. So, that means, that the investor cares about the ongoing consumption process in addition to of course, what is going to be the terminal or final wealth at the end of the investment period. So, accordingly let us be very specific now about this consumption process and the final wealth. So, we let $c(t)$ denote the amount of money that is consumed at time small t and let $X(t)$ denote the wealth level at time small t further. So, since we are talking about a final wealth, so obviously, we have to consider an investment period. So, further let the investments time window be $[0, T]$. So, that means, an initial investment is made at time 0 and it is terminated at time capital T. So, then the objective is the maximization of the following. So, earlier what we had? So, suppose that we take two utility functions U_1 and U_2; U_1 for consumption and U_2 for final wealth. So, an investor can have different utilities one for final wealth and one for consumption. Of course, they can be identical also, but we are just looking at the generalized setup. So, then in the original framework what did we have? So, the utility of the final wealth will be given by $U_2(X(T))$ and the utility of the consumption is going to be given by $U_1(c(t))$. Now, this $U_1(c(t))$. this happens at different times. So, these are values obtained at different times t and so, we bring it back to time 0 and this gives me $c(t)\beta^t$. So, this is some discount factor. So, for example, B beta so, this beta could be say 1 over 1 plus r. So, accordingly when you consider $U_1(c(t))$ then we have to discount it back to the initial time and this is from t equal to 0 to capital T, alright. So, this is the sum of all the discounted utility of consumption and this is the utility of the final consumption. There are other ways of course, expressing this mathematically. So, the objective is the maximization of this term as well as maximization of this term. Now, remember that these are random variables. So, $X(t)$ is going to be a random variable and so, is c of t. So, accordingly the maximization will involve the maximization of the expected value of the of these functions consisting of the utility of consumption and the utility of the final wealth level. So, here U_1 and U_2 these are two utility functions as I have already explained and this beta strictly lying between 0 and 1 is some sort of a discount factor.

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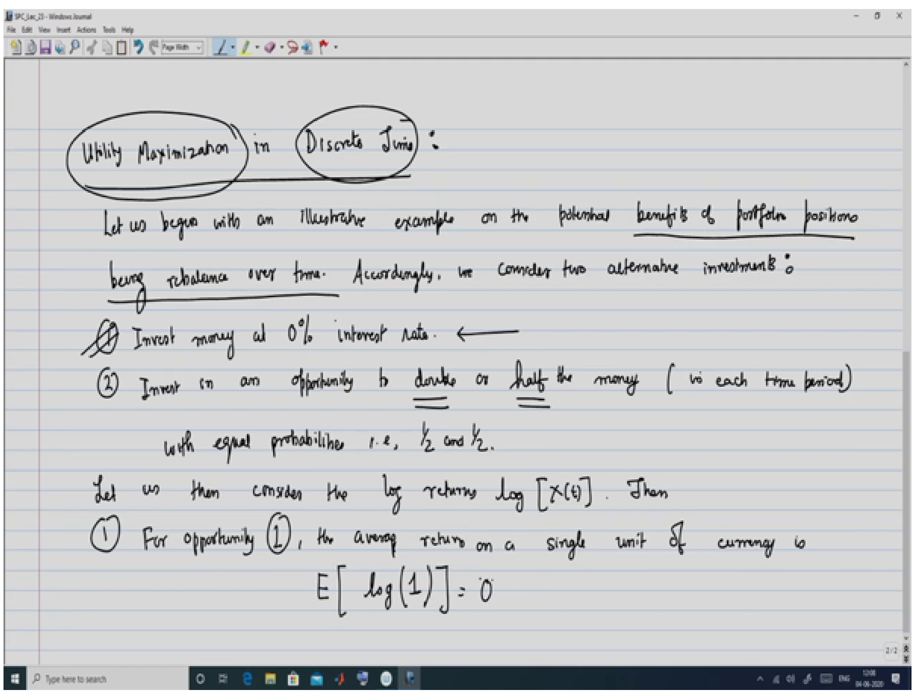

So, now, let us move on to utility maximization. So, how we are going to execute the objective function that we have listed above? So, that means, how are we going to actually solve this problem here. So, let us look at the broader topic of now a utility maximization in discrete time. So, let us begin with an illustrative example on the potential benefits of portfolio positions being rebalanced over time. And in accordance with that we consider two alternative investments. So, the first alternative is that invest money at 0 percent interest rate. So, of course, you know we do not have 0 percent interest rate in practice, but this is just to drive form the point that one could benefit from portfolio positions being rebalanced over time. So, utility maximization eventually will boil down in the discrete time setup to rebalancing your portfolio from time to time. And what you really need to look at is you have to look at this that how to accomplish this rebalancing from time to time say t equal to 0, t equal to 1, all the way to t equal to capital T. So, that eventually the end result is that you have achieved the maximal possible expected utility of the consumption as well as the wealth process. So, coming back to this example so, the first alternative is invest money at 0% and the second alternative is that you invest in an opportunity to double or half the money in each time period alright. So, this is just like the binomial model either you can go up by a factor of 2 or you can come by a factor of half with equal probabilities that is half and half. So, now, when we are talking about utility maximization obviously, we have to specify what is the utility function. So, accordingly we consider the log utility. So, let us then consider the log returns; so, that means, the utility to the final time point will be log of X of capital T. Then what happens in the first case? So, for opportunity 1, that is this opportunity the average return on a single unit of currency is, what is this going to be? You have invested so; you invest an amount of 1 since the interest rate is 0%. So, you end up with an amount of 1. So, in that case the final wealth level is 1 and the log of that is going to be of the final wealth level is going to be 1 and this expected value is simply going to be 0 because $\log(1) = 0$.

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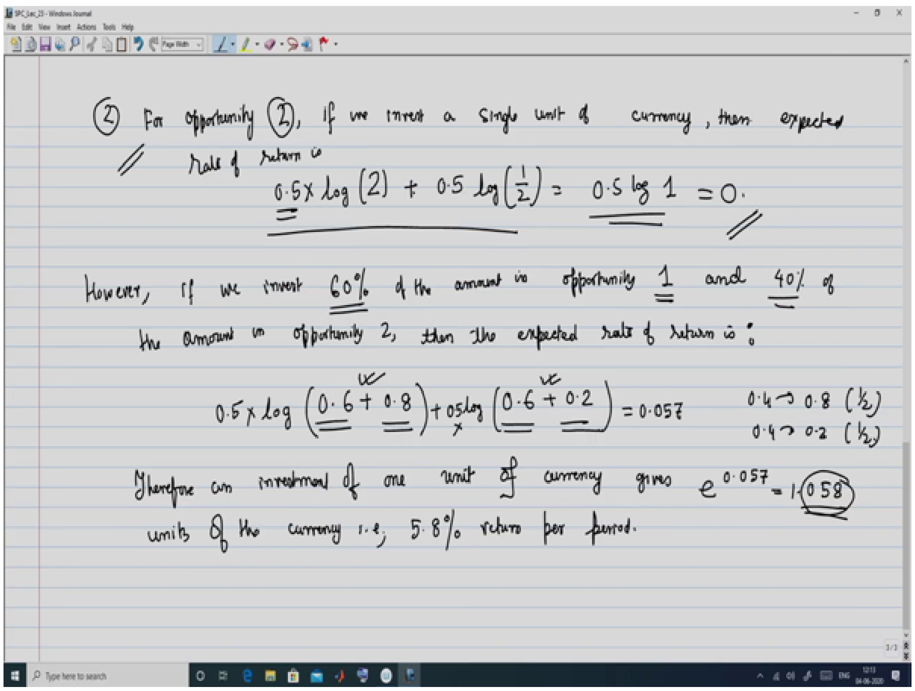

Now, let us look at what happens in for opportunity 2. So, in case of opportunity 2, if we invest a single unit of currency then what is going to happen? Then either the wealth becomes 2 or the wealth label becomes half. So, it is utility is going to be $\log(2)$ and $\log\left(\frac{1}{2}\right)$ and it has the probability of $\frac{1}{2}$. So, when you are trying to calculate the expectation you get half here into $\log(2)$ and $\frac{1}{2}\log\left(\frac{1}{2}\right)$. So, then we can say that the expected rate of return is this expression and this is nothing but $0.5\log(1) = 0$, which is 0. So, you see that so, this takes care of the second alternative. Now, you have see that if you individually invest either an opportunity 1, then the expected return or the average return is going to be 0; if you individually invest just an opportunity 2 then again the expected return is going to be equal to 0. So, however, now we need to start looking at other alternatives. So, remember that in the first case it is so, if we need to start now looking at portfolios comprising of opportunity 1 and 2. So, the two cases that you have considered, in the first case the weight for the first investment was 1 and the weight for the second investment was 0. And, in the second case the weight for the first investment is 0 and the weight for the second investment is 1. So, what you are going to do now is, consider another situation where you know these two extreme cases of complete investment in either of the opportunities is discarded and we now look at a possibility of taking a combination of investment in both of them. So, that means, you know we take w_1 and w_2 both of them to be equal to non-zero. So, however, let us see that if we invest 60 percent that is weight is 0.6 of the amount in opportunity 1; that means, investing at 0 percent rate and 40 percent of the amount in opportunity 2, then the expected rate of return is the following. So, if you invest an amount of the 60 percent in opportunity 1, the; that means, you get 0.6 unit of the currency. So, what you will get is that from opportunity 1 you will end up getting 0.6. Now, the remaining 40 percent means that you have invested an amount of 0.4 in or in in the opportunity 2, so, that 0.4 will go either to 0.8 with probability half or it will come down to 0.2 with probability half. So, that means, that for this you will have 0.6 plus 0.8 is the final amount; 0.6 coming from opportunity 1 and 0.8 from opportunity 2. Or you will get 0.6 from opportunity 1 plus 0.2 from opportunity 2. So, these two are the possible final wealth level. So, the utility of each of them are going to be log of 0.6 plus 0.8 or log of 0.6 plus 0.2. And, then you want to calculate the expected return. So, this one this return can happen with the probability of half so, I multiply this with 0.5 and this can also happen with probability half so, I multiply this with 0.5 and add them up. And, this turns out to be equal to 0.057. So, therefore, an investment of one unit of currency gives e raised to 0.057 equal to 1.058 units of the currency, that is, you are able to achieve looking at the 0.058, this term 0.058. So, that means, you have been able to achieve 5.8 percent return per period.

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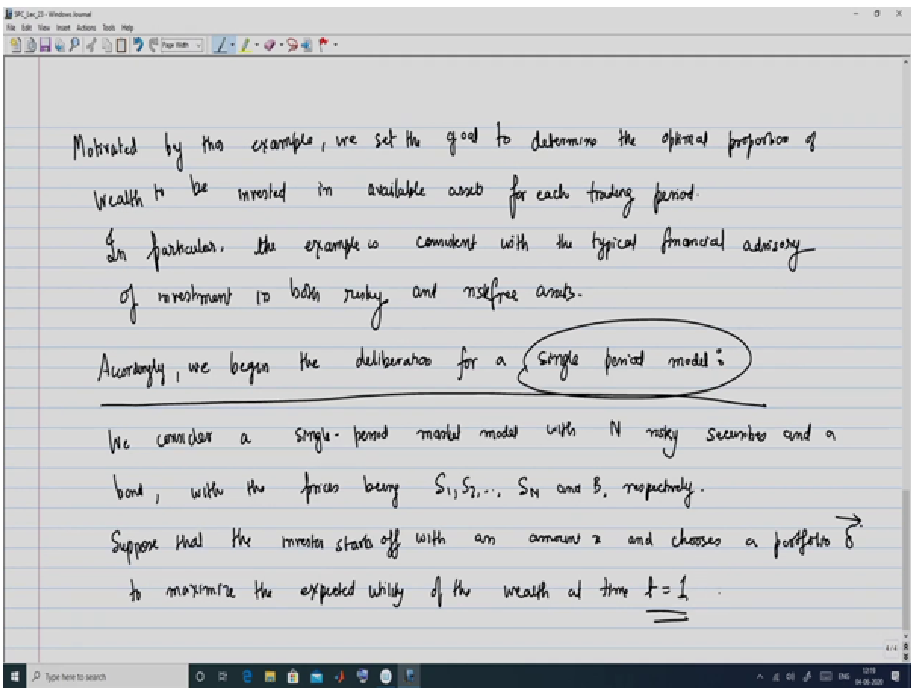

So, motivated by this example, we set the goal to determine the optimal proportion of wealth to be invested in available assets for each trading period. So, in particular, the example is consistent with the typical financial advisory of investment in both risky and risk free assets. Let me elaborate a little bit more on what I am trying to convey here in terms of this motivation. What I am saying here is that, initially you looked at two opportunities and in the first opportunity you had 0 percent interest and in the second opportunity you had a likelihood of going up by a factor of 2 or coming down by a factor of half with equal probabilities. And, you observe that in both the scenarios if you just invest completely in those opportunities exclusively then you end up with getting an expected return to be equal to 0. However, we choose a particular case of a combination of investment with 60 percent in opportunity one and 40 percent in opportunity 2; which ended up giving you a positive expected rate of return. So, if you run this kind of portfolio reshuffling just not in the 60 40 percent proportion, but all possible combination that is you know you look at all the opportunities set and then you can get a large class of expected utilities values. And your goal is to find out that combination of weights w 1 and w 2 that is going to give you the largest possible expected utility out of this investment strategy. And, this example is just to illustrate one case how you can achieve a greater amount of return by investing in a diversified manner rather than investing exclusively in individual assets. And, we start of this as the beginning point of our discussion on how actually we will determine what those optimal weights w 1 and w 2 are in the context of dynamic programming. So, what you do is so, accordingly, we begin the deliberation for a single period model and then we will extend this to the multi period model, where we will make use of the Bellmans dynamic programming principle ok. So, we begin with the setup for this and consider a single period model for the market with N risky securities and a bond, with the prices being S 1, S 2 all the way to S N for the risky securities and B for the bond respectively. Suppose that, the investor starts off with an amount x and chooses a portfolio delta vector remember the delta vector comprises of delta naught, delta 1, all the way to delta N to maximize the expected utility of the wealth at time t equal to 1, right. So, we are taking time t equal to 1 from time t equal to 0; because as you have said this is for the single period model.

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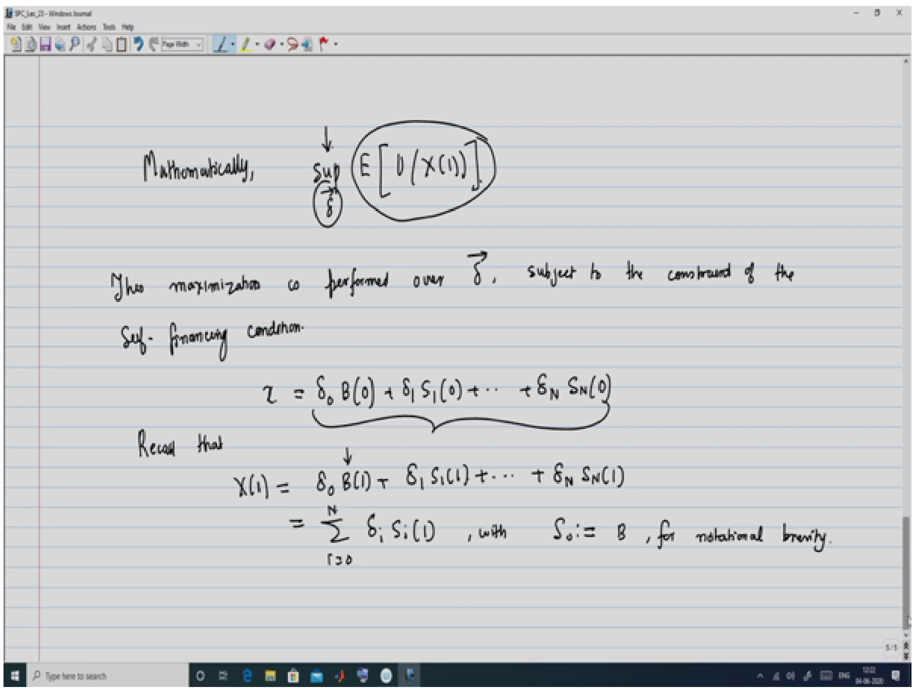

So, mathematically what is this going to be written as. So, in concrete mathematical terms this means that you are trying to find the supremum or the maximum to determine what is the best delta such that you achieve the maximization. So, figure out what is going to be your delta, so that the expected utility at time 1 is going to be maximized or you obtain the supremum. So, this maximization is performed over the vector delta subject to the constraint of the self-financing condition. What is the self-financing condition? See you start off with an amount of x and

$$x = \delta_0 B(0) + \delta_1 S_1(0) + \cdots + \delta_N S_N(0).$$

And, this two total investment must exclusively be funded by the exact amount that you have been had that is your amount of x. So, just recall that; so, in this context what I want is that we need to recall that when we are discussing this wealth process earlier in a single step model. So, we had

$$X(1) = \delta_0 B(1) + \delta_1 S_1(1) + \cdots + \delta_N S_N(1).$$

So, this is basically exactly the same expression except at time 1 plus delta N S N of 1 and this can be now rewritten, so, I am introducing this compact formulation of delta i S i of 1 i equal to 0 to N and this is done with just one change and that is with S naught being defined as B. So, this is for notational brevity.

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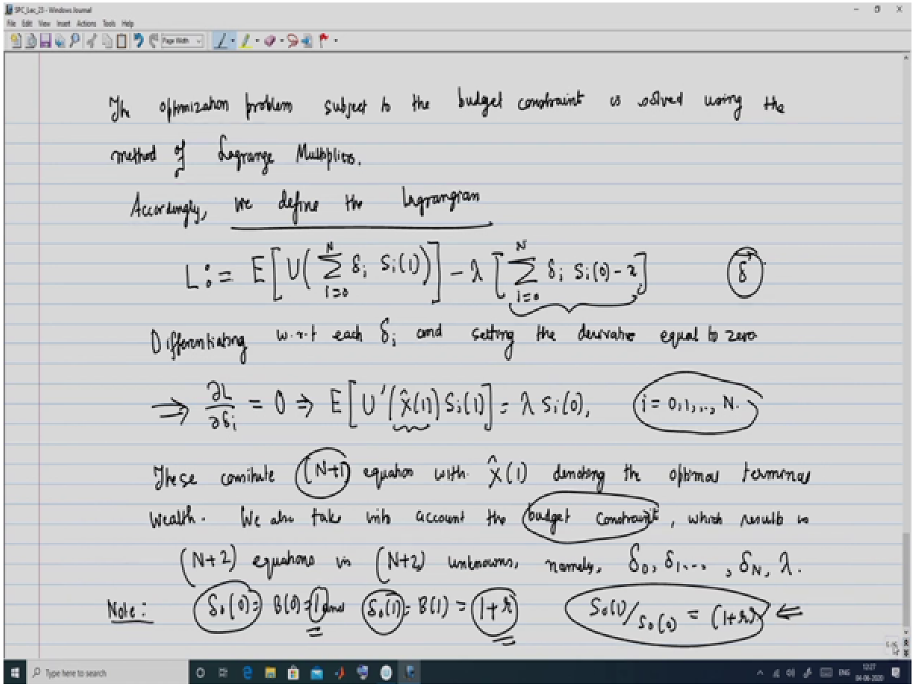
So, the optimization problem subject to the budget constraint is solved using the method of Lagrange multipliers that you have already encountered, when you looked at a multi asset portfolio optimization in the mean variance framework. So, accordingly, we define the Lagrangian remember what do you want to



maximize, you want to maximize the expected utility of $X(1)$. So, what is the what is $X(1)$? $X(1)$ is nothing but this expression. So,

$$L := E\left[U\left(\sum_{i=0}^{N}\delta_i S_i(1)\right)\right] - \lambda\left[\sum_{i=0}^{N}\delta_i S_i(0) - x\right].$$

And, so, I define this to be my Lagrangian L. So, what you do? So, we basically want to do the optimization with respect to delta. So, accordingly what we do is, so, delta is nothing, but a delta naught, delta 1 all the way to delta N. So, accordingly in order to determine the optimized delta is we start by differentiating L with respect to each delta i and setting the derivative equal to 0. So, we have

$$\frac{\partial L}{\partial \delta_i} = 0 \implies E[U'(\hat{X}(1))S_i(1)] = \lambda S_i(0), \ \ i = 0, 1, 2\cdots, N.$$

Now, since $i = 0, 1, 2\cdots, N$, so this means that this set of equations constitute $N + 1$ equations with the newly introduced variable $\hat{X} + 1$ denoting the optimal terminal wealth. We also take into account, the budget constraint that means, this relation being equal to 0, which results in, in addition to this $N + 1$ we have this budget constraints. So, we end up having $N + 2$ equations in $N + 2$ unknowns and what are this $N + 2$ unknowns? So, these are delta naught, delta 1 all the way to delta N and of course, lambda. So, here I just want to note that S naught of 0 is equal to B of 0 is equal to 1 and S naught of 1 is equal to B of 1 is equal to 1 plus r. Now, it is not necessary that I should take this to be equal to 1, but for simplicity of taken because we are not going to use this one value, but rather we will just make use of the fact that S naught of 1 over S naught of 0 this is just going to be equal to 1 plus r. So, we so, in that sense of course, you could have taken this to be some constant and then we could have taken S naught of 1. So, I could have taken S naught of 0 equal to some constant and I could have taken S naught of 1 to be that constant into 1 plus r. So, we do not lose the generality because this is the result that we are going to use.

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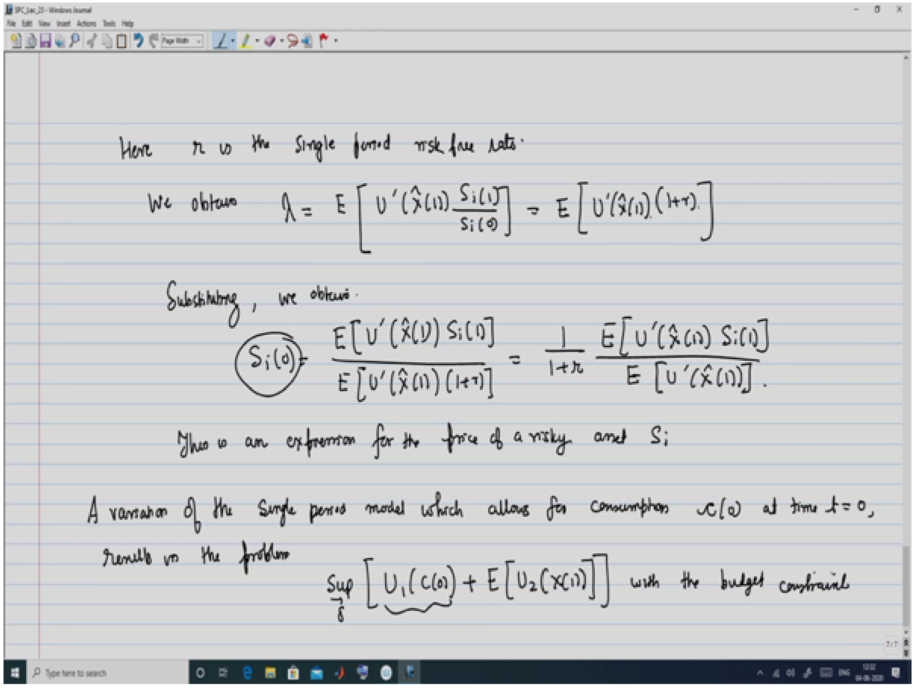
So, here r, so, this newly introduced r that you have here r is the single period; remember, we are just considering the single period model. So, this is the single period risk free rate. Now, going by the above equation so, if you observe carefully here this relation; so, from here we obtain the value of lambda. What is the value of lambda?

$$\lambda = E\left[U'(\hat{X}(1))\frac{S_i(1)}{S_i(0)}\right] = E\left[U'(\hat{X}(1))(1 + r)\right].$$



And, remember that this is going to be; so, S i of 1 over S i of 0 this is 1 plus r. So, that is what I was referring to as we will just make use of the ratio into U prime of X hat of 1 into 1 plus r. So, now, that we have obtained lambda. So, we can then obtain by substitution that S i of 0 this is going to be E. So, what we do is that we substitute the value of lambda again in this expression and get the lambda in the denominator here to obtain S i of 0.

$$S_i(0) = \frac{E\left[U'(\hat{X}(1))S_i(1)\right]}{E\left[U'(\hat{X}(1))(1+r)\right]} = \frac{1}{1+r}\frac{E\left[U'(\hat{X}(1))S_i(1)\right]}{E\left[U'(\hat{X}(1))\right]}.$$

And, this is an expression for the price of a risky asset. So, price I mean the price at time t equal to 0 of a risky asset S 1. So, just one last observation is that a variation of the single period model which allows for consumption a c of 0 at time t equal to 0, results in the problem:

$$\sup_{\delta}\left[U_1(c(0)) + E[U_2(X(1))]\right],$$

with the budget constraint.

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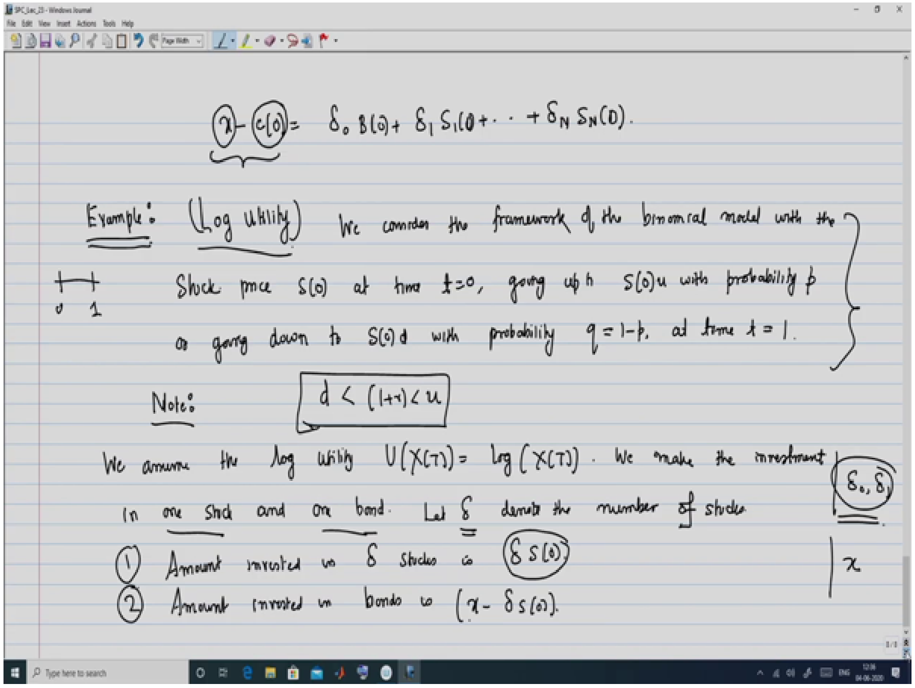
$x - c(0)$, remember that we start off with wealth x, but you can invest only the amount that is left after consuming $c(0)$. So, that means,

$$x - c(0) = \delta_0 B(0) + \delta_1 S_1(0) + \cdots + \delta_N S_N(0).$$

So, now, let us come to an example to illustrate, how this maximization of expected utility is accomplished and for this purpose we again make use of the log utility. So, for this we consider the framework of the binomial model, but we are still we have the single period setup with the stock. So, we have again we have time t equal to 0 and time t equal to 1. So, with the stock price $S(0)$ going so, this stock price $S(0)$ at time t is equal to 0 going up to $S(0)u$ with probability p or going down to $S(0)d$ with probability q which obviously, is $1-p$ and, these values can happen at time $t = 1$. Now, note that we have the restriction that

$$d < 1 + r < u.$$



So, now, once we have this set up for the asset prices, so, we assume the log utility that is $U(X(T)) = \log(X(T))$, and we denote the investment in one stock. So, we actually not denote, but rather make the investment in one stock and one bond. So, this is a very simple example, wherein I am considering the investment in one bond as the general market model. But, instead of investing in a capital N number of stocks we just invest in one stock and let us see how the optimization problem in this case plans out. So, we have made an investment in one stock and one bond. So, accordingly I will use the notation; so, that means, I have here I have delta naught and delta 1. But, what I am going to do that since you know we just have δ_0 and δ_1, so, I will let delta denote the number of stocks ok. So, as a consequence of this, what happens? We observe that if you buy delta number of stocks. So, then the total amount invested in delta stocks is $\delta S(0)$. And, remember that we had an initial amount of X and out of which we have invested delta S naught in the stocks. So, as a result the remaining amount invested in bond is $x - \delta S(0)$.

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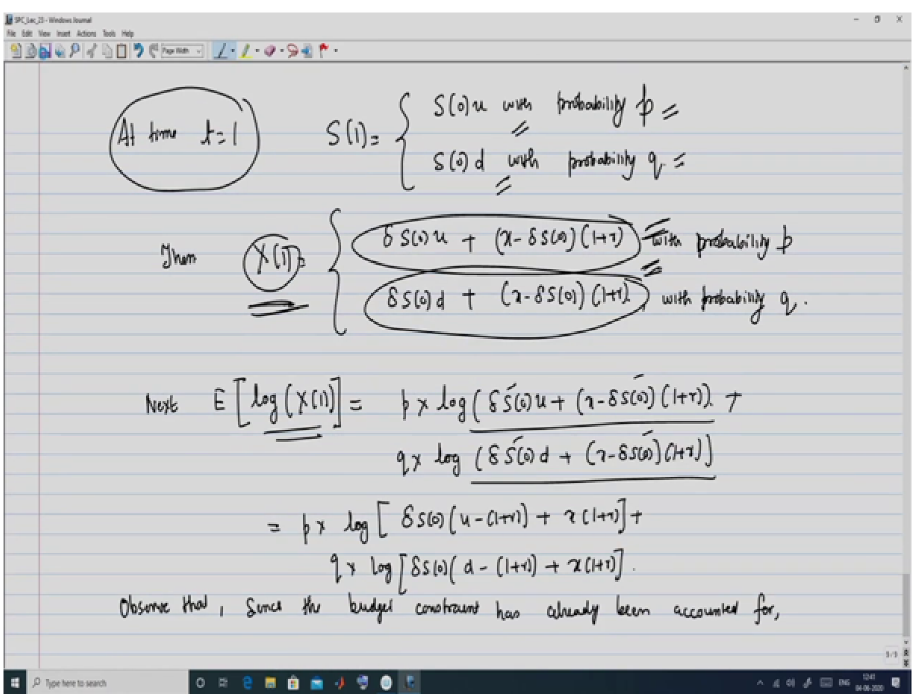

So, now at time $t = 1$, what happens? So, according to the binomial model at time $t = 1$,

$$S(1) = \begin{cases} S(0)u & \text{with probability } p \\ S(0)d & \text{with probability } q \end{cases}$$

So, then what happens? Then, what are the possible values of $X(1)$? See the your investment in bond what was your investment in bond? So,

$$X(1) = \begin{cases} \delta S(0)u + (x - \delta S(0))(1+r) & \text{with probability } p \\ \delta S(0)d + (x - \delta S(0))(1+r) & \text{with probability } q \end{cases}$$

So, that means, your total wealth will be the sum of these two with the first scenario having the probability p and the second scenario it has the probability q, alright. So, now you have this random variable X taking two possible values, the one here with probability p or the one here which is the probability q. So, next we need to calculate the expected utility or of terminal wealth, so, that means, it is going to be

$$E[\log(X(1))] = p\log(\delta S(0)u + (x - \delta S(0))(1+r)) + q\log(\delta S(0)d + (x - \delta S(0))(1+r))$$

$$= p\log[\delta S(0)\{u - (1+r)\} + x(1+r)] + q\log[\delta S(0)\{d - (1+r)\} + x(1+r)].$$

Now, observe that, since the budget constraint has already been accounted for and how is it been accounted for because we have already taken this $x - \delta S(0)$ term here?

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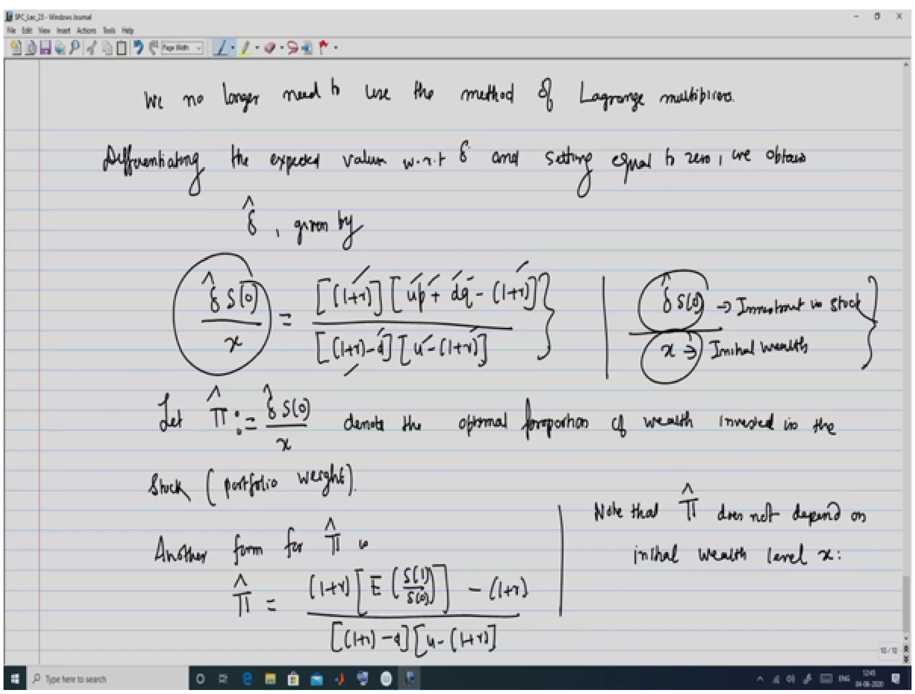

So, therefore, we no longer need to use the method; now, with the constraint gone so, we no longer need to use the method of Lagrange multipliers. So, differentiating; so, this essentially reduces to a single variable calculus problem. So, differentiating the expected value so, we have this expected value here. So, we differentiate this expected value here. Remember that here p and q are given to you know what is u, d and 1 plus r and of course, you know what is x and S 0. So, the only factor that you need to determine is going to be the delta and you need that delta for which this expected utility is maximized. So, essentially it boils down to then maximizing this function here as a function of delta. So, accordingly we differentiate the expected value with respect to delta because that is what we are trying to optimize and setting equal to zero, we obtain; so, after some simplification you get delta hat we obtain delta hat given by the expression delta hat. So, I obtain

$$\frac{\hat{\delta}S(0)}{x} = \frac{(1+r)[up + dq - (1+r)]}{[(1+r) - d][u - (1+r)]}.$$

So, observe here that the term on the right hand side 1 plus r 1 plus r these are known, u and d are known and so are p and q. So, this is essentially independent of x. Now, also observe carefully that if delta hat is the optimal value of delta, so, the delta hat S naught is the investment in in stock and then x is the initial wealth. So, this fraction is going to be just the weight of the investment in the stock. So, accordingly so, I will define that

$$\hat{\pi} := \frac{\hat{\delta}S(0)}{x}.$$

This will denote the optimal proportion of wealth invested in the stock or what is known as the portfolio weight. And, you see that this is independent of the initial wealth level. And, another form for $\hat{\pi}$ is

$$\hat{\pi} = \frac{(1+r)\left[E\left(\frac{S(1)}{S(0)}\right)\right] - (1+r)}{[(1+r) - d][u - (1+r)]}.$$

And, the last observation is that note that $\hat{\pi}$ that is the optimal portfolio weight $\hat{\pi}$ does not depend on initial wealth level x. So, this brings us to the end of this lecture. Just to do a brief recap, what I did today is that we started looking more specifically into the problem of maximizing the expected utility. And, for that purpose we started off talking about what is going to be the expected utility of the terminal wealth in case of a single period model; say, we have time t equal to 0 at time t equal to 1 and we explained the general principle of that. And, finally, we illustrated this through an example where we had the utility function to be the log utility. And, you calculated the expected utility of the wealth at the final time point that is t is equal to 1. In order to determine what is going to be our optimal number of stocks that we need to purchase that which you denoted by delta hat in order to achieve our goal of maximization of the expected utility of the wealth at the final time point. So, in the next class, we will continue this discussion and we will move on to from the single period-setup to the multi-period setup and we will look at an example of how this can actually be accomplished in case of specified utility functions.

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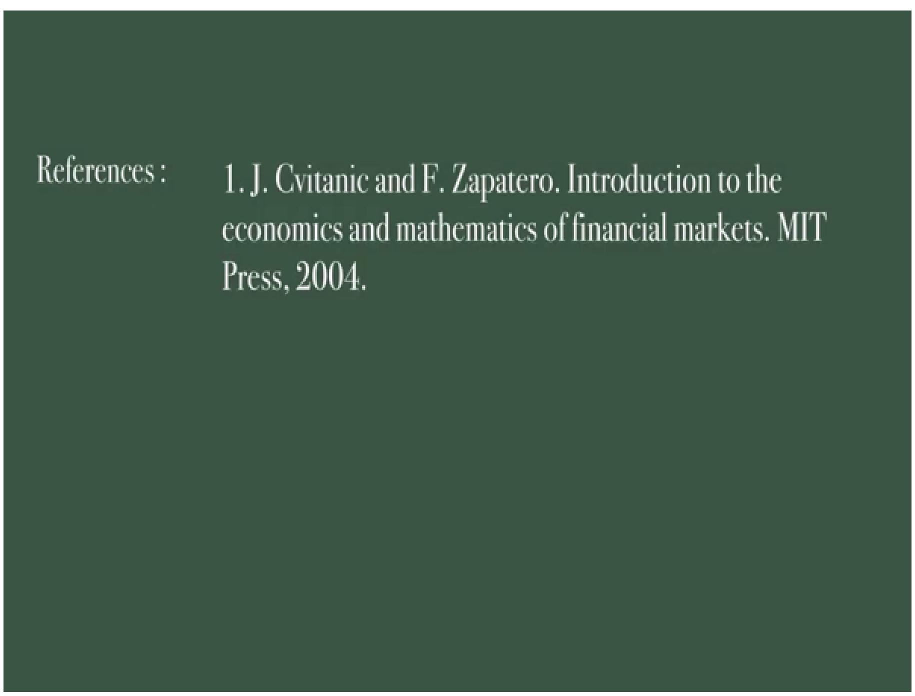

Thank you for watching.