Mathematical Portfolio Theory

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

Module 05: Optimal Portfolio and Consumption
Lecture 01: Discrete time model and utility function

Hello viewers. Welcome to this lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. You would recall that we had spent the last few lectures discussing on the non-mean variance framework. And, in todays class we start off on a new broader topic namely an optimal portfolio and consumption. And, in this we will talk about both the optimal portfolio as well as consumption in the set were set up for discrete time models as well as continuous time models. And, this is the point we will go back and refer to one of our earlier lectures, when he had talked about the binomial model and we have talked about the geometric Brownian motion model.

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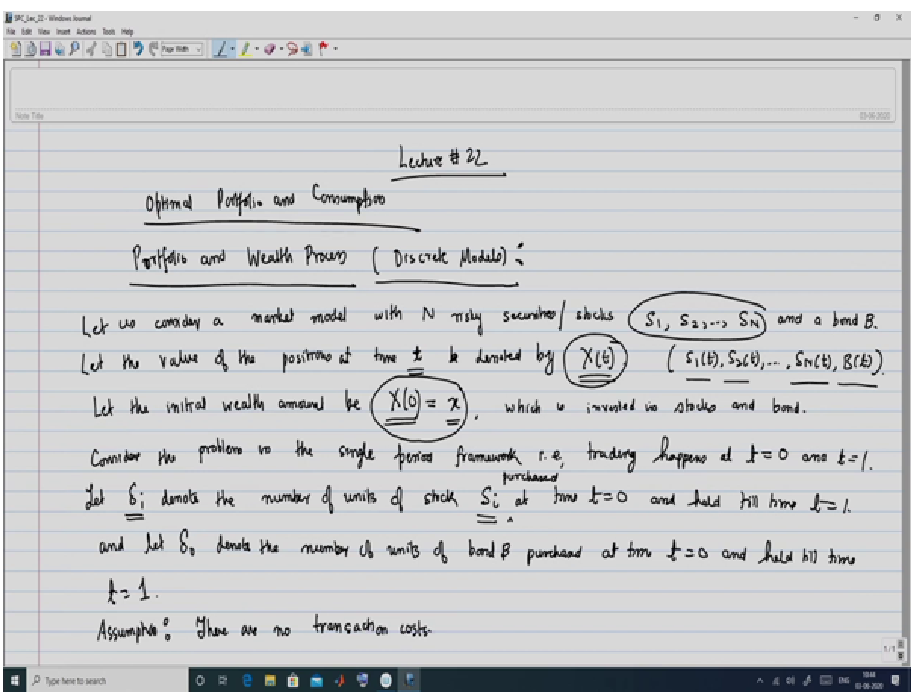

So, let us begin this lecture. And, the first topic that you will do is so, this broader topic is optimal portfolio and consumption. So, we begin with the first topic that is portfolio and wealth process. And, as I said that we will discuss this in the discrete and continuous time framework. So, we will start off with discrete models. So, let us consider a market model with n risky securities or a stocks which are designate by $S_1, S_2, \cdots, S_N$ and the market has a bond B. Further, let the value of the positions at time t be denoted by X of t. So, this means I will have things like $(S_1(t), S_2(t), \cdots, S_N(t), B(t))$. Further, the assumption is that the initial wealth level or amount be $X(0) = x$. So, what do I mean is basically now to be more explicit here X of t will denote the wealth level at time t, when you have started off at an initial time t equal to 0 with a wealth level of x. And, in a synonymous manner we will use $(S_1(t), S_2(t), \cdots, S_N(t), B(t))$ to denote the price of the stocks and the bond at time t. So, let us start get focused on what is going to be my wealth process $X(t)$? Sub subject to as having started off with a initial amount to be invested as being some little x. And, this initial amount is invested in the a capital L number of stocks and bond. So, we now consider the problem in the single period framework, that is trading happens at say time t equal to 0 and t is equal to 1. Now, let delta i denote the number of units of stock S_i at time t equal to 0 so; that means, at time t equal to 0 you purchase delta i number of the stock S i and this is held till. So, I should say that this is purchased at a time t equal to 0 and held till time t equal to 1. And, let delta naught denote the number of units of bond B purchased at time t equal to 0, and held till time t is equal to 1. So, the assumption is that that there are no transaction costs, all right.

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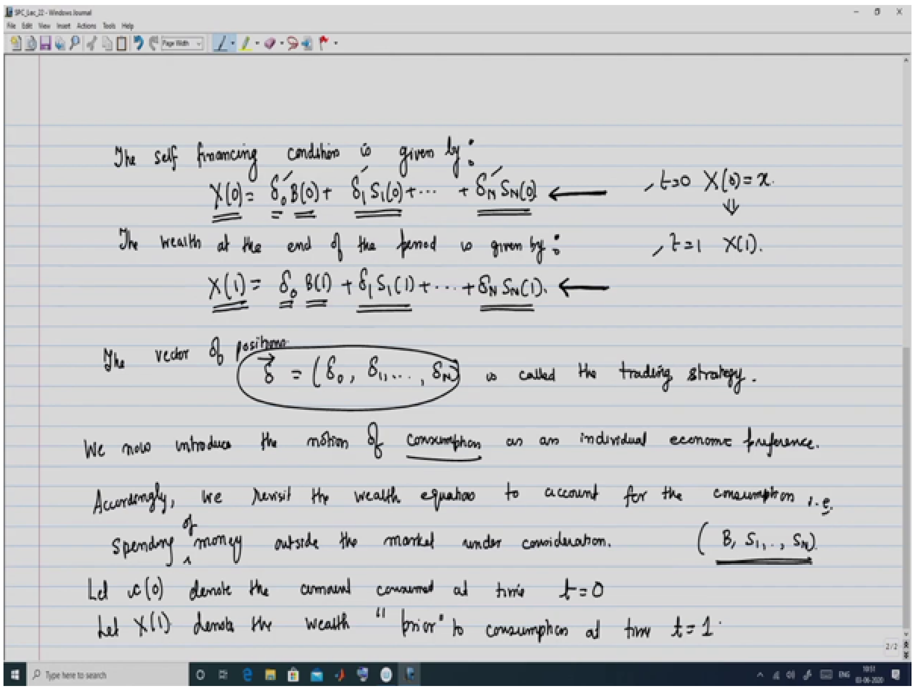

So, now, once we have this set up what you can do is that we now can introduce what is known as the self-financing condition. So, accordingly the self-financing condition is given by

$$X(0) = \delta_0 B_0 + \delta_1 S_1(0) + \cdots + \delta_N S_N(0).$$

And, the wealth at the end of the period is given by

$$X(1) = \delta_0 B_1 + \delta_1 S_1(1) + \cdots + \delta_N S_N(1).$$

So, what you have here? So, basically we start off at time t equal to 0, we start off with the wealth level X 0 equal to X and this at time t equal to 1 has the value of X of 1 given by this relation. So, how do you get X 0? So, I remember that you have invested in delta naught number of bonds at time 0. And, you have invested in delta i number of stocks S i at time 0. So; that means, that this expression here, this is going to be the total investment that you have at time t equal to 0. And, this must exactly match the amount of money that you have been had namely X of 0. Now, remember that this delta naught delta 1 all the way to delta N, these are positions that you get into at time t equal to 0 and hold on to until time t equal to 1. So, accordingly the wealth at the end of the period is given by so, delta naught into the price of the bond at time one, which is B 1 and then all these delta is with the price of the stock S_i being $S_i(1)$. So, you add this up and this will give you the value of your investment or the value of the portfolio at time t equal to 1. So, this is called the self-financing condition because that you can only invest the total investment must exactly match the amount that is available in hand. So; that means, that the whole investment of delta naught delta 1 to delta N must be completely self-financed by the money that is hand without any external support. So, now, what I have introduced is we introduce a notation of this to club together delta naught, delta 1 all the way to delta N and we call this vector delta. So, this is the vector of positions delta as given here this is called the trading strategy. So, what do you do now is we now introduce the notion of consumption as an individual, economic preference. So, accordingly as a result of the introduction of the concept of consumption, we revisit the wealth equations to account or incorporate the consumption. That is consumption means spending money outside the market under consideration. So, remember what is the market under consideration? The market under consideration was the bond S 1 all the way to S N and consumption means that, say if we are in the market under consideration; that means, that your investment will be in the bond or one of the bond or S_1 or S_2,· · ·,S_N or a combination of that. So, consumption is basically spending money not on any of these what we call securities, that is the bond and capital and number of stocks, but spending the money somewhere outside the market, for the purpose of your individual consumption. So, here this basically means that, the amount that is consumed is no longer in circulation in the market under consideration all right. So, let then we introduce a notation. So, let us c 0 denote the amount consumed at time t is equal to 0. And, another notation we introduced is let X of 1 denote the wealth prior to consumption at time t is equal to 1.

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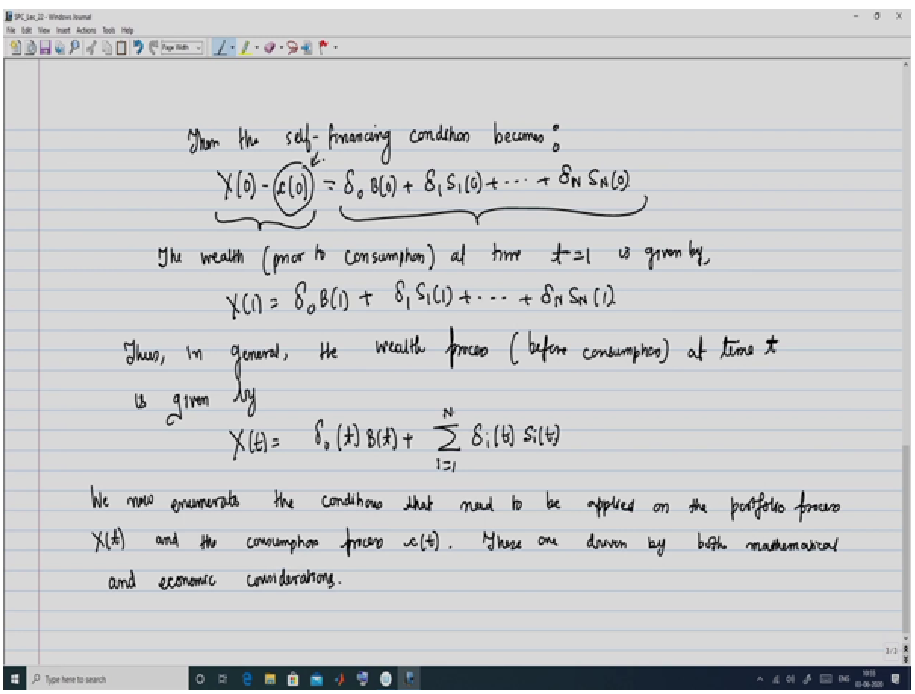

Now, let us now reexamine the self-financing condition after consumption. So, accordingly the self-financing condition becomes. So, earlier what we had? We had

$$\delta_0 B(0) + \delta_1 S_1(0) + \cdots + \delta_N S_N(0).$$

And, earlier this was equal to X of 0 is the wealth that is available at time 0. However, now that you have made a consumption of C 0; that means this amount of money has gone outside the market. So, the remaining amount that you have is the one which you can invest in the market. And, then X 0 minus 6 C 0 must be equal to the amount invested in the market after the consumption C 0 has happened. And, also the wealth prior to consumption as indicated in the preceding line of the previous page. So, the wealth prior to consumption at time t equal to 1 is given by as before

$$X(1) = \delta_0 B(1) + \delta_1 S_1(1) + \cdots + \delta_N S_N(1).$$

. So, thus in general the wealth process before consumption at time t is given by

$$X(t) = \delta_0(t)B(t) + \sum_{i=1}^{N} \delta_i(t)S_i(t).$$

So, we now enumerate the conditions that need to be applied on the portfolio process $X(t)$ and the consumption process. So, remember the consumption is also a process, you can consume at the time generic time t. And, these conditions are driven by both mathematical and economic considerations. So, let us enumerate these conditions.

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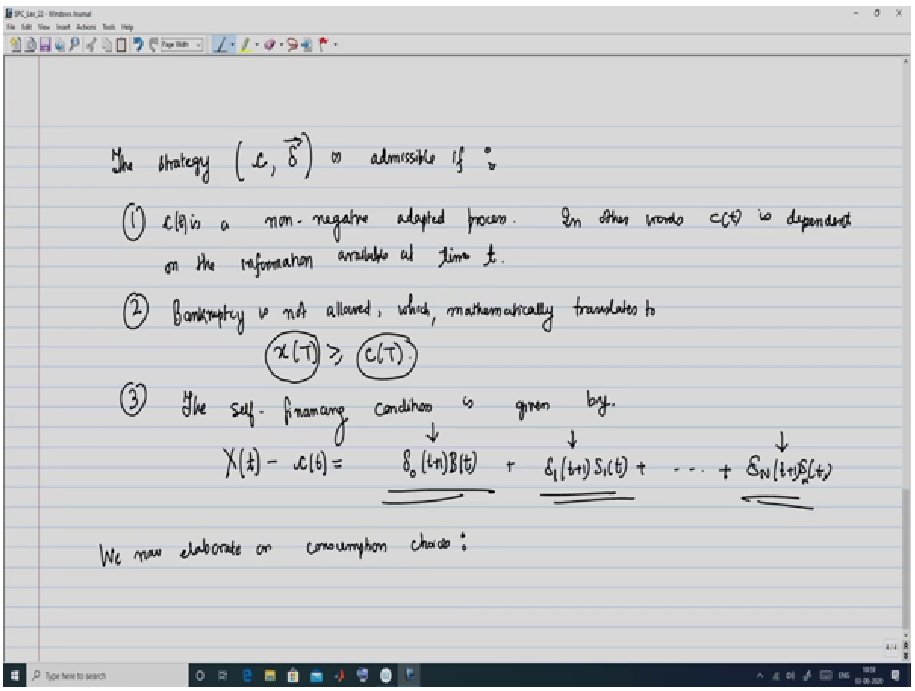

So, the strategy and by strategy I mean here C as well as the vector delta of the investment, this is admissible or that is allowed. If, so, this is something like opportunity set that we have done if C of t is a non-negative adapted process. And, let me explain what adapted process is. So, this means that, in other words C of t is dependent on the information available at time small t. Secondly, bankruptcy is not allowed which mathematically translates to the following, that the wealth at time capital T must be greater than or equal to C of capital T. So; that means, your consumption at time capital T cannot exceed your wealth level at time capital T. And, thirdly the self-financing condition is given by the following. So, let us look at time t at time t you use have a wealth level $X(t)$. And, then you have consumed C of t and the remaining amount you invest in bonds delta naught bond, δ_1 is in this first talk all the way to delta of a S N of t. And, this delta naught will have the argument t plus 1 to indicate as that this is an investment made at time t and held up to time t plus 1. So, likewise $\delta_1(t+1), \cdots, \delta_N(t+1)$. So, what we had actually here is that this delta naught of t, this is basically the investment made at time t minus 1 and this is the investment made at time t minus 1. And, this is the number of units that you hold at time t till time small t and then the value becomes delta naught of t into B t. And, summation of delta i of t S i of t and this ends up being the value of your holdings at time t given by X of t. So, please understand that this is the investment made at time t minus 1 held until time t. So, likewise here this one is the investment that is made at time t as likewise this values, but held until time t plus 1. So, for delta S the argument will indicate the time point till which the investment is actually being held. So, now what we do is that, we now elaborate on consumption choices.

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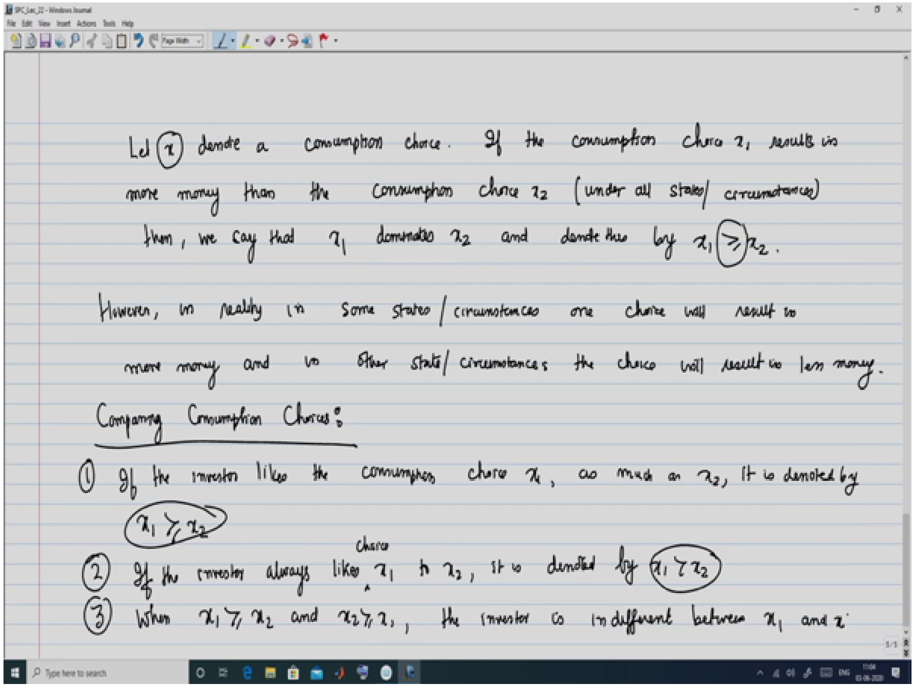
So, accordingly let x denote a consumption choice. And, x could be basically a different alternative investment that you have or a different consumption alternatives that you have. Now, if the consumption



choice x_1 results in so, this x should not be confused to the self-financing condition. So, this should be this is synonymous with C. So, this results in more money being generated, then the consumption choicex_2 under all states or circumstances, then we say that x_1 dominates x_2 and we denote this by x_1 being greater than or equal to x_2. So, this is the notation it is not literally greater than, but this is the notation. Now, unfortunately the reality is slightly different as we had seen in case when you are discussing SSD, FSD and TSD in the previous class. So, in reality what happens is that in some states? So, this is similar to when we looked at the cumulative distribution of one being greater than other in some range of R and the cumulative distribution of the one being less than the other in some other range of the returns. So, a similar kind of argument will now result in this following statement, that in reality in some states or a circumstances one choice will result in more money, and in other state or circumstance the same choice will result in less money. So; obviously, this brings us to the question of how you can make a consumption choice. So, the next thing we will look at is comparing consumption choices. So, the first topic is the first sort of axiom or a comparative criteria is the following, that if the investor likes the consumption choice x_1 as at least as much as x_2, this is denoted by this notation. Secondly, if the investor always likes x_1, choice x_1 to x_2 it is denoted by this notation. And, thirdly when the investor likes choice x_1 as much as x_2 and x_2 as much as x_1 the investor is indifferent between the two choices x_1 and x_2.

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And, this is denoted by x_1 similar to x_2. So, this brings me to the properties of preference relation and will enumerate four properties. So, the first property is completeness or some sort of exhaustiveness. And, this is stated as given any two strategies x_1 and x_2 we have either so, there are only three possibilities. That x_1 is at least preferred to x_2 or x_2 is preferred to x_1 at least or possibly both. Next, is transitivity. So, if the consumption or under x_1 is preferred over x_2 or at least you know preferred as much as x_2. And, x_2 is at least preferred as x_3, then this preference will hold or for x_1 over x_3. Thirdly, this is monotonicity. So, if x_1 is preferred over $x_2$if in numerical terms, then the consumption x_1 is preferred over x_2. And, finally, we have the condition of convexity. So, in this case what we do is that we start off with two consumption x_1 and x_2. And, we consider the consumption choice x_3 and that is given by some linear combination alpha x_1 plus 1 minus alpha x_2. Of course, alpha strictly lies in the open interval $[0, 1]$. Then, we have x_3 is preferred over x whenever we have x_1 is preferred over x and x_2 is preferred over x.

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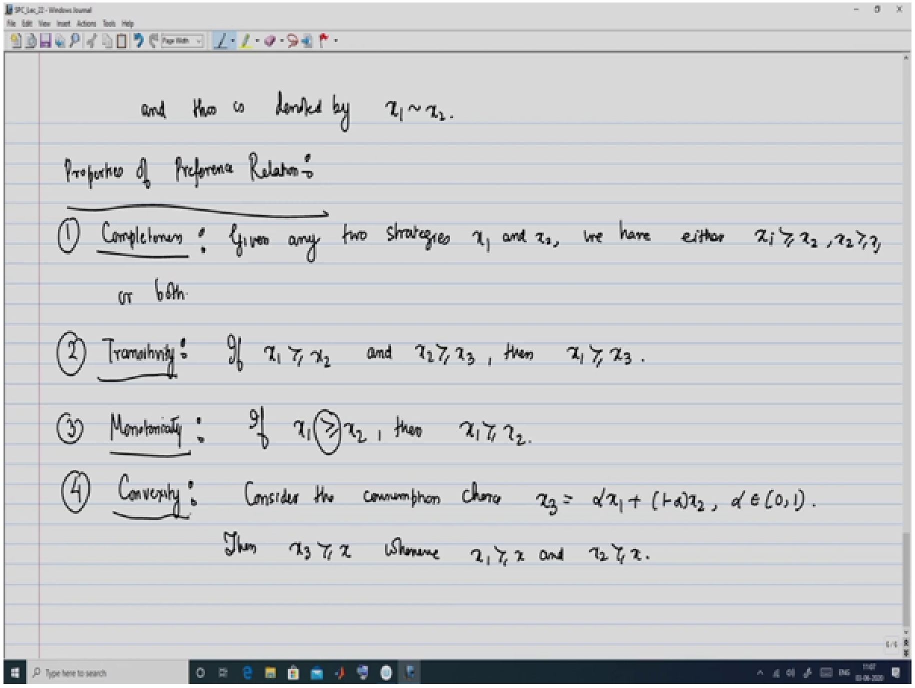
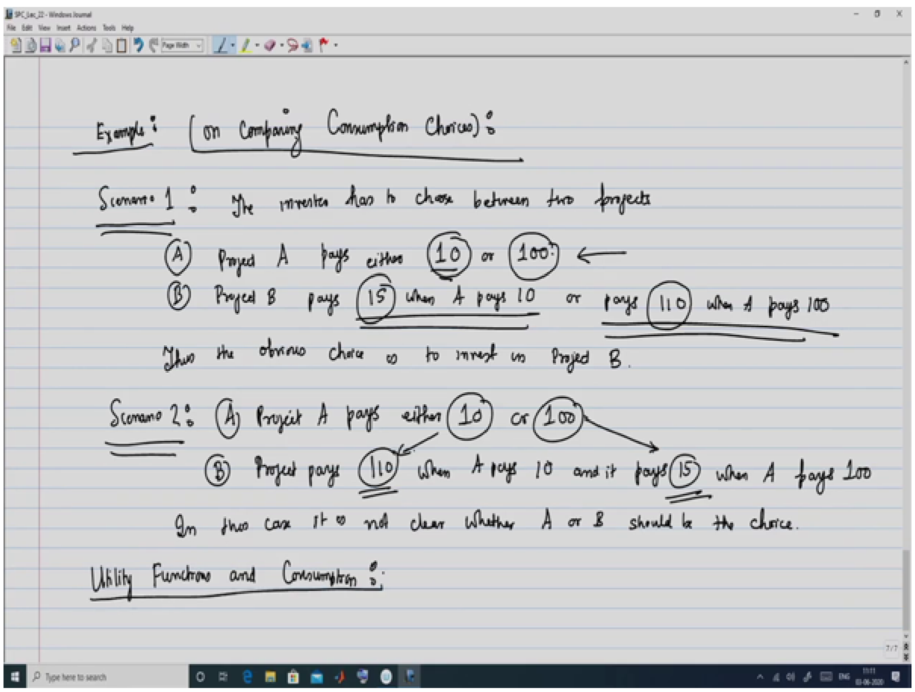
Let us now consider, an example of a comparing consumption choices. And, you look at two scenarios.





So, scenario 1 is the enumerated as follows and in scenario 1 the investor has to choose between two projects. The first project is that so; project a pays either 10 or 100. And, there is a project B pays 15 when A pays 10 or it pays 110, when A pays 100. So, there are so, under the circumstance that A pays 10, then B will pay 15 which is a higher amount. And, in the circumstance when A pays 100, then B will pay 100. So, in both the cases B both the scenarios B will pay a more. So, thus the obvious choice is to invest in project B, all right. So, let us look at another scenario a slightly different where you were caught in a dilemma. So, here again project A pays as before here. It pays either 10 or 100, but project B pays 100 and 10, when A pays 10. So; that means, it pays more and it pays 15 when A pays 100. So, in this case B pays much less. So, under 1 circumstance B will pay more in another circumstance B will pay less. So, accordingly in this case, it is not clear whether A or B should be the choice. So, this brings us to revisiting utility functions, in the context of consumption.

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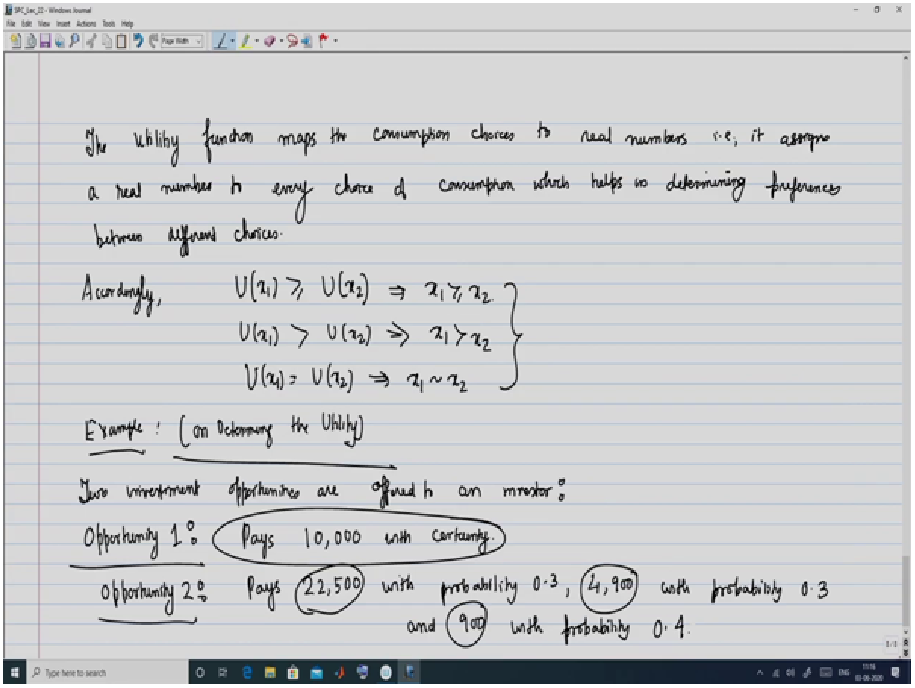

So, recall that the utility function maps the consumption choices to real numbers. Remember we had introduced the term called utility and that is it assigns a real number to every choice of consumption which helps in determining preferences between different choices. So, accordingly if $U(x_1)$, that means, the $U(x_1)$ is greater than or equal to that of x_2. This will imply that x_1 is preferred at least as much as x_2. If, $U(1)$ is strictly greater than $U(x_2)$, then it implies that x_1; obviously, is preferred over x_2. And, if $U(x_1)$ is equal to $U(x_2)$, this means that there is there investor is indifferent to the two choices x_1 and x_2, all right. So, when we talked about utility functions we had sort of assumed that you know we have a wide range of choices of utility functions, we talked about logged utility quadratic utility and so on and we had used this in several context. So, what are you going to do now is going to look at as a very simple example to identify how to figure out, what is going to be the utility function for an each individual investor. Now, this is actually a non-trivial exercise, but we just give an illustration of a very simple way of doing in a very particular case as mostly as an illustrative example. Of course, you know in reality it is going to be very difficult to extend such kind of a situations in general, when you are looking at a personal specific utility function. At which point you have to mostly rely on whether the person is risk averse risk neutral or risk loving and accordingly you can make a choice of utility functions. But, what you are going to do now is look at a very a specific example of how one can determine the utility functions based on very specific inputs from the investor in terms of their preferences. So, accordingly we look at this example on determining the utility. So, what happens here is two investment opportunities are offered to an investor. So, what is the first opportunity? In the first opportunity and the investment pays an amount of 10, 000 with certainty and in the second opportunity it pays. So, the second opportunity the first one is a deterministic quantity and it pays a certain amount. So, this is like a bond. And, the second opportunity is going to pay out random variables. So, it pays out a three possible values 22, 500 with probability 0.3, 4900 with probability 0.3 and an amount of 900 with probability 0.4.

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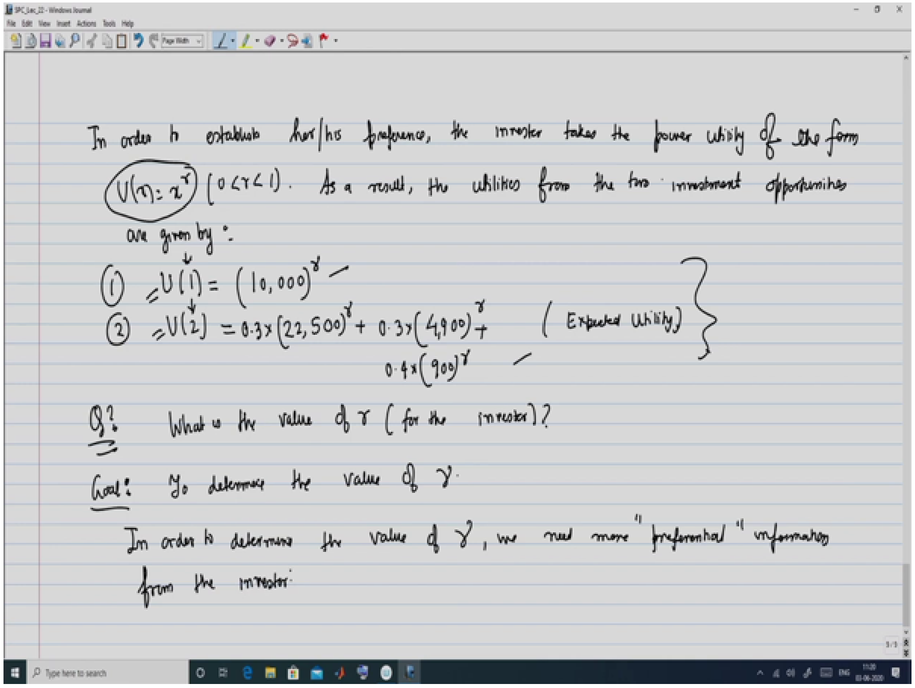
So, in order to establish her or his preference, the investor takes the power utility of the form $U(x) = x^{\gamma}$, a 0 less than gamma less than 1. As a result the utilities from the two investment opportunities are given by the following. So, the utility $U(x) = x^{\gamma}$. So, then we look at both the opportunities and what is going to the utility. So, in the first case for opportunity one I will denote the utility as $U(1)$ and this is going to be simply. So, here if one does not indicate the value, but this is going to indicate the opportunity 1, and this is going to be nothing, but the wealth level 10 000 raised to gamma. And, in the second case $U(2)$ that is



the utility for opportunity 2. This is going to be given by the expected value. So, in this case it is going to be the expected utility and this will be given by. So, we had the values 22,500 and the utility is this value raised to gamma, multiplied by the probability of 0.3 plus we had a 49 hundred. So, we had 4900 raised to gamma multiplied by the probability 0.3 plus we had 900 the utility of which is raised to gamma multiplied by the probability 0.4. So, these are going to be the utilities for the first opportunity and the utility for the second opportunity. So, this brings me to the question as to what is the value of gamma. So, unless I know the value of gamma for the investor I cannot make a comparison between $U(1)$ and $U(2)$ and hence cannot establish a preference criteria. And, the goal is so; accordingly the goal is to determine the value of gamma. Now, in order to determine this value of gamma, we need more preferential information from the investor.

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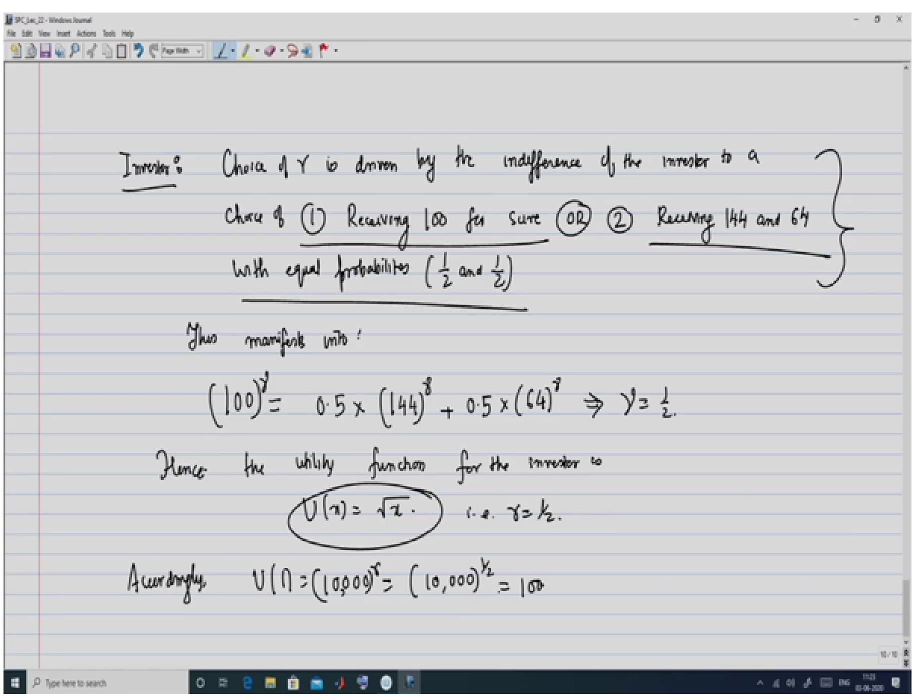

So, then the investor says that, that they are indifferent so; that means, in this case the choice of gamma is driven by the indifference of the investor to a choice of either receiving an amount of 100 for sure, or receiving 144 and 64 with equal probabilities, that is half and half. So, when I have a said that the in the investors indifference so; that means, their utility or expected utilities are going to be identical. So, this manifests into the following. This so, the utility for 100 is going to be 100 raised to gamma. And, the expected utility for the second choice is going to be

$$(144)^{\gamma} \times 0.5 + (64)^{\gamma} \times 0.5$$

and this gives you that $\gamma = \frac{1}{2}$. So, hence based on this preference indicated by the investor, we get the utility function for the investor is given by e of x is equal to square root of x all right. So, once you have got U of x equal to square root of x, that is gamma is equal to half you can put this back in gamma equal to half here and gamma is equal to half here. So, accordingly what we get is

$$U(1) = (10000)^{\gamma} = (10000)^{\frac{1}{2}} = 100.$$

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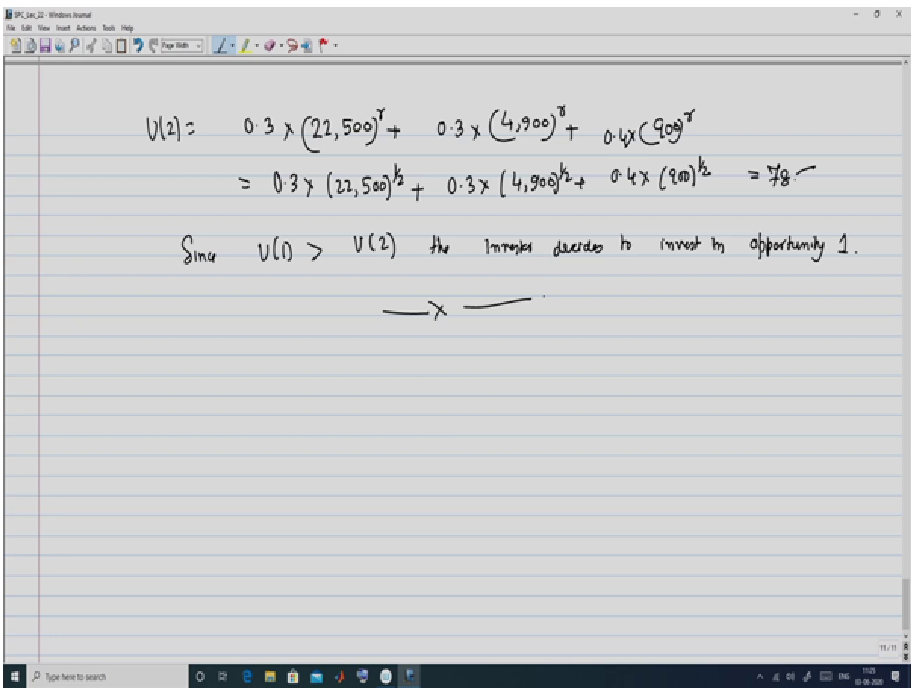

And,

$$U(2) = (22500)^{\gamma} \times 0.3 + (4900)^{\gamma} \times 0.3 + (900)^{\gamma} \times 0.4.$$

And this turns out to be so, what does it this raised to gamma. So, this is

$$(22500)^{\frac{1}{2}} \times 0.3 + (4900)^{\frac{1}{2}} \times 0.3 + (900)^{\frac{1}{2}} \times 0.4.$$

And, this turns out it to be equal to 78. So, since U 1, which is equal to 100 is greater than U 2 which is 78. The investor decides to invest in opportunity 1. So, just to do a recap of whatever we have done in todays class in this class, we started off with a new topic on optimal portfolio and consumption. And, we mainly emphasized our discussion today on the notion of consumption and how we can make a preference of a consumption choice over another or how rather we can decide on, how to make a preferential determination of which choice is better for us. And, then we connected this to the utility functions, and we looked at a couple of examples involving the consumption choice as well as how to determine the utility function. And, more importantly we looked at a discrete time model in a market, which comprises of a bond and capital N number of stock. And, we looked at a 2 important aspect of this; one is what is known as the self-financing condition. And, then the other is the wealth process as a result of this investment in a bond N in capital N number of stocks. So, the next class we will continue with this discussion of the discrete time framework. And, we will started looking at how to make an optimized portfolio as a result of this process. And, first we will talk about the single period model and then we will move on to a multi period model, in order to determine what is going to be my optimal portfolio and then what is going to be my optimal consumption. And, for that we will resolved to an approach called dynamic programming due to Richard bellman.

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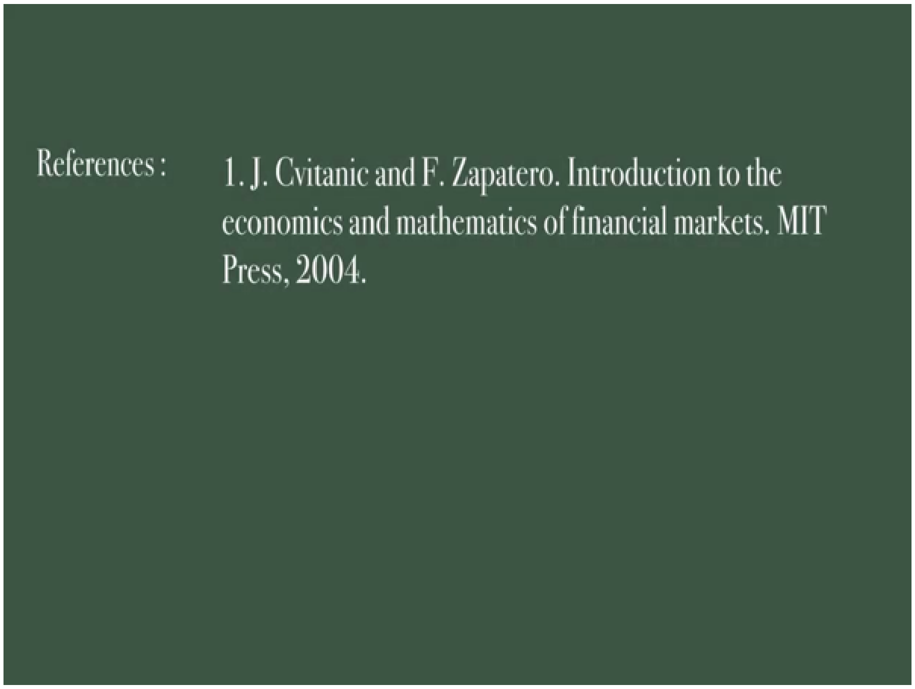

So, thank you for watching.