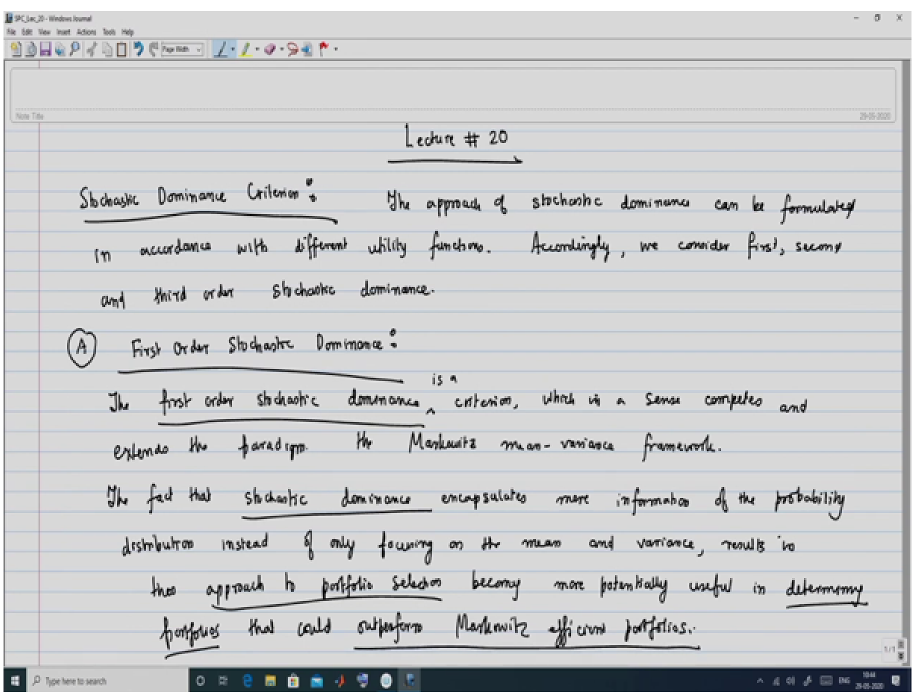
Mathematical Portfolio Theory

Prof. Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

Module 04: Non-Mean-Variance Portfolio Theory
Lecture 08: Stochastic dominance; First order stochastic dominance

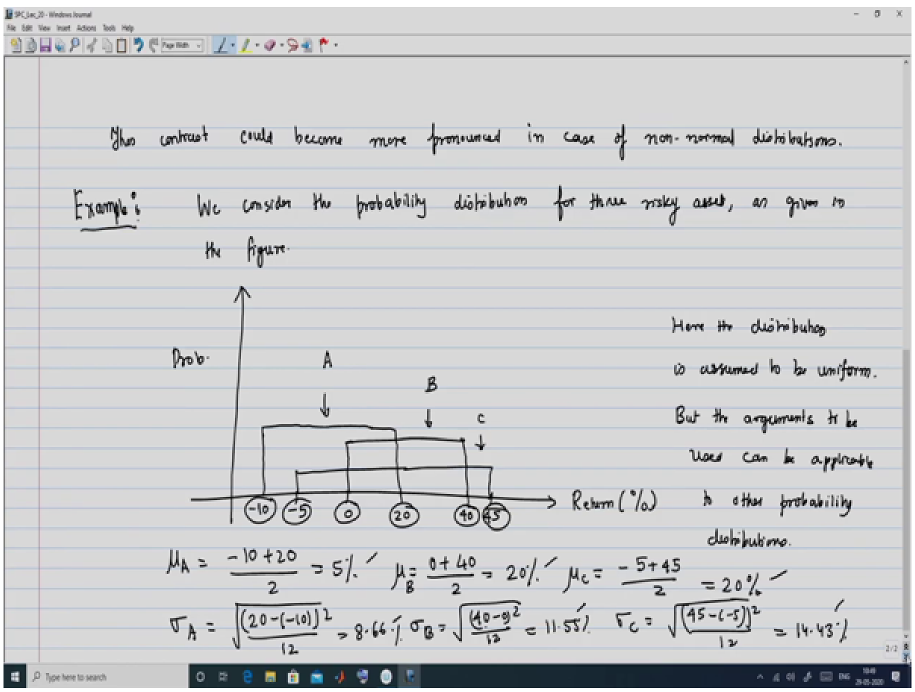
Hello, viewers. Welcome to this next lecture on the NPTEL-MOOC course on Mathematical Portfolio Theory. You would recall that we had continued our discussion on the Non-Mean-Variance Analysis in the last class, where you talked about semi-variance and semi-deviation. And, you looked at the portfolio analysis while using semi-variance or semi deviation as the measure of risk and accordingly we also extended this in case of the capital market line and the security market line again replacing the standard deviation with the semi deviation as the measure of risk. So, now in todays lecture and the next lecture what we will do is that we will look at one another important concept in the non-mean-variance framework and that is what is known as the stochastic dominance, and we will look at the three forms of stochastic dominance. And, todays lecture we will discuss in detail on the first order stochastic dominance.

(Refer Slide Time: 01:27)



So, according to we start this lecture on the topic of stochastic dominance criteria. So, this approach of stochastic dominance can be formulated in accordance with different utility functions and you will explain this in details in the subsequent discussion. And, accordingly, what we will do is that we consider first, second and third order stochastic dominance. So, let me start with todays discussion that is on a first order stochastic dominance. So, the first order stochastic dominance criteria. So, we observe that which in a sense competes and extends the paradigm of the Markowitz mean-variance framework. So, we have extensively looked at the Markowitz mean-variance framework and now, we are going to introduce the first order stochastic dominance. And, this should be viewed more as some sort of extension of the mean-variance framework while at the same time competing with the notion of the mean-variance framework. So, the fact that stochastic dominance encapsulates or captures more information of the probability distribution instead of only focusing on the mean and variance. So, instead of just being restricted to mean and variance stochastic dominance also focuses on capturing more information about the probability distribution, alright. So, this fact results in this approach to portfolio selection namely, stochastic dominance becoming more potentially useful in determining portfolios that could outperform Markowitz efficient portfolios alright. So, this means that if you take this approach of portfolio selection it potentially could end up determining portfolios that performs better than the portfolios on the efficient frontier that we saw using the Markowitz mean-variance framework.

(Refer Slide Time: 06:48)

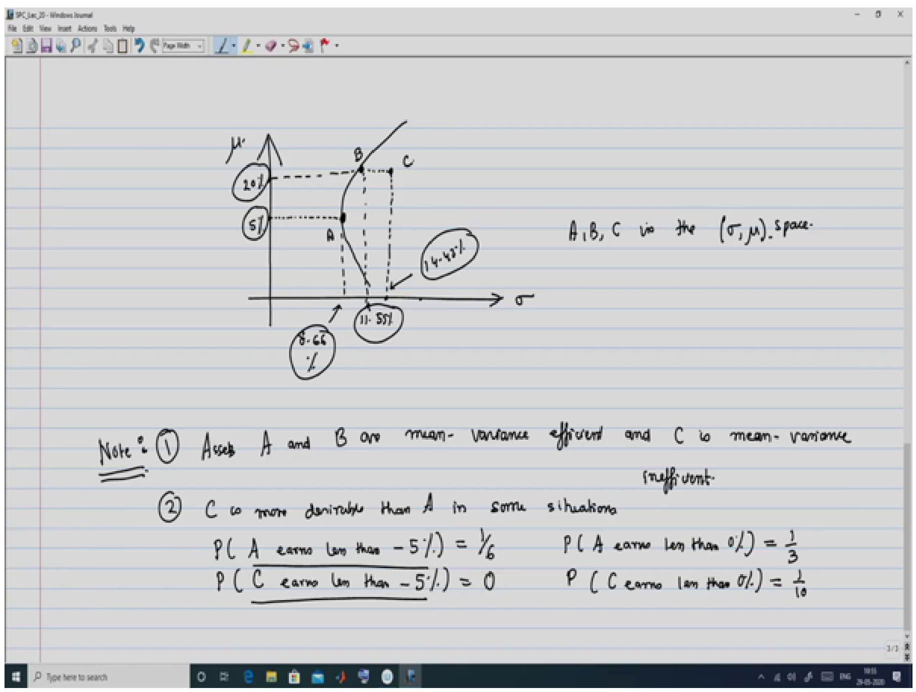


Now, this contrast that means, this difference between the stochastic dominance portfolio and Markowitz mean-variance portfolio could become more visible or pronounced in case of non-normal distributions, alright. So, to motivate this what we do is that now we consider an example. So, we take the probability distribution for three risky assets, as given in the figure. So, let us look at how the probability distribution is going to look like. So, it is going to look something like this. On the x-axis we have returned as percentage and on the y-axis we have the respective probability. And, then we consider so, let us consider the returns to be minus 10, minus 5 percent, 0 percent, 20 percent, 40 percent and 45 percent. Now, we take this portfolio A from minus 10 to 20 percent we take portfolio B. So, this is going to be my portfolio A; then, portfolio B is something that is going to be from 0 to 40 percent. So, this is portfolio B and portfolio C is something from that will go from minus 5 to 45 percent. So, here the distribution is assumed to be uniform and we do this for simplicity of understanding, but the argument to be used subsequently can be applicable to other kinds of probability distributions ok. So, now we have this uniform distribution. So, now, you observe that in case of A portfolio A it is a uniform distribution lying in the range minus 10 to 20; in case of portfolio B this is a uniform distribution in the range 0 to 40, and in case of C it is going from minus 5 to 45. So, we remember that for a uniform distribution the mean is going to be A plus B over 2. So, accordingly the return on portfolio A is simply going to be

$$\mu_A = \frac{-10+20}{2} = 5\%, \quad \mu_B = \frac{0+40}{2} = 20\%, \quad \mu_C = \frac{-5+45}{2} = 20\%.$$

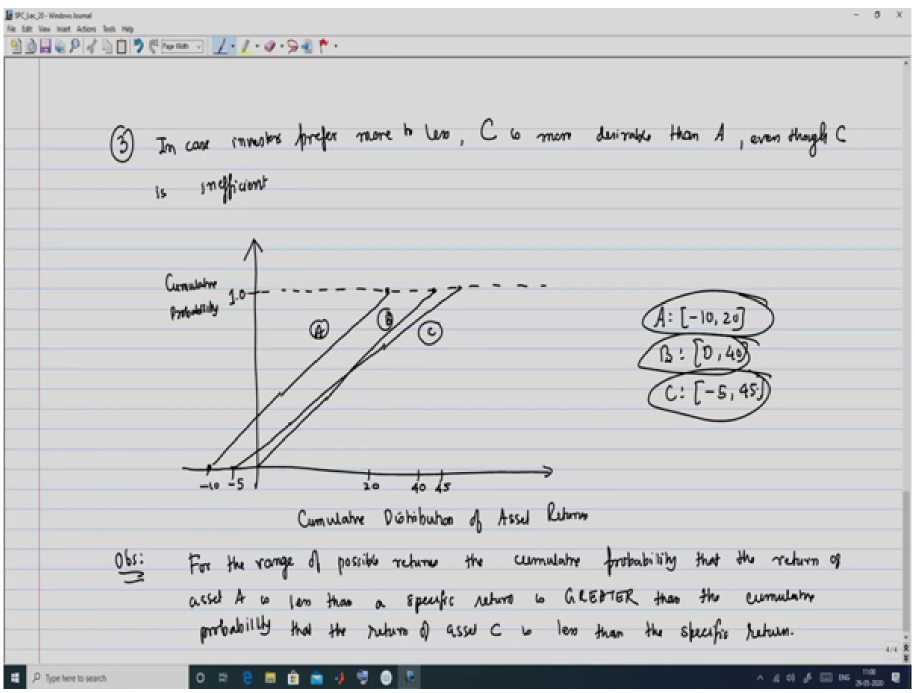
And, in a similar way as sigma a remember sigma A is going to be square root of B minus A whole square by 12. So, this is going to be 20 minus of minus 10 square over 12. So, this is going to be equal to 8.66. Sigma B is going to be square root of 40 minus 0 square upon 12. So, this is going to be equal to 11.55 percent and sigma C is going to be square root of 45 minus of minus 5 square over 12 and this turns out to be equal to 14.43 percent. So, mu A is 5 percent, mu B is 20 percent, mu C is 20 percent sigma is 8.66 percent sigma B is 11.55 percent and sigma C is 14.43 percent.

(Refer Slide Time: 11:49)



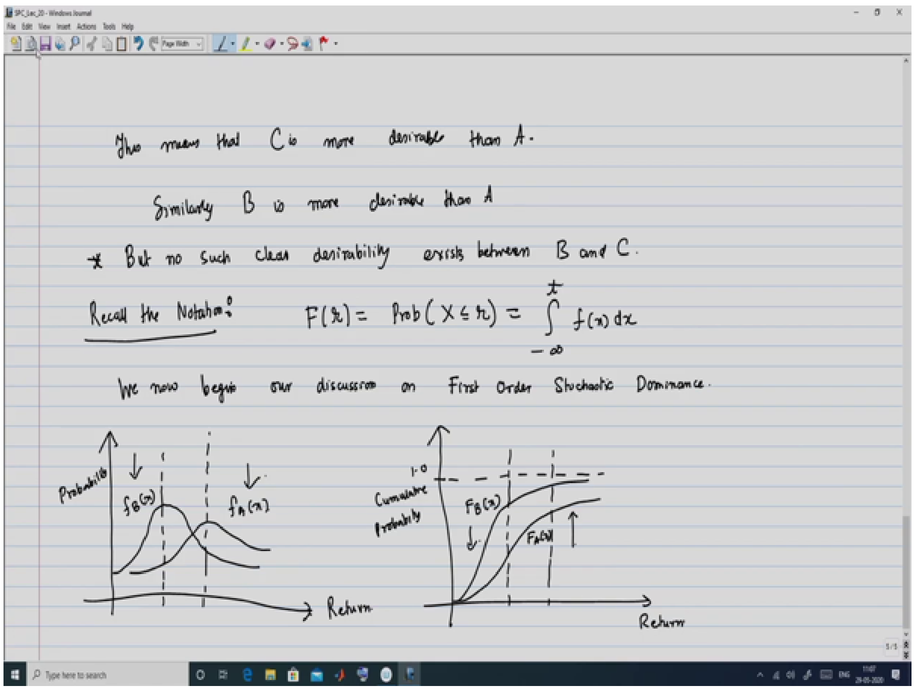
So, what you do now? We will look at this in the mu sigma plane. So, let us consider mu sigma plane. So, in the mu sigma plane and this is going to be the efficient frontier and say this is your 5 percent and this is your 20 percent. So, this portfolio here and on the x-axis sigma is I take this to be 8.66 percent. So, this pair of 5 percent and 8.66 percent this is going to be portfolio A and then for 11.55 percent and so, I could have. So, let me put this here ah. This is 8.66 percent. Let this point here be 11.55 percent and let this point here be 14.43 percent. So, I take this 11.55 percent here and the corresponding mu would be 20 percent. So, mu 20 percent and sigma 11.55 percent this is going to be portfolio B and the portfolio C is again going to be 20 percent value for mu with 14.43 percent for sigma. So, this is going to be by portfolio C. So, A is identified as the one with mu 5 percent and sigma 8.66 percent; B is the one with mu 20 percent and sigma 11.55 percent and C is the one again with mu 20 percent and sigma 14.43 percent. So, this is basically a portfolios A, B, C in the sigma mu space, alright. So, now, what we are going to do is we are going to make a few observations out of this. So, here you observe that A and B lie on the efficient frontier. So, this means that these assets or portfolios A and B these are mean-variance efficient and C since it does not lie on the efficient frontier so, this is going to be mean-variance inefficient. So, this means that in the Markowitz framework only A and B are going to be desirable and a C is not going to be desirable because C is a portfolio that does not lie on the efficient frontier, alright. So, now, this is one observation that we have made from the point of view of the Markowitz mean-variance framework. The next observation that we make is that that C however, is more desirable than A in some situations. So, even though C is inefficient and A is efficient, even then C is going to be desirable more desirable than A in certain situations. So, for example, to illustrate this situation; so, what is going to be the probability that A earns less than minus 5 percent? This probability is going to be 1 over 6, but what is the probability that C earns less than minus 5 percent? So, that is going to be 0. So, you see the probability that C falls below 5 minus 5 percent is actually this probability is less than the probability that a falls below minus 5 person. So, that means that C actually is performing better than A in the scenario that the returns are following below minus 5 percent. Now, another scenario could be that probability that the portfolio A earns less than 0 percent. So, in this case this is going to be equal to one third, but the same probability that C earns less than 0 percent this is going to be equal to 1 over 10. So, again you know the probability that A earns the negative return one third and this is equal to one third and that is actually higher than the probability of C earning a negative return which in that case is going to be one tenth. So, in terms of you know these are two specific examples where you see they are under certain circumstances if you set the criteria whether your portfolio returned can fall below minus 5 percent or whether it can fall below 0 percent in both the scenarios, we see that C even though it is not an efficient portfolio, but A is even then C can actually be more desirable than A in that particular context, alright.

(Refer Slide Time: 17:20)



So, one more observation that I want to make and that is so, the 3 rd observation that I want to make here is that in case investors prefer more to less C is more desirable than A, even though C is in efficient alright. So, now, let us look at what is going to be the cumulative distribution of asset returns for each case. So, remember that A is uniformly distributed in the range minus 10 to 20; B is uniformly distributed in the range 0 to 40, and C is uniformly distributed in the range minus 5 to 45 alright. So, let us identify this point. So, I have a minus 10 here, then I have minus 5, then I have 20, we have a 40 and we have a 45. And, here on the y-axis we have a cumulative probability, the maximum can be 1. So, for portfolio A, the it is basically the cumulative probability will be the straight line from minus 10 all the way to 20. Now, in case of B it goes from 0 to 40. So, the 40 is going to be here. So, from 0 to 40 the graph is going to look like this and C is from minus 5 to 45, so 45 is here. So, I start off with minus 5 and then I go all the way to 45. So, this is A, this is going to be B and this is going to be C. So, this is the cumulative distribution, alright. So, now, we can make and some observation. So, you see that in the observation that for the range of possible returns for each case, the cumulative probability that the return of asset A is less than A specific return is greater than the cumulative probability that the return of asset C is less than the same specific return. So, what it means is that if you look at the entire range of values then the cumulative probability; that means, the value on the y-axis and you know that cumulative probability basically gives you the probability of the random variable taking a value less than or equal to certain value. So, we take that to be the threshold value. So, if we fixed a certain threshold value, then the probability or the cumulative probability of the returns being below that certain threshold level in case of A. And if we look at the cumulative probability of C of certain that same threshold, and you compare this two cumulative probabilities of the returns being below that certain threshold you will observe that this cumulative probability in case of the asset A is going to be greater than that of the asset C because you observe here that the graph A or the cumulative probability distribution always lies above C.

(Refer Slide Time: 22:18)

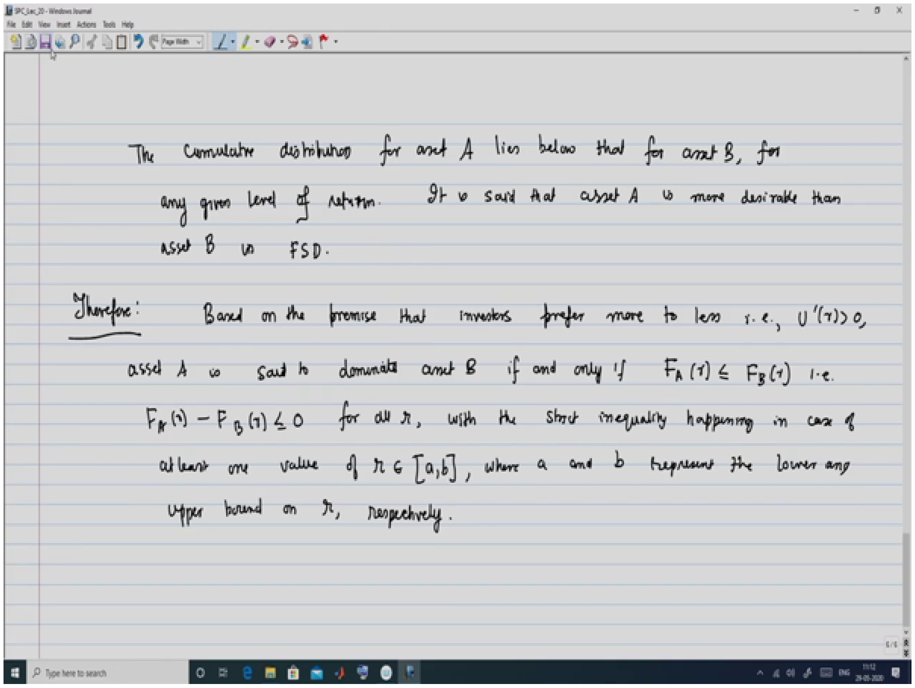


So, accordingly we can say that this means that C is more desirable than A. So, because the probability of falling below a certain threshold is consistently higher than in case of A as compared to C so, obviously, for that certain threshold the loss likelihood as given by the probability or the equivalently the cumulative probability is going to be less in case of C and that is why from a loss avoiding point of view C is going to be more desirable than your asset A ok. However, a if you go back and observe carefully so, now, you see that in case of A this is very clearly distinguishable in case of A and C and in a similar manner if you compare A and B. So, similarly B is more desirable than A. But since there is a intersection of B and C so, no such clear desirability exists between B and C ok. So, now, it is time to just recall the notation for cumulative distribution that you had obtained you know at the beginning of the course. So, their notation F of r and I am using r to indicate that we are looking at the return random variable. This probability

$$F(r) = Prob(X \leq r) = \int_{-\infty}^{t} f(x)dx,$$

where f of x is the probability density function of x. And, accordingly we now begin our discussion on first order stochastic dominance so, FSD with an illustrative example, alright ah. So, now, what we do is that we are going to look at the graphs of the probability density function. So, so we consider first the graph of the probability density function and on the x-axis we have the return and the y-axis we have the probability. So, here we have the probability density function of B, and then we have the probability density function of A. And, the corresponding cumulative probabilities so, for that again we have the return on the x-axis and cumulative probability on the y-axis. So, on the top we have ah the cumulative distribution of B and below that we have the cumulative distribution of A, alright. So, now, what you can do is we can make an observation here that. So, you observe that here we have the distribution f B and f A where f A is slightly to the right of f B and consequently what you have is that the cumulative distribution of F B is consistently going to be at a higher level than the cumulative distribution of F A. So, this graphical representation of the specific case is something which will motivate our discussion on stochastic dominance.

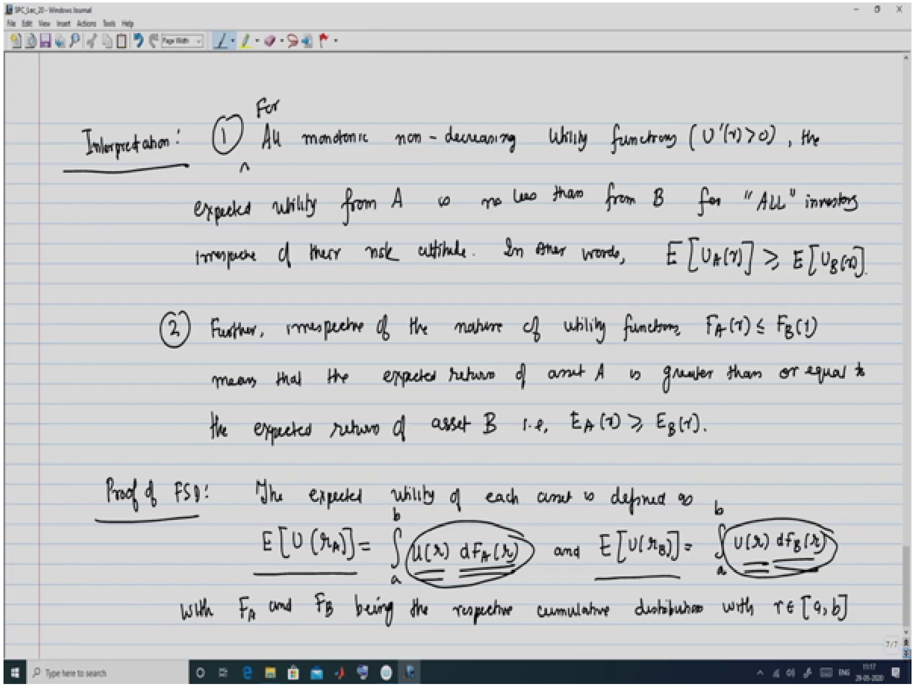
(Refer Slide Time: 26:51)



So, we now make the following observation that the cumulative distribution for asset A as you have already seen in the graph, this lies below that for asset B for any given level of return. So, it accordingly it is said that asset a is more desirable than asset B in FSD that is First order Stochastic Dominance. So, therefore, based on the premise or the assumption that investors prefer more wealth to less or more returns to then less that is mathematically it is u prime of r greater than 0 asset A is said to dominate asset B if and only if. So, it is a two way condition you have F A of r being less than or equal to F B of r as seen in the graph. So, that is F A of r minus F B of r being less than or equal to 0 for all r and with the strict inequality that is F A r being strictly less than F B of r happening in case of at least one value. So, at one of at least one point of r in the interval a, b the strict inequality must hold and where your a and b represent the lower and upper bound on the return r the random variable r which is the return respectively.

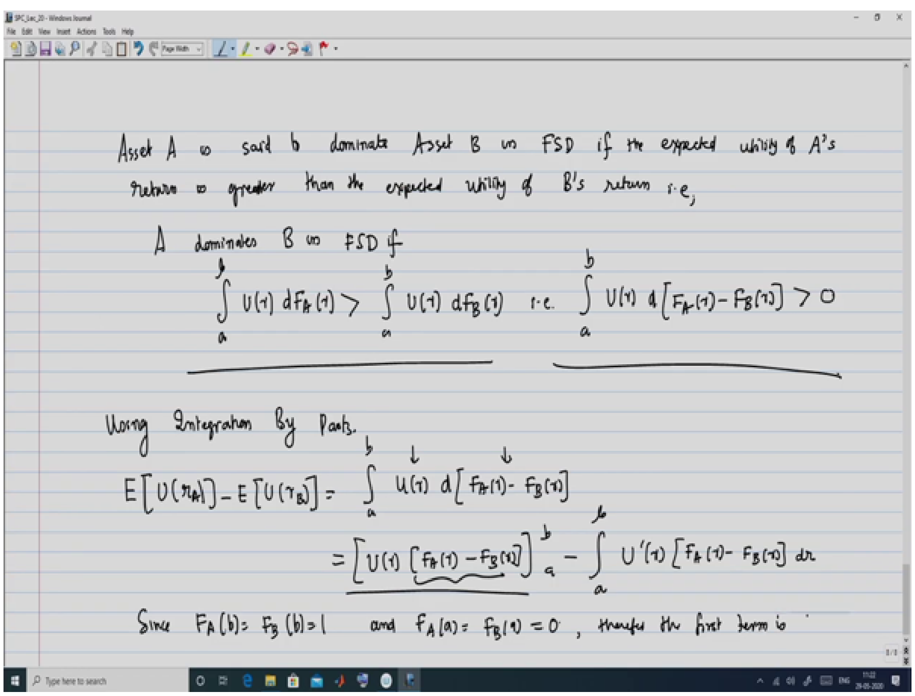
(Refer Slide Time: 29:51)

So, let us now give an interpretation of what we have discussed so far. So, the 1 st interpretation is the following that all monotonic, non-increasing non-decreasing rather utility. So, I should write for all monotonic non-decreasing utility functions that means, where we have U prime of r greater than 0, the expected utility from A is no less than from B for all investors irrespective of their risk attitude. So, mathematically this means that the expected utility of all investors in case of the first asset or the return of the first asset. So, expected utility A of r is going to be greater than or equal to expected utility in case of B. And, a 2nd interpretation that we have here is that irrespective of the nature of utility functions being used F A of r less than or equal to F B of r means that the expected return of asset A is greater than or equal to the expected return of asset B that is E A of r is greater than or equal to E B of r. So, we now give the proof of FSD or the First order Stochastic Dominance. So, accordingly what we do is that we first identify the expected utility of each asset. So, the definition r as follows the expected utility of return on the first asset this by definition of expectation is going to be u of r into d F of A r in the interval a b. So, that is the range of your returns. And, E of U of r B in a similar manner this is going to be integral of U of r d F B of r from a to b with F A and F B being the respective cumulative distribution with as I have said the range of returns



is a, b. So, you have the cumulative distribution d F of A and d F of B and then you have the utility U of r. So, the expected utility in the first case will involve this integral with respect to the cumulative distribution F A and in the second case it is going to be with respect to the cumulative distribution F B. So, we now have settled down in what the expected utilities are in case of A and B so, two generic portfolios A and B.

(Refer Slide Time: 34:38)

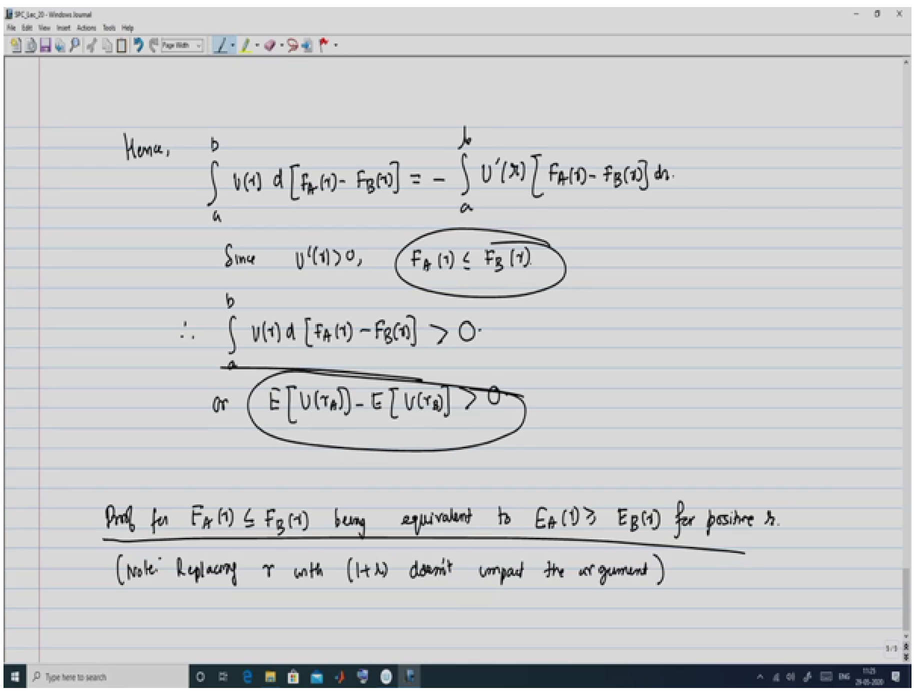


So, now we come to what is known as the dominance ah. So, accordingly asset A is said to dominate asset B and remember that this is in the context of FSD and this domination is defined as if the expected utility of the return of asset A is greater than the expected utility of return from B. That is, in other words, we say that A dominates B in the sense of FSD if integral a to b of U of r, so, I am just using the definition of expected utility done previously d F A of r this is greater than integral a to b U of r d F B of r, that is, we can write this as integral a to b U of r into d of F A r minus F B r and this is greater than 0. So, we can use either this form or this equivalent form, alright. So, now, using integration by parts, what we will have? So, we will have

$$E[U(r_A)] - E[U(r_B)] = \int_a^b u(r)d[F_A(r) - F_B(r)].$$

If we if you use the integration by parts so, we see that so, we will get this as the first function I will take this as the second function. So, it is first function U of r into integral of second which is F A r minus F B r in the range a to b minus integral of derivative of the first function that is U prime of r into integral of the second function that is F A of r minus F B of r into d r and this integral is from a to b. So, this is now simply going to be equal to. So, now, here we just make the observation regarding this term. So, now, since of b is equal to F B of b is equal to 1. So, remember that b is the largest value of the return. So, this it is the cumulative value is going to be equal to 1 and accordingly similarly F A of a is equal to F B of a this is going to be equal to 0. So, this means that since this is this holds therefore, the first term is 0.

(Refer Slide Time: 38:10)



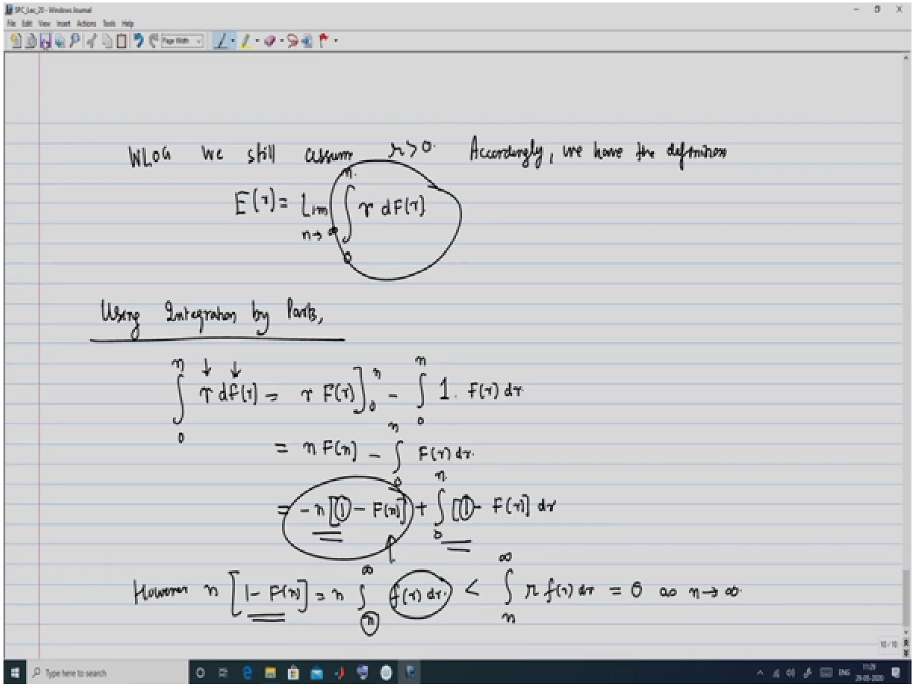
Hence we can conclude that

$$\int_a^b U(r)d[F_A(r) - F_B(r)] = -\int_a^b U'(r)[F_A(r) - F_B(r)]dr.$$

Now, since U prime of r is greater than 0 and F A of r is less than or equal to F B of r and the inequality strict inequality holds in one case. Therefore, this integral a to b U of r d of F A r minus F B r this is going to be greater than 0. And, since this integral you will remember this is nothing, but expected value of U of r A minus expected value of U of r B this is greater than 0. So, that means, that it is because of this condition we have been able to arrive at this condition in terms of the expectation to show the first order stochastic dominance ah. So, what I want to do is now I want to do another proof and this is the proof for F A of r less than or equal to F B of r being equivalent to E A of r greater than or equal to E B of r for positive r and we just make a note before we start the proof that replacing r with 1 plus r does not impact the argument. So, this is just to accommodate the case that ah mathematically one r can be actually between 0 to minus 1 also.

(Refer Slide Time: 40:41)

So, in order to prove this so, we without loss of generality in order to prove this we still assume r is greater than 0. So, accordingly we have the definition of E of r and what is this going to be? It is going to be integral of r d F of r as per the definition of expectation 0 to infinity remember that your r is positive and



instead of 0 to infinity we take that this integral from 0 to n with the limit that n tends to infinity. So, again using integration by parts what do we get? So, look at this integration. So, when you are doing integration by parts so, we will look at integral 0 to n of r dF of r. So, I take r to be the first function and F dF as the second function. So, it is first function r into second integral of the second function F of r from 0 to n, minus integral of 0 to n of derivative of first function which is 1 into integral of second that is F of r dr. So, here this is going to be n into F of n; of course, the second term is going to be 0 minus integral 0 to n F of r dr. Now, what you can do is that we can add and subtract one term. So, that is minus n. So, I will have minus n minus 1 into 1 minus F of n and accordingly, here I will have a plus 1 minus F of r dr integral 0 to n. So, you see here we from this term I will have minus n and this integral dr of from 0 to (Refer Time: 42:54) and they cancel out. So, I add a minus n here and plus n here. Now, that we have this form so, let us now focus on this first term. So, however, n into 1 minus F of n, what is this going to be? So, this is going to be equal to n into integral n to infinity of f of r dr ok. So, this is 1 over F of n. So, it is 1 minus cumulative distribution from 0 to n. So, this means that this is 1 minus F of n is nothing, but what happens from n to infinity. So, accordingly this is going to be an integral from n to infinity of f of r dr. Now, we remember that here what we have done is that so, here we take the r as in this case you know since the integral is from n to infinity. So, obviously, your r is going to be greater than n; that means, n is less than r. So, then I can write this as integral since n is less than r. So, I can write this as integral n to infinity r f of r dr and this is going to be equal to 0 as your n tends to infinity. So, that means, that we have been able to get rid of this first term here and the only thing that remains is the second term.

(Refer Slide Time: 44:21)

So, therefore, what I can say is that so, looking at the above relation therefore,

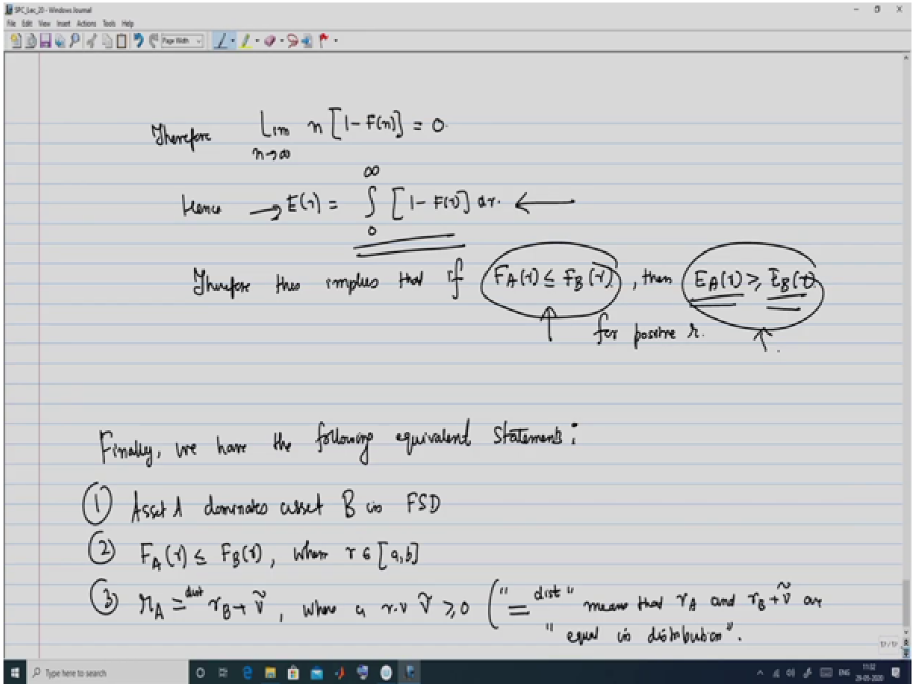
$$\lim_{n\to\infty} n[1 - F(n)] = 0.$$

And, hence the integration this integration which is the expectation of r this simply becomes this expression as

$$E(r) = \int_0^\infty [1 - F(r)]dr.$$

Therefore, this implies that if

$$F_A(r) \leq F_B(r),$$



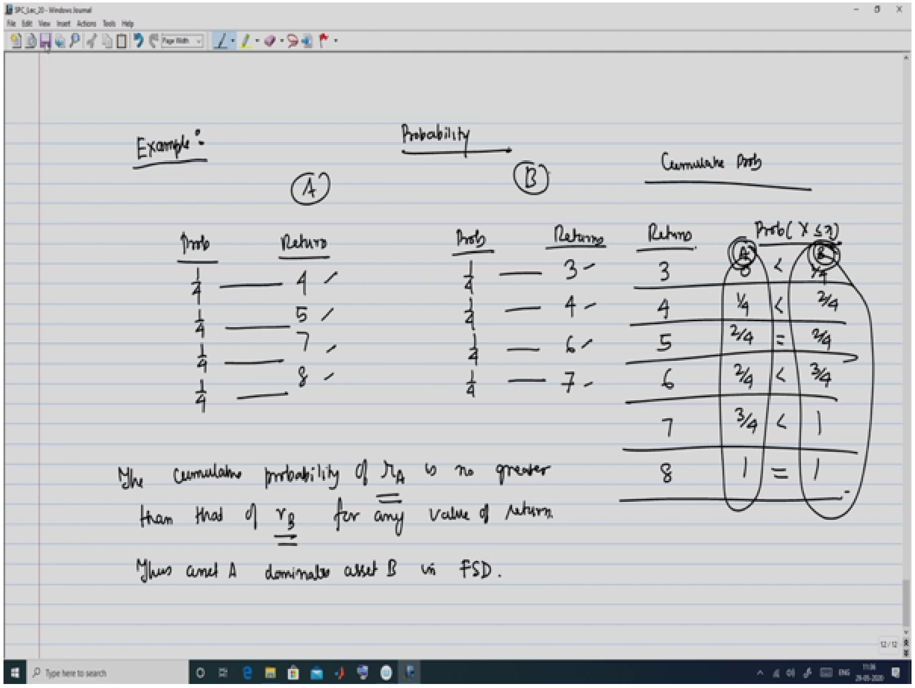
then we can use this relation to show that

$$E_A(r) \geq E_B(r),$$

for positive r. So, for this all you have to do is you essentially take this definition of E A of r and E B of r here and you subtract them and that will become the integral involving these two and using this property you can show that this relation holds. So, in finality, we have the following equivalent statements; one is that asset A dominates asset B in FSD. This is equivalent to the basic concept that F A of r is less than or equal to F B of r where r belongs to some range a, b. And, this is equivalent to that the return of asset A is equal in distribution to return of asset B plus v tilde, where a random variable v tilde is greater than or equal to 0. And, here I have introduced a new notation equal superscript dist. So, this notation means that r A and r B plus v tilde are what is known as equal in distribution.

(Refer Slide Time: 47:19)

So, let me conclude by one example to illustrate FSD. So, let us consider the probabilities of two assets A and B. So, let the probability and the returns of asset A. So, the returns could be 4, 5, 7, 8 each with identical probability of $\frac{1}{4}$ and in case of asset B the returns could be 3, 4, 6, 7; again, each with identical probability of $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$ and $\frac{1}{4}$. So, from here what you get is we will get the cumulative probabilities. So, for this we have to look at what are the possible values of return. So, the possible values of returns are 3, then we have 4, then 5, then we have 6, then we have 7 and then 8. And, what is going to be the cumulative probability of X less than or equal to x. So, in case of A what is this going to be and in case of B what is this going to be. So, in case of A these values are going to be a 0, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ and 1. And, in case of B this is going to be $\frac{1}{4}$, $\frac{2}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, 1 and 1. So, now we have the cumulative probabilities for both A and B. So, now, you observe that the cumulative probability of r A alright, so that means, this values is no greater than that of r B that is given in this column for any value of return. So, observe carefully that here 0 is less than $\frac{1}{4}$, $\frac{1}{4}$ is less than $\frac{2}{4}$. So, this is less than, this is so, this is strictly less than, strictly less than equal to strictly less than, strictly less than equal to. So, you see that in each case the cumulative probabilities either strictly for A is either strictly less than that of B or is equal to that of B. So, that means, that at under no circumstance, the cumulative probability of A can exceed that of the cumulative probability of B. So, that means, that if you draw the graph of capital $F_A(r)$ and $F_B(r)$, then $F_B(r)$ is always going to be either coinciding with $F_B(r)$ or going to be below $F_B(r)$. So, in accordance with the first order stochastic dominance definition that we have already done we can conclude that thus asset A dominates asset B in FSD, alright. So, this



brings us to the conclusion of this lecture. In this lecture what we did is that we started off extending the concept of the mean-variance framework and what we did is that we started looking at what is going to be the definition of the stochastic dominance. And, we observed that the stochastic dominance can be of first, second or third order and in todays class we just discussed the first order stochastic dominance. And, we explained this from the point of view of that that a portfolio A dominates a portfolio B, if the cumulative distribution of F_A for asset A will be less than or equal to the cumulative distribution for the returns of asset B, $F_B(r)$. And, we what we did is that we looked at two proofs in terms of the expected values and in terms of the utility. So, in the next class, we will continue our discussion and complete the second order stochastic dominance and the third order stochastic dominance.

Thank you for watching.