Mathematical Portfolio Theory

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Module 04: Non-Mean-Variance Portfolio Theory
Lecture 07: Semi-variance framework

Hello viewers! Welcome to this lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. So, far we have spent several classes discussing utility functions and in the context of portfolio theory, we looked at how we are looking at the maximization of the expected utility. And, then we talked about the non-mean variance analysis and we have looked at some cases, or approaches for non-mean variance analysis. And, the last couple of classes we were focused on talking about the safety first criteria and we have identified and discuss three different safety first criteria's. So, in this class we will now move on to an important step of transiting from the mean variance framework to a non-mean variance framework. And, we will talk about a new risk measure that is what is known as the semi variance.

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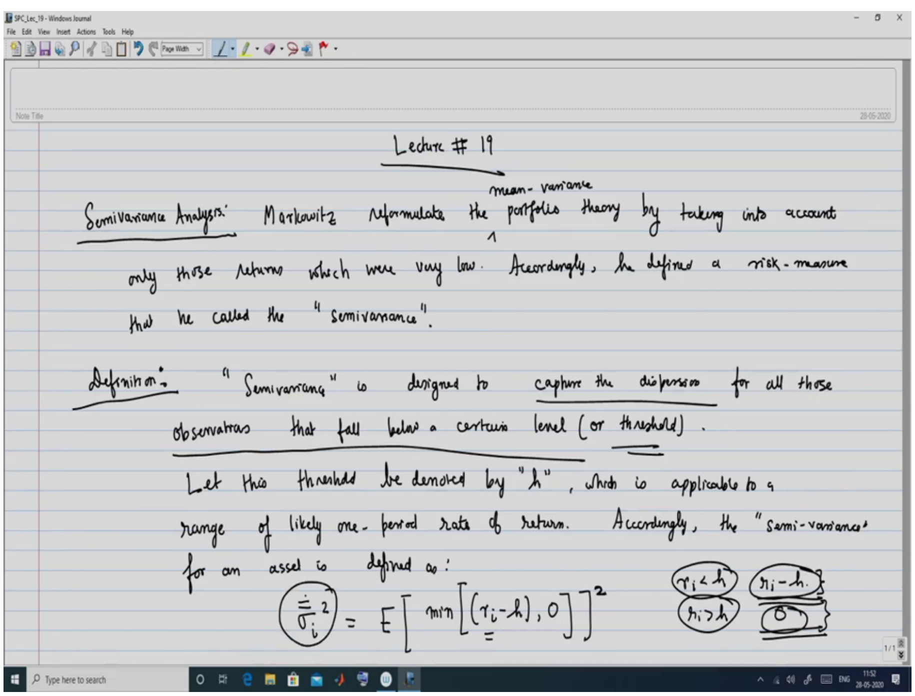

So, according to we start this lecture with semi variance analysis alright. So, we start off with a prelude to it by observing that Markowitz, who actually proposed the mean variance analysis. Decided to reformulate the portfolio theory by so, the mean variance portfolio theory by taking into account only those returns, which were very low. And, accordingly he defined a risk measure that he called the semi variance. So, start off with the definition of semi variance. So, the "semi variance" is designed to capture the dispersion for all those observations that fall below a certain level or threshold. So, semi variance by this statement we have said that it tries to capture the dispersion. So, this designed to capture those dispersion for observations, which follow fall below a certain level. So, earlier when we were talking about the mean variance framework, what I looked at is that I looked at the dispersion about the mean on either side. So; that means, those dispersion which are a higher valued than the mean and those which are below the mean or the expected return. And, accordingly we had defined the risk measured to be the variance, which was an indicator of dispersion of all the values. But, in case of semi variance we take a more selective approach and we you do not take those returns, which are above this predefined threshold, but rather which are below a certain threshold. So, this is sort of the motivated by the fact that you are more interested, in penalizing those return instances which follow below a certain minimum level. And, not penalize those which are actually higher than at that certain level, as was the case in case of the risk measure being the variance. So, accordingly so, this threshold we now that you have talked about this threshold. So, let this threshold be denoted by letter “h”, which is applicable to arrange or collection or random variables of likely 1 period rate of return. So, accordingly the semi variance for an asset is defined as follows. So, what you do is that in the semi variance? What we will do is that we will take r i minus h, that is the return on the asset i. And, we take the minimum of $\{r_i - h, 0\}$ and then we take the square of this and we calculate it is expectation. And, we define this as the semi variance. So, the variance is σ_i^2. So, semi variance will be σ_i^2 with 2 bars at the top. So, if you observe carefully, what we have here is that you see here the motivation is if your r i is less than h; that means, it falls below a threshold. So, we take that dispersion $r_i - h$. And, if your $r_i > h$ you know then we do not take the dispersion. So, as accordingly we only take into account when it follows the dispersion only if it falls below h, and ignore the scenarios when the return is above h. And, then we take the so, this is written as the minimum of this square. So, we take the square of the dispersion of this and then we take the minimum of $\{r_i - h, 0\}$. So; that means, that if r i is less than h you captured the dispersion $r_i - h$, but when $r_i > h$, the dispersion is taken to be 0. And, you take the square of this just as you had done in case of the variance, and you calculate the expectation of this because it involves a random variable r_i. And, i define this expectation to be equal to the semi variance σ_i^2, alright.

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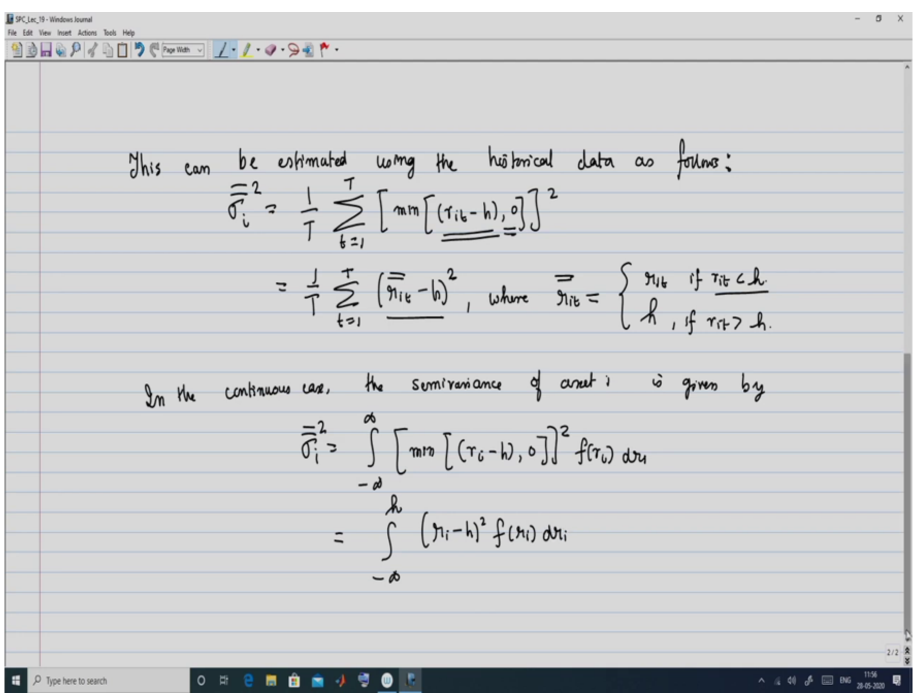

So, how do you estimate it? So, using historical data as we had done in case of mean variance and covariance earlier. So, we will give the estimate using the historical data as follows and this is sigma. So, σ_i^2 that is the semi variance, this based on historical data will be given by the expectation.

$$\bar{\bar{\sigma}}_i^2 = \frac{1}{T}\sum_{t=1}^{T}\left[\min\{r_{it} - h, 0\}\right]^2$$

$$= \frac{1}{T}\sum_{t=1}^{T}\left[\bar{\bar{r}}_{it} - h\right]^2,$$

where

$$\bar{\bar{r}}_{it} = \begin{cases} r_{it} & if\ r_{it} < h \\ h & if\ r_{it} > h. \end{cases}$$

So, now, in the continuous case, the analogous definition for the semi variance of asset i is given by.

$$\bar{\bar{\sigma}}_i^2 = \int_{-\infty}^{\infty}\left[\min[r_i - h, 0]\right]^2 f(r_i)dr_i$$

$$= \int_{-\infty}^{h}(r_i - h)^2 f(r_i)dr_i$$

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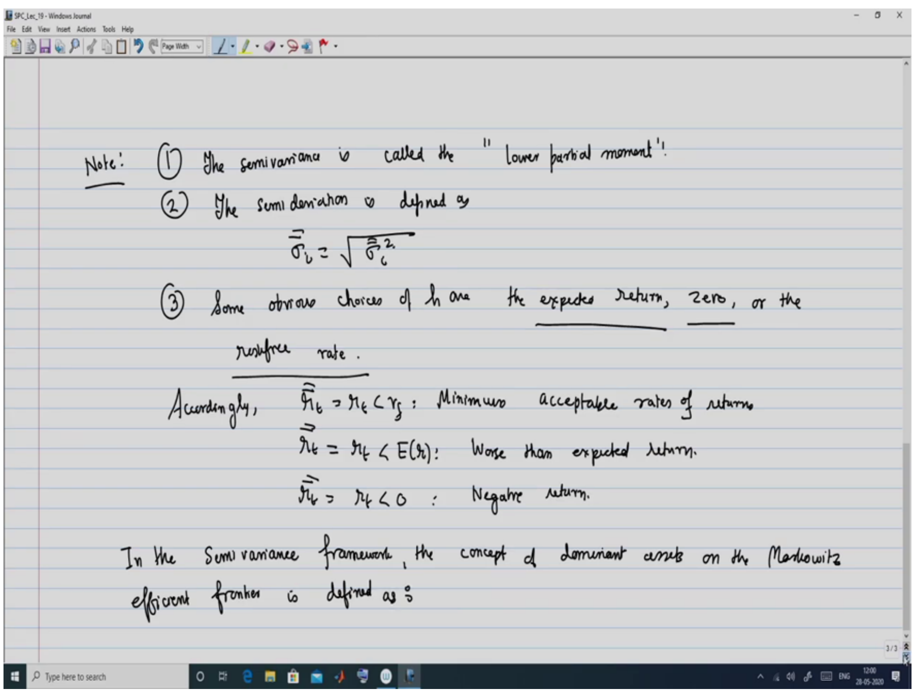

So, now, I make a few observation; the first observation that I want to make is that the semi variance is called the“lower partial moment”. Secondly, the semi variance leads us to the definition of semi deviation and this is defined as sigma i double bar. So, which; obviously, is square root of sigma i double bar square. And, thirdly some obvious choices of the threshold h are the expected return so, that is one obvious choice. In case you are just worried about no loss that this could be zero, or it could simply be the risk free rate. So, these are just some common choices, but not necessarily the complete exhaustive list. So, there could be other ways of choosing your rates. So, accordingly based on these three choices; $\bar{\bar{r}}_t = r_t < r_f$. So, this is going to be a minimum acceptable rates of returns. Then, $\bar{\bar{r}}_t = r_t < E(r)$, that is worse than expected return and $\bar{\bar{r}}_t = r_t < 0$, so, this is negative return. So, this corresponds to the falling below risk free rate, falling below the expected return and the falling below 0 or earning negative return respectively for the three cases. So, once we have defined what is a semi variance? And, our equivalently what is the semi deviation? We now need to start looking at the concepts from the Markowitz mean variance framework, in the context of the semi variance or equivalently the semi deviation. And, so, accordingly what you start is that will start looking at the most important concept that came from the Markowitz framework namely the efficient frontier and what are going to be dominant portfolios. So, accordingly I can make the observation that in the semi variance framework, the concept of dominant assets on the Markowitz efficient frontier is defined as the following.

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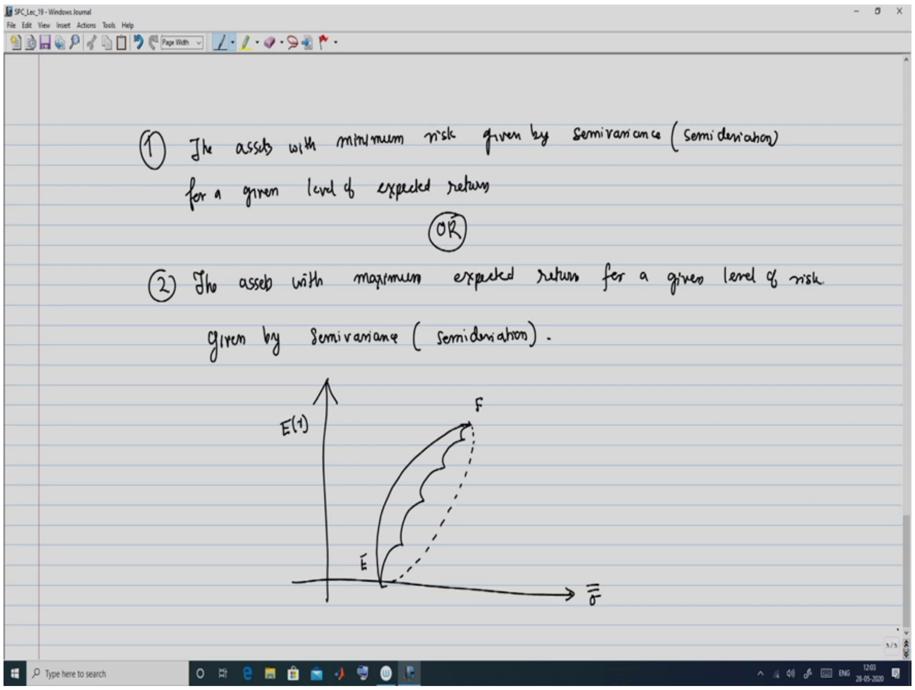

So, there so, here, so, the efficient frontier you would recall that in case of the Markowitz mean variance setup, what was the efficient frontier? The efficient frontier was given by that, by those portfolios such that for a given level of risk it is the portfolio that has the highest return, or for a given level of return is the portfolio that has the minimum risk, or simply the portfolio that has the minimum risk. So, now, when you are trying to define this in the context of the semi variance or semi, semi deviation, then we can analogously extend this concept drawn from this definition of efficient frontier or equivalently from dominant portfolios. So, accordingly the efficient frontier in this context I will make the following observation. That it is defined as those assets with minimum risk given by semi variance, or semi deviation for a given level of expected return. Or the assets with maximum expected return for a given level of risk given by so, the given level of risk will be given by again, the semi variance or semi deviation. So, it is the same definition in the mean as was in a mean variance framework. I accept that instead of variance and standard deviation, we respectively just change it to a semi variance or a semi deviation. So, graphically the efficient frontier is now going to be something which is on not on the sigma E r plane anymore, but is going to be sigma double bar E r plane. And, then the efficient frontier is going to look something like this. So, E F, so, E F is for efficient frontier alright.

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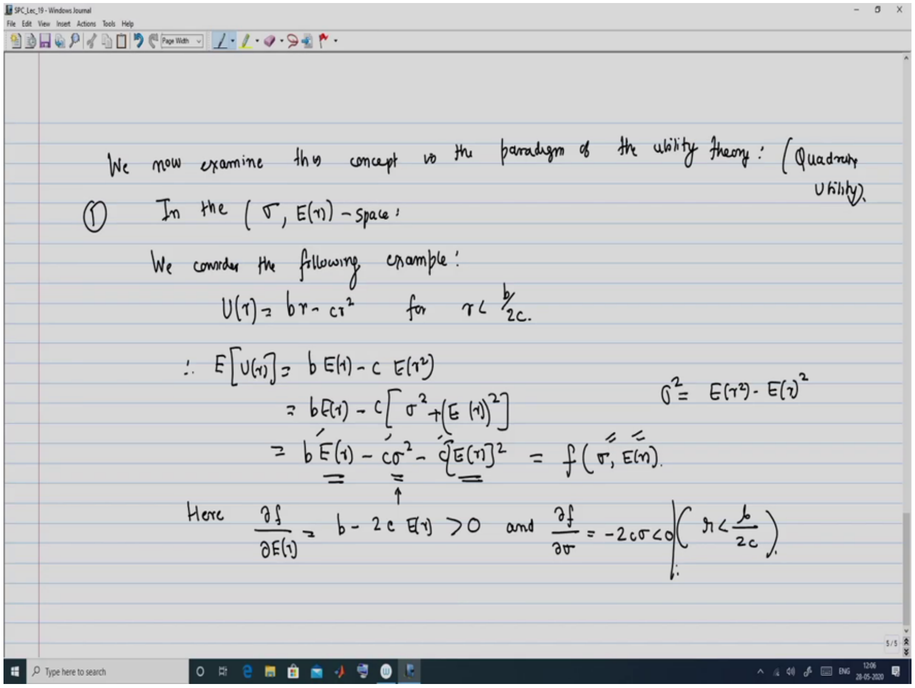
So, now, what we do is? We now examine. So, now, in the non-mean variance framework of course, you have to bring utility function into the picture. So, we now examine this concept in the paradigm of the utility theory ok. So, in order to do this let us just look at you know will see this in the form of the quadratic utility. And, do a comparison of what happens in the sigma E r plane and sigma double bar E r plane? So, we start off with first the sigma E r space or the sigma E r plane. So, we consider the following example. So, this is just for illustrative purposes the quadratic utility. So, here we take the quadratic utility

$$U(r) = br - cr^2, \text{ for } r < \frac{b}{2c}$$



. We will see why this condition is necessary. So, accordingly what is going to be the expected utility? The expected utility is simply going to be

$$
\begin{aligned}
E[U(r)] &= bE(r) - cE(r^2) \\
&= bE(r) - c[\sigma^2 + (E(r))^2], \;\; \sigma^2 = E(r^2) - (E(r))^2 \\
&= bE(r) - c\sigma^2 - c[E(r)]^2 \\
&= f(\sigma, E(r))
\end{aligned}
$$

So, here

$$\frac{\partial f}{\partial E(r)} = b - 2cE(r) > 0$$

and

$$\frac{\partial f}{\partial \sigma} = -2c\sigma < 0, \;\; r < \frac{b}{2c}.$$

So, that means, as your sigma increases, then your expected utility diminishes; that means, with an increase in the volatility your expected utility is diminished. So, this is what you see that is a characteristic of this particular example of quadratic utility, when you are using the mean variance framework. Now, let us see how we are going to extend this in case of a quadratic utility for the non-mean variance framework. So, for that purpose the same utility function is not going to work, but we are going to have to modify this utility function in a slightly different way, but it should be such that it is actually it qualifies as a utility function. And, subsequently we then look at what is going to be the sensitivity of f with respect to E r and sensitivity of f with respect to your semi deviation, that is sigma double bar.

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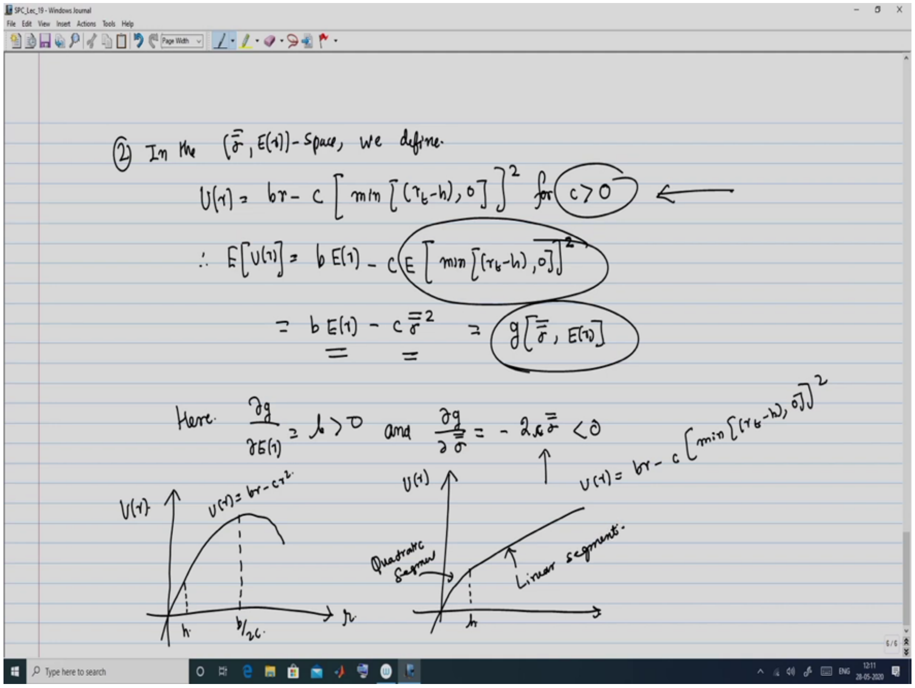
So, for this purpose what you have to do is now we move on to the sigma double bar E r space, we have to define a new utility function U of r. And, this I will define as

$$U(r) = br - c[\min\{r_t - h,\ 0\}]^2, \;\text{ for }\; c > 0$$

$$\therefore \;\; E[U(r)] = bE(r) - cE[\min\{r_t - h,\ 0\}]^2$$

And, remember this by definition is the semi deviation. So, this is going to be b of E r minus c, this is going to be so, there is a square at the top. So, this is actually semi variance sigma double bar square. And, I will



call this. So, you see that this is a function of E r and sigma double bar. So, I will call this a function g of sigma double bar and E r. So, here what we will have is? So, here we will have that, the sensitivity of this expected return g, that is,

$$\frac{\partial g}{\partial E(r)} = b > 0$$

and

$$\frac{\partial g}{\partial \bar{\bar{\sigma}}} = -2c\bar{\bar{\sigma}} < 0$$

So, remember that this is something that you have already also observe, in case of sensitivity of f with respect to the semi deviation. So, graphically this is going to look something like this. So, here so, this was the first case and you look at U of r, what is U of r? U of r is b r minus c r square and this is my threshold h and this is going to be my b over twice c. And, the graph for the newly defined so, this is more important for this newly defined quadratic utility. This is going to look something like this and then it is going to be straight line. So, here

$$U(r) = br - c[\min\{r_t - h, 0\}]^2.$$

And, your h will be here and this part this is going to be the quadratic segment of the definition, and this is going to be the linear segment alright. So, now that we have talked about what is the definition of a semi variance and semi deviation and looked at it and the paradigm of a quadratic utility function. Now, let us come to the main context of this course that is portfolio theory. And, let us see how we can actually do a portfolio analysis in this framework of semi deviation or equivalently semi variance.

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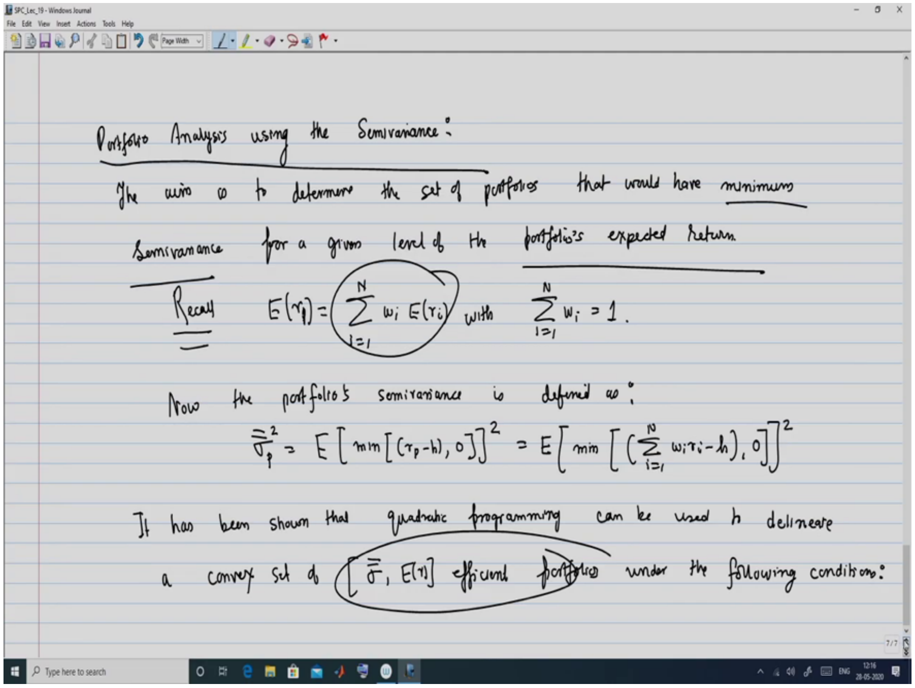
So, we start off with this topic now of portfolio analysis using the semi variance. So, the aim is to determine the set of portfolios, that would have minimum semi variance. So, this is basically determining the efficient frontier, in the semi variance framework. So, it is the minimum semi variance for a given level of the portfolios expected return. So, for this purpose we again go back to the drawing board. And, we recall that the expected return on the portfolio p is given by and I consider capital N number of assets. So, it is going to be

$$E(r_P) = \sum_{i=1}^{N} w_i E(r_i) \text{ with } \sum_{i=1}^{N} w_i = 1.$$



So, now first I need to so, I have said that what I want to do is I want to minimize the semi variance for the portfolio for a given level of the portfolios expected return. So, accordingly, the portfolio semi variance so, far I have defined the semi variance of an asset. So, now, the portfolios semi variance is defined as

$$\bar{\bar{\sigma}}_P^2 = E[\min\{r_P - h, 0\}]^2 = E[\min\{\sum_{i=1}^{N} w_i r_i - h, 0\}]^2.$$

So, now, this problem is not exactly trivial or straightforward as it was the case in the mean variance framework. And, will require a numerical computation to ascertain what is going to be the portfolio that will lie on the efficient frontier. So, I will just give a sketch of how you know the motivation behind this and how it can be accomplished. So, accordingly we observe that several authors have showed that, quadratic programming can be used to delineate a convex set of sigma double bar E r efficient portfolios, under the following given conditions. So, under certain conditions you can actually get this set efficient portfolios in the sigma double bar E r plane.

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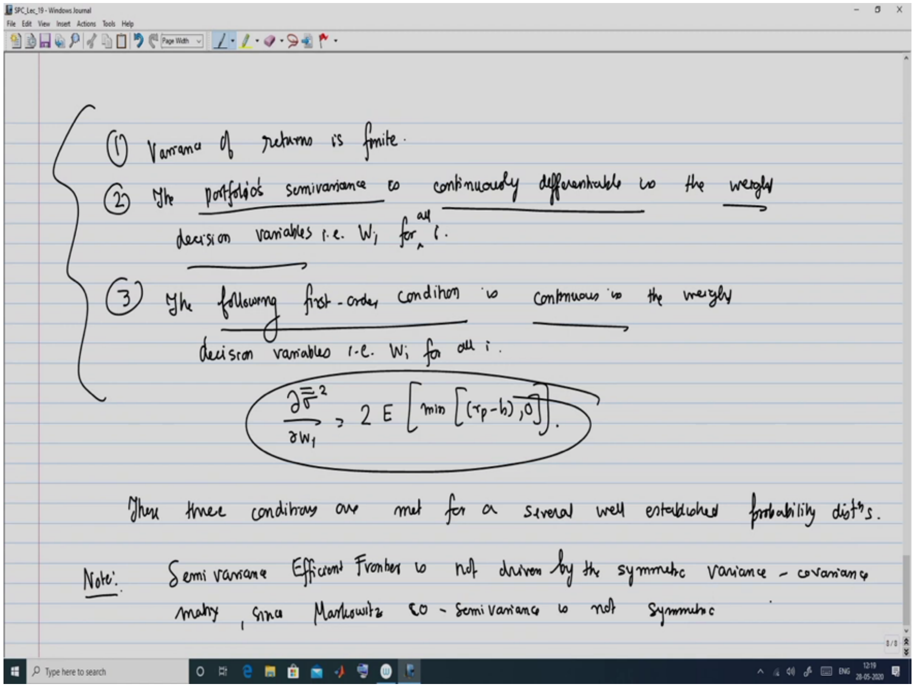
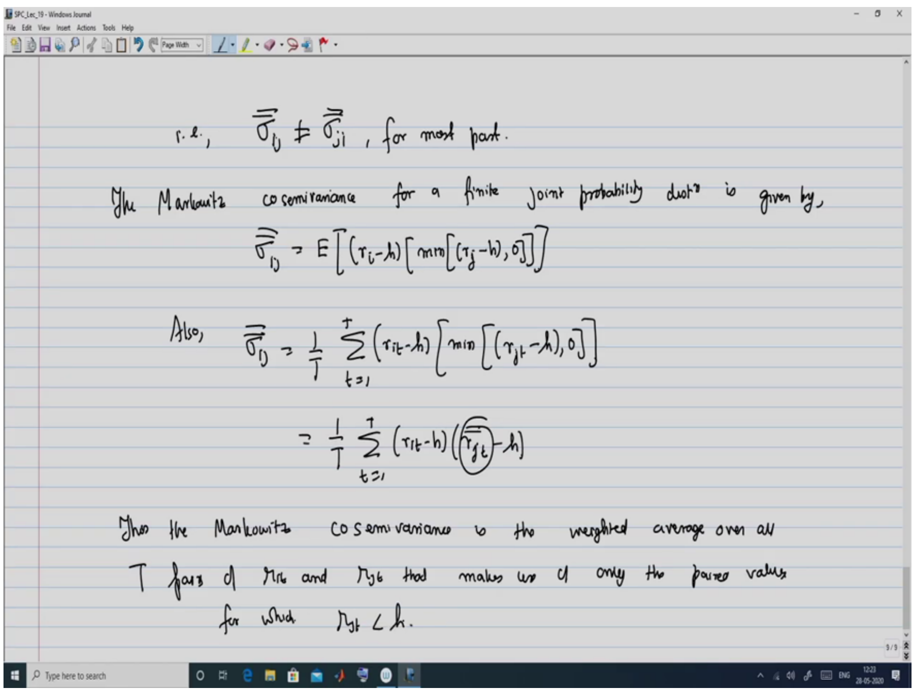
So, I will enumerate these conditions one by one. So, the first condition is that the variance of returns is finite. The second condition is that the portfolios semi variance is continuously differentiable in the weight decision variable, that is W i for all i. And, the third condition is that the following so, I will list a first order condition. So, the following first order condition is continuous in the weight decision variables, that is W i for all i. And, what is this condition? This is going to be

$$\frac{\partial \bar{\bar{\sigma}}^2}{\partial w_i} = 2E[\min\{r_P - h, 0\}].$$

So, what you need is that we need that the semi variance should be continuously differentiable with respect to i and the following first order condition is continuous so; that means, this is continuous. So, typically these three conditions are met or achieved for several well established probability distributions, alright. So, just to make a note, that the semi variance so, semi variance efficient frontier as we have enumerated now. So, this semi variance efficient frontier is not driven by the symmetric variance, covariance matrix. Since, and the reason for this is that, this semi variance defined by Markowitz. And, results in what is known as the co semi variance is not symmetric.





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That is sigma i j double bar and sigma j i double bar these are not equal for most part. So, now, I have abruptly brought about this sigma i j double bar and sigma j i double bar. So, what we have done is so, far we have extended the definition of variance and standard deviation to a semi variance and a semi deviation. Now, in the original Markowitz framework we had this covariance matrix, which was symmetric, but the analogous matrix or analogous entities that were part of this matrix, they are not symmetric to each other as per the definition that we are about to present. So, accordingly the Markowitz so, I just want to be more specific about this co semi variance. So, the Markowitz co semi variance for a finite joint probability distribution so, there are two variables involved. So, we have to have a finite joint probability distribution is given by the following.

$$\bar{\bar{\sigma}}_{ij} = E[(r_i - h)\min\{r_j - h, 0\}]$$

And, from the empirical point of view this is estimated as

$$\bar{\bar{\sigma}}_{ij} = \frac{1}{T}\sum_{t=1}^{T}[(r_{it}-h)\min\{r_{jt}-h,0\}] = \frac{1}{T}\sum_{t=1}^{T}(r_{it}-h)(\bar{\bar{r}}_{jt}-h)$$

So, in summary we can say that thus the Markowitz co semi variance is the weighted average over all T pairs of r i t and r j t, that makes use of only the paired values for which r j t is less than h.

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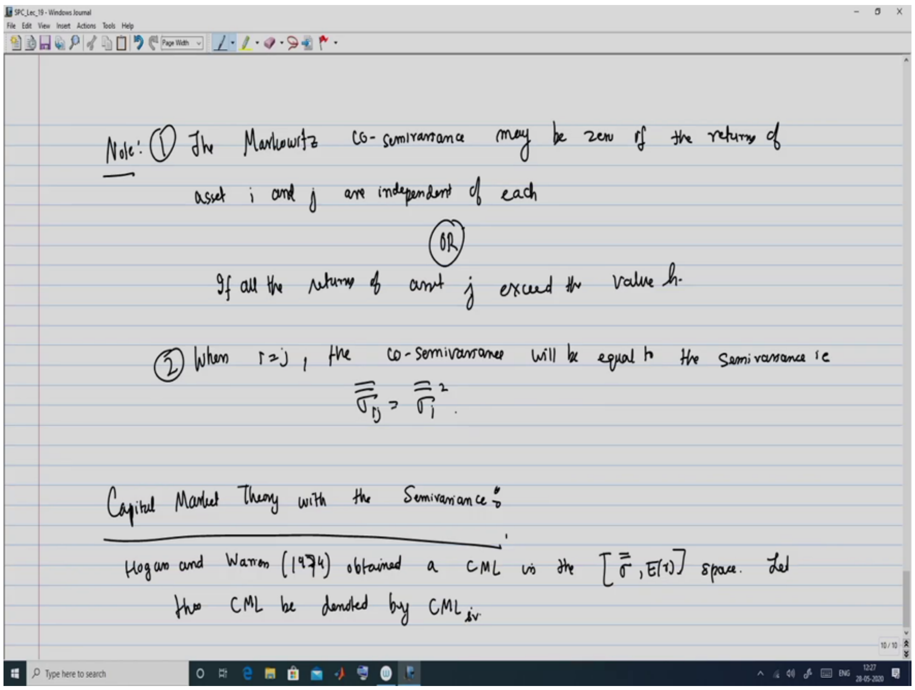

So, just an observation that the Markowitz co-semi variance as we have defined just now. May be 0 if the returns of asset i and j are independent of each other. So, this anyway is sort of obvious from the fact that the covariance of 2 independent random variables is equal to 0. And, so, this can in turn be visualized or intuitively seen in case of the co semi variance. The other way that the co semi variance can actually be 0 is that if all the returns of asset j exceed the value of h. So, what this is going to do is then in that case it is going to render this term to be equal to 0, if it is exceeded in all cases. So, also the other observation is that when i equal to j. So, I will let me numerate this as 2 separate observations. So, when i equal to j the co semi variance will be equal to the semi variance, that is sigma i j double bar is going to be equal to sigma i square double bar alright. So, now, what I want to look at is that since we are drawing analogies with whatever we have done in case of the Markowitz framework. That is the semi the mean variance framework and we started off by talking about the utility functions, and then we talked about the efficient frontier. And, also made certain observations about the analogous version of the covariances, namely in this case it is called the co semi variance. So, one key topic from the mean variance framework, that needs to be now examined in the context of semi variance is what is the capital market line or the capital asset market capital asset pricing model. So; that means, you have to start looking at the 2 aspects that we did when you are talking about the CAPM framework namely the CML and SML. When the risk measure is no longer the variance or standard deviation, but is instead replaced by the semi variance or the semi deviation. So, accordingly we start off with the concept of the capital market theory with the semi variance. So, this goes back to a paper by Hogun and Warren, which came in 1974. And, what they did was that they obtained a capital market line in the sigma double bar E r plane instead of the sigma E r plane, that is there in case of the mean variance framework. Now, would let this CML be denoted or named as CML of s v just for semi variance alright.

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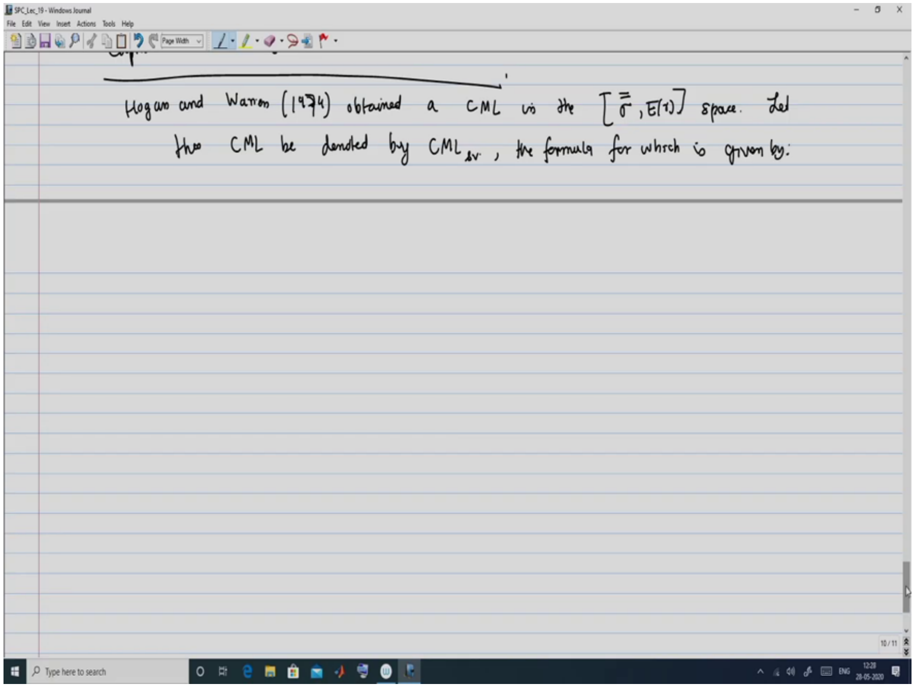

And this CML the formula is given by the following.
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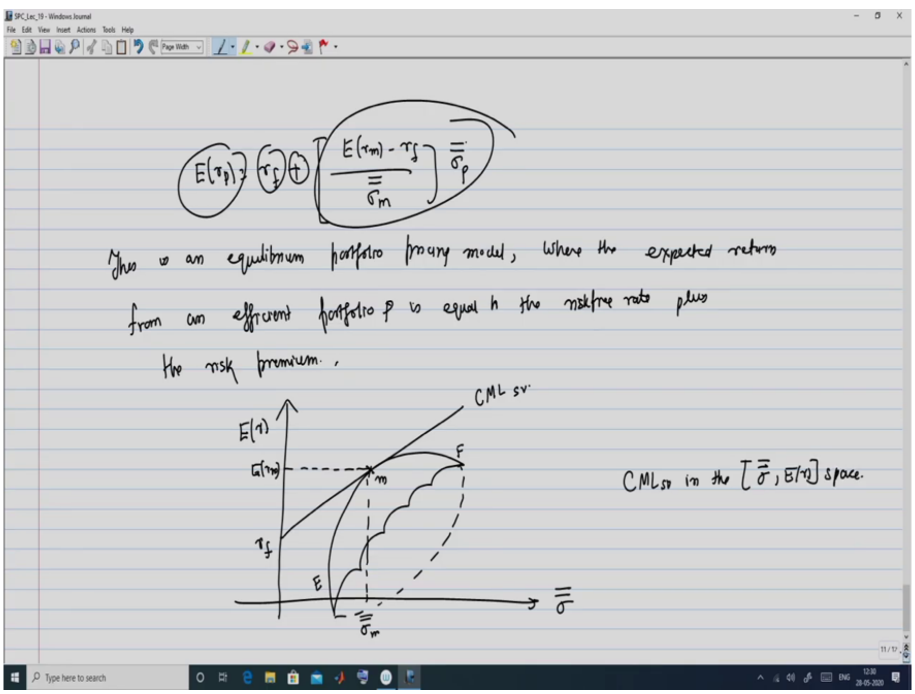

So, here we will have as before we will have

$$E(r_P) = r_f + \left[\frac{E(r_m) - r_f}{\bar{\bar{\sigma}}_m}\right] \bar{\bar{\sigma}}_P.$$

So, the interpretation is that this is an equilibrium portfolio pricing model, where the expected return from an of efficient portfolio P; that means, E of r P. This is equal to the risk free rate plus the risk premium, which is given by this term. So, graphically let us have a look at how this is going to look like. So, here we are in the sigma double bar E r plane. And, and so, if so, I start off with r f and this point of tangency as before is the market portfolio. So, this will give me the corresponding E of r m and sigma m double bar.

And, this curve here is E F for efficient frontier and this line which is what is known as the CML in the semi variance framework? So, this is nothing, but the CML in the semi variance framework in the sigma double bar E r space.

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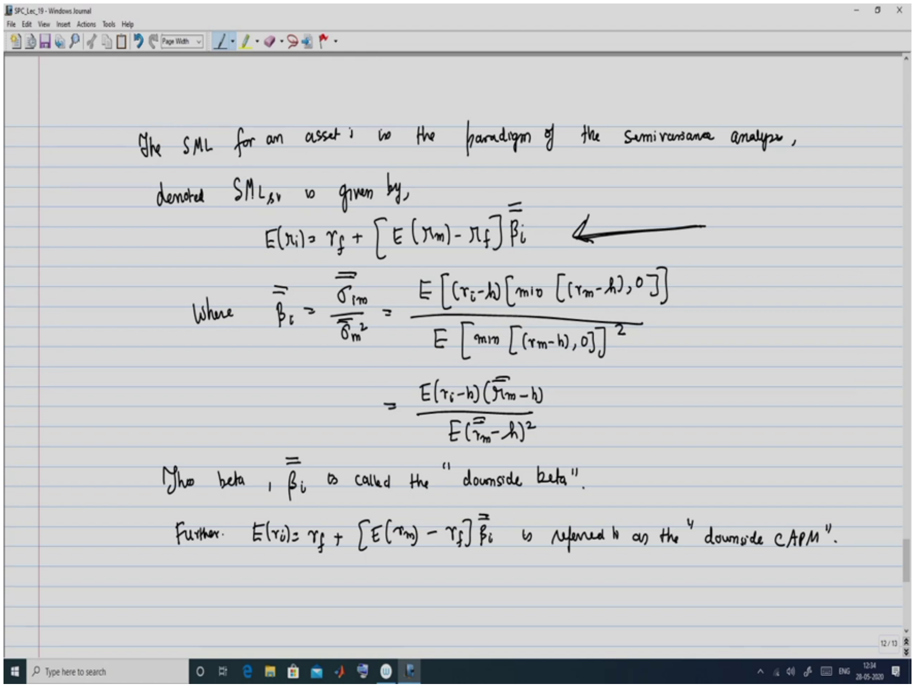

So, now we come to the last topic that is the SML in the semi variance framework. So, the SML for an asset i in the paradigm of the semi variance analysis, which is denoted by CML sv is given by,

$$E(r_i) = r_f + [E(r_m) - r_f]\bar{\bar{\beta}}_i.$$

So, remember that this was the original SML. So, again with the only thing that changes in the context of semi various is that the beta i is replaced by beta i double bar. And, now I have introduced what is beta i double bar. So, I have to define that beta i double bar this is going to be sigma i m double bar, over sigma m square double bar. And, the numerator according to the definition is going to be the expected value of r i minus h into minimum of r m minus h comma 0. And, the denominator is simply going to be the definition of semi variance for the market portfolio, which is expected value of minimum of r m minus h comma 0 a whole square. And, this can be reduced to E of r i minus h into r m double bar minus h over E of r m double bar minus h square. And, this beta which is been newly introduced that is beta i double bar, this is called the downside beta. The reason why we use the term downside beta here is that because, it captures our the sensitivity of the asset with respect to the market portfolio in the context of the downside risk; that means, the risks associated from the losses driven by the returns falling below a certain threshold level. So, further what we can write is that this relation, that we had E of r i is equal to r f, this SML this r f plus E r m minus r f into beta i double prime. This relation that we have derived here, this is referred to as the downside SML or as the downside CAPM remember CAPM is synonymous with SML.

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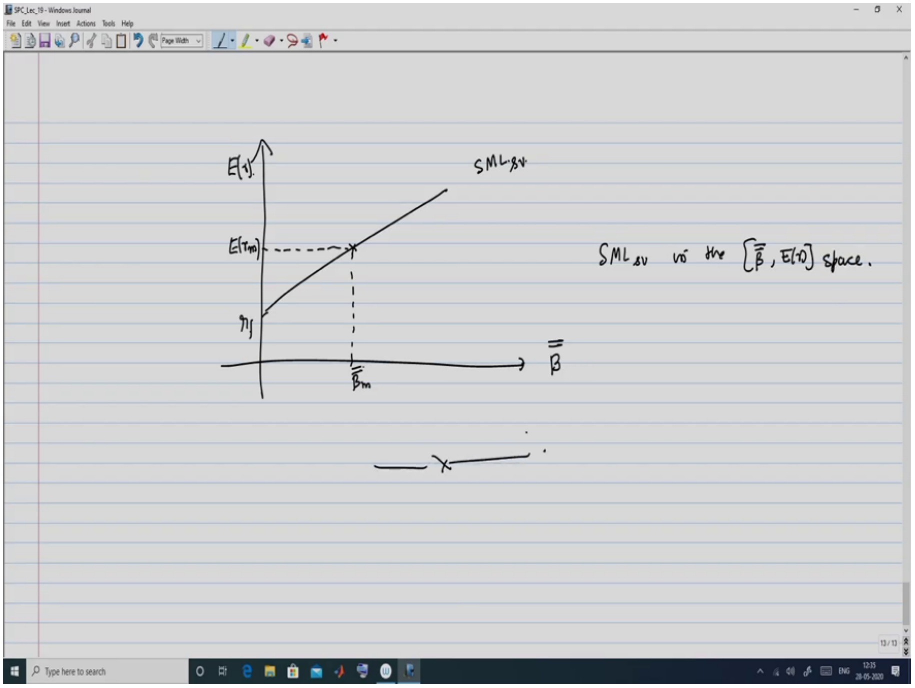
So, graphically this is what it is going to look like?. So, I will have to consider this in the beta double prime double bar plane, against E of r and this SML, which I will denote it as SML. So, here please make a correction this should be SML sv. So, here this SML sv this is the line which emanates from r f and this passes through m. So, I get the corresponding E of r m and here I get beta m double prime. So, this graph is nothing, but this SML sv or the SML for the semi variance framework in the beta double bar E r space alright. So, this brings us to the end of this lecture just to do a recap we have mainly focused on this lecture in dealing with various aspects of semi variance or equivalently semi deviation. Semi variance is designed



to capture all only those returns that follow below a certain threshold. And, the standard deviation of that or the square root of that is what is the standard deviation equivalent namely it is called the semi deviation. And, then we looked at this from three perspective this entire discussion. And, what we had looked at here is we first looked at from the point of view of the utility and the expected utility, then we also talked about the efficient frontier. And, finally, we also included and discussed the extension of the capital market line and the security market line, in the context of making the transformation from the mean variance framework, and getting the equivalent CML and SML in case of the semi variance or semi deviation framework. In the next class we will continue our discussion on this non mean variance theory and we will introduce a new topic which is called the stochastic dominance.

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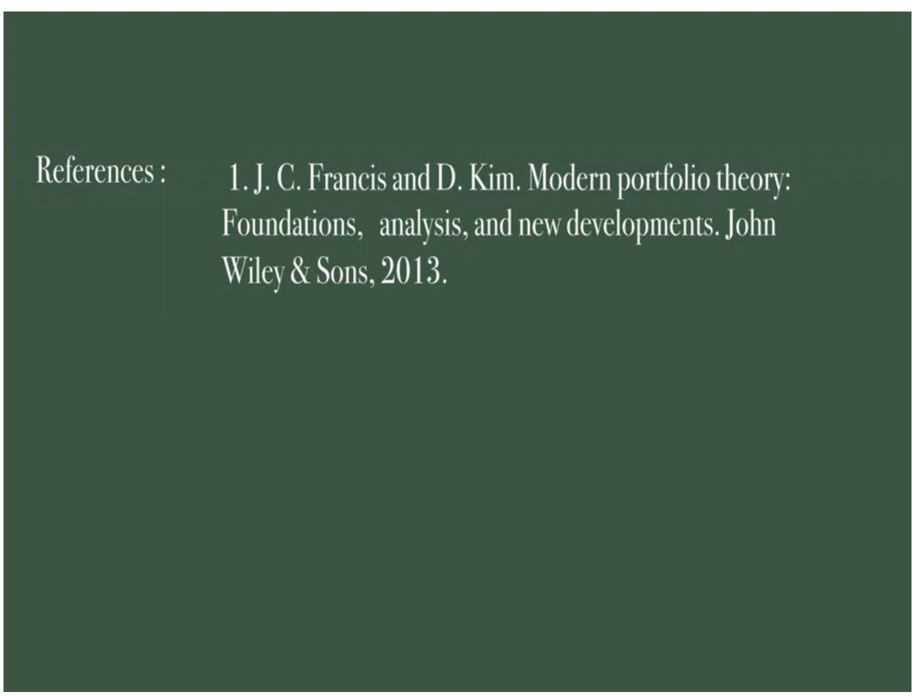

Thank you for watching.