Mathematical Portfolio Theory
Module 04: Non-Mean-Variance Portfolio Theory
Lecture 17: Geometric Mean Return and Roy's Safety-First Criterion

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Hello viewers. Welcome to this next lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. You would recall, that so, far we have been talking about utility functions in the context of portfolio management. And, we identified that the eventual goal is to maximize the expected utility of an investor. And, we started talking about the context of Non-Mean-Variance Analysis.

So, we will start, in this today's lecture we start off with digging a little deeper into this notion of a non-mean-variance analysis. We will look at a particular example and then we will talk about the first of the several criteria's, that can be used in the non-mean-variance framework, namely the safety first criteria.

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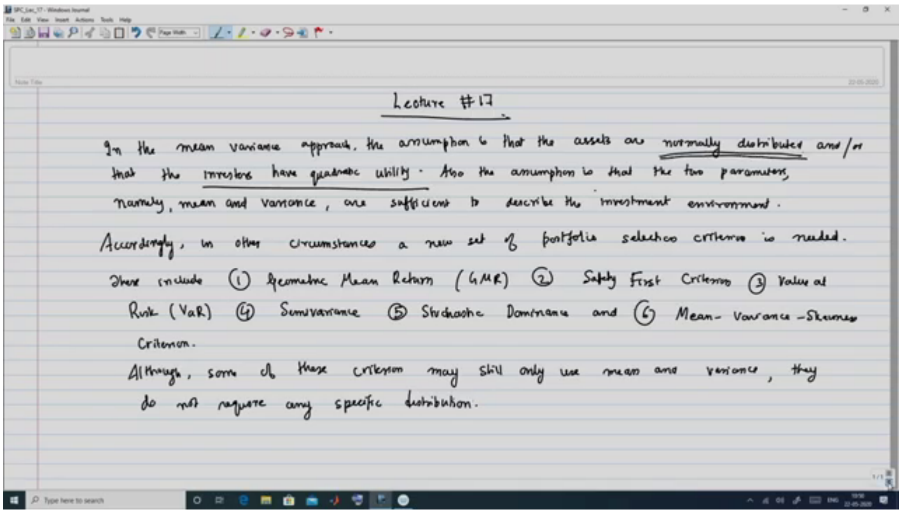

So, we start the lecture. So, we make the statement that in the mean variance approach, the assumption is that the assets are normally distributed, and that the investors have quadratic utility. Also, the assumption is that the two parameters, namely mean and variance, remember this is the mean variance framework. These are by themselves sufficient to describe the investment environment. So, accordingly so, once these circumstances do not hold so; that means, in other circumstances; that means, when they are not normally distributed or when you do not have quadratic utility. Those under circumstances a new set of portfolio selection criteria is needed. Now, some of these cases are so, these include. So, for example, geometric mean return, which we will discuss in today's class. Then, we have the safety first criterion. The third is value at risk, this is something we will discuss in detail in the later half of the course or VaR. Then, we have something called the semi variance, you have stochastic dominance, and something which is known as the mean-variance-skewness criterion ok.

So, just an observation that although some of these criterion may still only use mean and variance, they do not require any specific distribution. So, he could have a more generalized setup where you are not restricted only just to the assumption of normally distributed random variables, ok.

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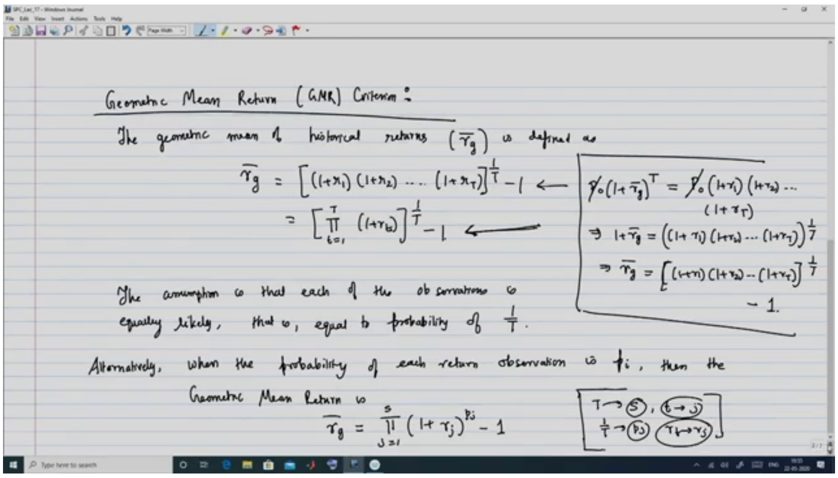

So, let me start of with the first of this criterion and this is what is known as the geometric mean return criterion.

So, the geometric mean of historical returns. So, we will denote this by r_g for geometric mean and the bar at the top for the mean is defined as

$$\bar{r_g} = [\Pi_{t=1}^{T}(1+r_{tk})] - 1$$

Alternatively, when the probability of each return observation are not equal, but rather are denoted by p_i. Then, the geometric mean return is given by

$$\bar{r_g} = \Pi_{j=1}^{s}(1+r_j)^{p_j} - 1$$

So, here you observed that instead of T, I have replaced to this S and instead of i small t I have replaced this with j. So, basically I am considering S number of period. Just to distinguish it from the notation for the notation I have here. And, here of course, you know my 1 by T; this is being replaced by my p_j. And, your r_t is replaced by r_j ok.

So, now the next thing we need to do is look at what is the maximization of the wealth at the terminal point that is T. And, so, accordingly what you will do is that we will get the formulation for W of T in the context of the geometric mean return, and then we will calculate what is going to be the expected final wealth?

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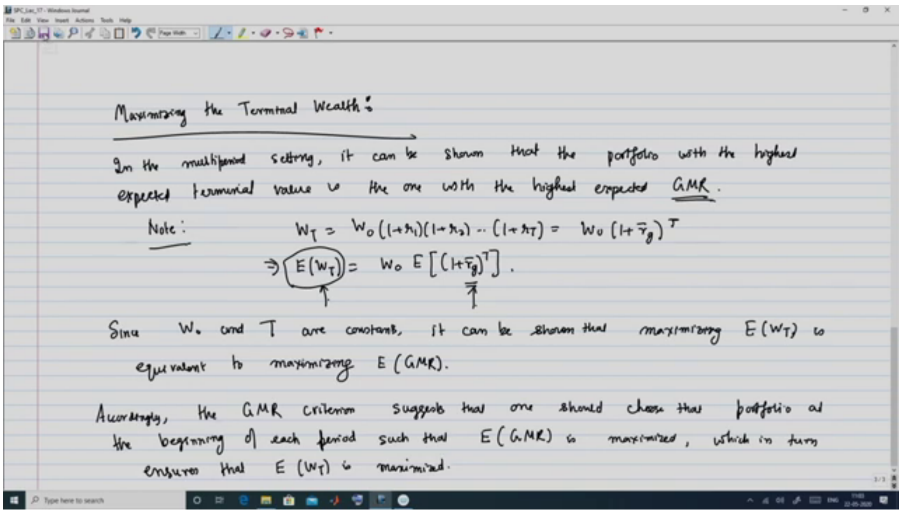

So, we start off with maximizing the terminal wealth. So, in the multi period setting, it can be shown that the portfolio with the highest expected terminal value is the one with the highest expected GMR, that is Geometric Mean Return.

$$W_T = W_0(1+\bar{r_g})^T$$

So, here we can state that since W naught and T are constants, it can be shown that maximizing $E(W_T)$, that is this term here is equivalent to maximizing $E(GMR)$. So that means, maximization of this. So, accordingly the GMR criterion suggests that, one should choose that portfolio at the beginning of each period such that, E the expected value of GMR is maximized, which in turn ensures that $E(W_T)$ is maximized.

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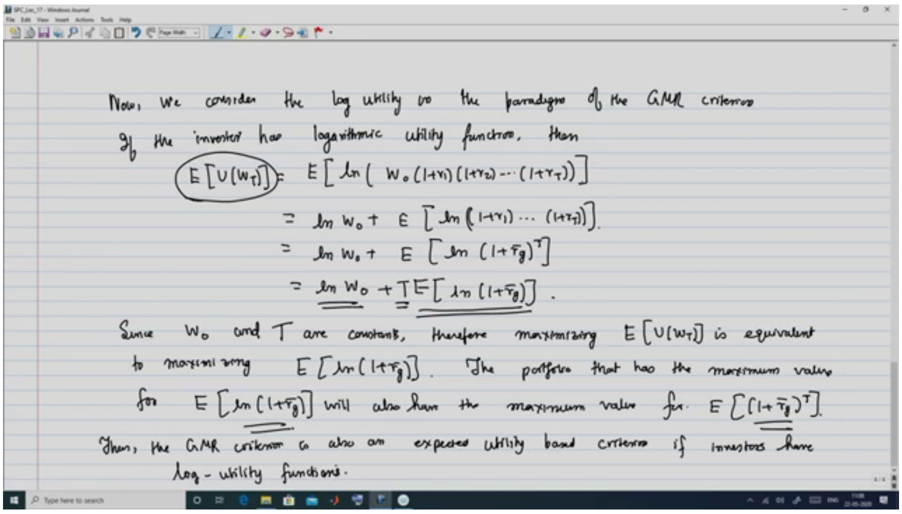

Now, we consider the log utility in the paradigm of the GMR criterion. So, if the investor has logarithmic utility function, then

$$E(U(W_T)) = \ln W_0 + TE[\ln(1+\bar{r_g})]$$

Now, since W naught and T are constants therefore; maximizing the expected utility of the final wealth level is equivalent.

So, this is a constant and T is constant. So, accordingly maximization of expected utility of the final wealth amounts to maximization of the expected value this. So, this is equivalent to maximizing the expected value of natural log of 1 plus rg bar.

So, the portfolio that has the maximum value for the expected value of natural log of 1 plus the GMR will also have the maximum value for expected value of 1 plus rg bar raised to T. Because, this is essentially equivalent to this. So, thus the GMR criterion is also an expected utility based criterion, if investors have log utility functions alright.

So, now we move on to once that we are done with the GMR criterion and how we showed that in case the utility function is a log utility? So, under the circumstances the maximization of the expected utility of the final wealth level is equivalent to maximization of the expected GMR value.

So, now, once this is done we move on to our next criteria and this is what is known as the safety cross first criteria. And, it is some sort of a conservative criteria and we will look at three different safety criteria's. So, let us now first begin with motivating why safety criteria's? Safety first criteria's are used in the first place.

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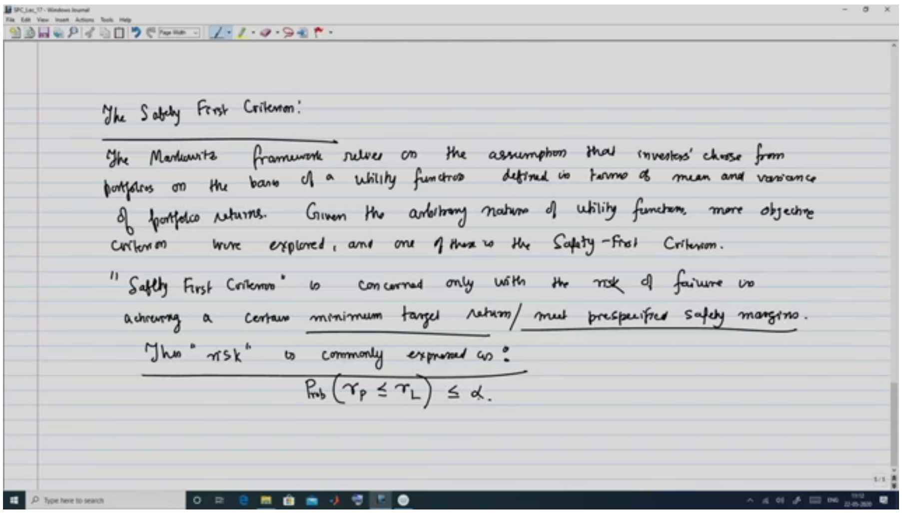

So, accordingly, we start this discussion on the safety first criterion. So, the motivation is as follows that the Markowitz framework, this relies on the assumption that investors choose from portfolios on the basis of a utility function defined in terms of mean and variance of portfolio returns. So, given the arbitrary nature of utility functions more objective criterion were explored, and one of this is the safety first criterion ok.

So, this "safety first criterion" is driven by the motivation, that we are only concerned with the risk of failure in achieving a certain minimum target return, or the failure to meet some pre-specified safety margins. So, this risk so, please understand I am not talking about the criterion here, but I am identifying the "risk". This risk is commonly expressed as the following. That so, if you have the portfolio return given by r_P and you are worried that this will be less than or equal to some minimum target or prespecified return that you want which is r_L.

So, you want to check that what is going to be the probability of r_P being less than or equal to r_L. And, this is risk is commonly expressed as that this probability must be less than or equal to α.

$$P(r_P \leq r_L) \leq \alpha$$

So, this means that if you were start of with a portfolio of P and the return of the portfolio is given by r_P and it is a random variable. And, then you fix a minimum level of return that you want, that is some sort of a safety threshold which you denote by r_L and you are you want to always stay above this r_L. So; that means, from the point of view of your risk, you are worried about what are the events? Under which you have what are the events for which you will have $r_P \leq r_L$.

And, what is going to be the probability of that? Just to get an indication of the percentage chances that you will not be able to beat your target return of r_L. And, you basically you were worried that you want this probability should be less than or equal to some level say alpha, which you can choose may be 0.05 or 0.1. If, you choose to be 0.05 this means that you do not want the chance of missing your target of r_L by more than 5 percent.

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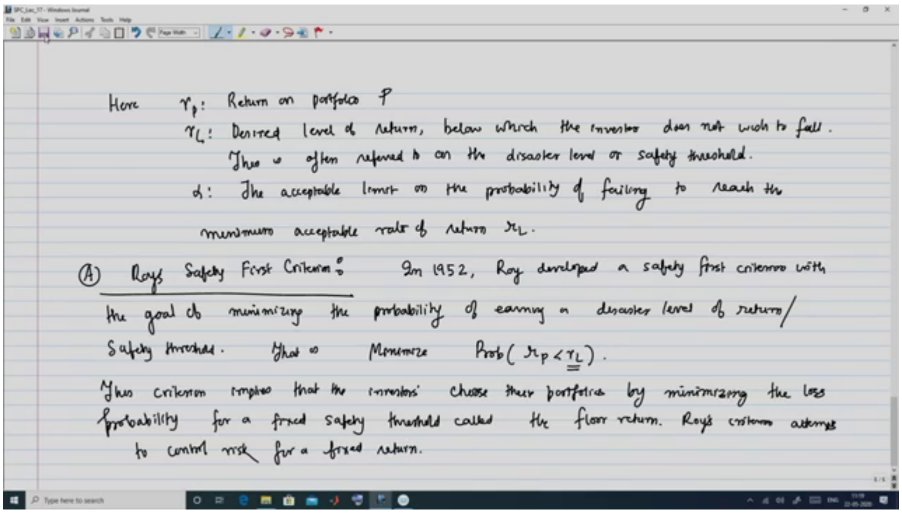

So, here we have introduce certain notations. So, the first one is r_P. So, this is nothing but the return on portfolio P your r_L which was your target return, this is your desired level of return, below which the investor does not wish to fall.

So, this is often referred to as the disaster level or safety threshold. And, finally, alpha is going to be the acceptable limit. That means, the maximum possible percentage term that you are willing to accept, on the probability of failing to reach the minimum acceptable rate of return r_L.

So, this criteria

$$P(r_P \leq r_L) \leq \alpha$$

is a very generic criteria. And, of course, you all want ideally your $\alpha = 0$, that there is no likelihood that you will miss your target; but since r_P is a random variable so, that is not likely to happen. So, an eventual goal is to meet the target in a manner, that the probability that you miss the target is as small as possible.

Now, keeping this general requirement of this

$$P(r_P \leq r_L) \leq \alpha$$

We will now enumerate and describe in detail three different such criteria's. And, in today's class we start of and describe the first of those safety criteria's, which is known as the Roy's safety first criterion.

So, accordingly we start with Roy's Safety First Criterion. So, they it is a fairly classical approach in 1952 a Roy developed a safety first criterion with the goal of minimizing the probability of earning a disaster level of return or what is known as the safety threshold.

So, the goal is to minimize, this probability that your portfolio return is below the your safety threshold or the disaster level ok. Now, what is this criteria implies? So, this criterion implies, that the investors choose their portfolio by minimizing the loss probability for a fixed safety threshold called the floor return.

So, this is called the floor return is a because this is the bare minimum return, that the investor wants to achieve and anything below this is not acceptable to the investor. So, accordingly Roy's criterion, the Roy's criterion attempts to control risk for a fixed return.

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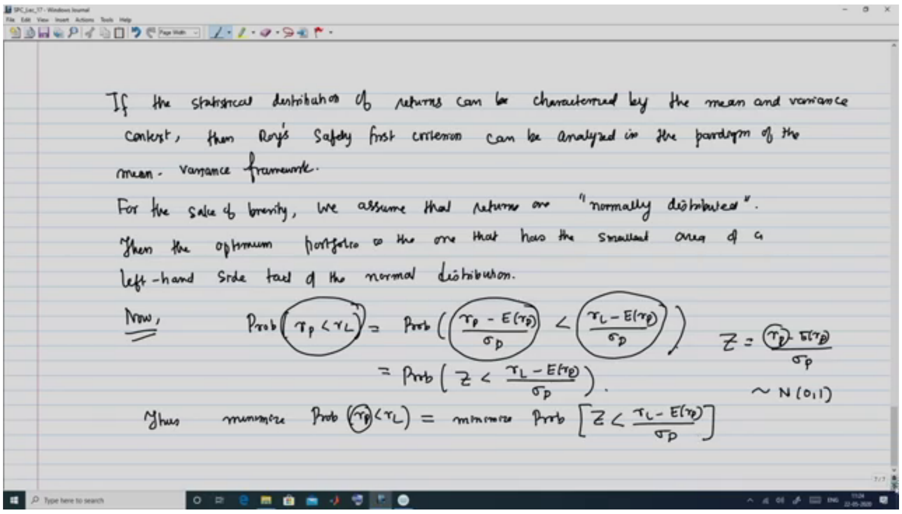

So, now let us look at and a specific situation. That, if the statistical distribution of returns can be characterized by the mean and variance. So, are no other moments are required or parameters are required. Then, Roy's safety first criterion can be analyzed in the paradigm of the mean variance framework.

So, for the sake of brevity, we assume that returns are "normally distributed". Just for the for a simplistic case, we assume that returns are normally distributed. Then, the optimum portfolio is the one, that has the smallest area of a left hand side tail of the normal distribution.

So, in order to see this we do a little calculation. So, what do you want? You essentially want to minimize this probability so,

$$\text{minimize } \; P(r_P \leq r_L) = \text{minimize } P[Z < \frac{r_L - E(r_p)}{\sigma_p}]$$

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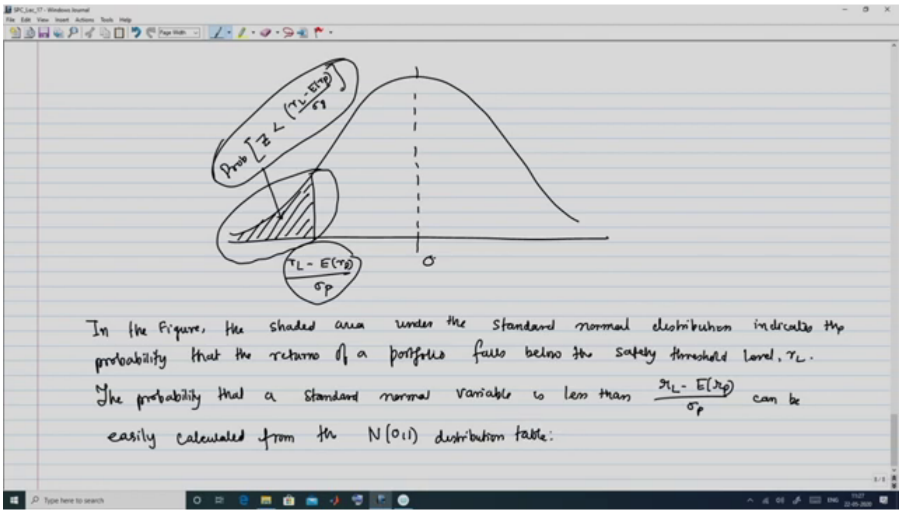

So, let us see this graphically how is this going to look? So, I have this normal distribution. So, I have this here and this is 0. And, on the x axis we have $\frac{r_L - E(r_p)}{\sigma_p}$. So, this is my random variable. Now, here you see so, Z follows a standard normal random variate. So, accordingly what you will have is that, the probability that $Z < \frac{r_L - E(r_p)}{\sigma_p}$. This probability is going to be this area alright.

So, this means that you want to minimize this condition and you see that you have to look at the smallest area. So, the minimization of this condition here is equivalent to the smallest area that has been shaded. So, that is the graphical interpretation of the Roy's safety first criterion. In case the returns of the portfolio are assumed to be normally distributed with the mean of $E(r_P)$ and variance of σ_{r_P}, so; that means, that in this context the goal is to settle for a portfolio. So, you could have a whole bunch of portfolios.

For each of those portfolios you will have $\frac{r_L - E(r_p)}{\sigma_p}$, which will essentially set the area under the shaded curve. So, the area which has been shaded here, and you want to find that amongst those which of the one is going to give you the smallest area that is actually shaded alright.

So, to be more explicit, we say that in the figure the shaded area under the standard normal distribution indicates the probability that the returns of a portfolio falls below the safety threshold level r_L. So, the probability that, a standard normal variable is less than $\frac{r_L - E(r_p)}{\sigma_p}$ can be easily calculated from the $N(0,1)$ distribution table.

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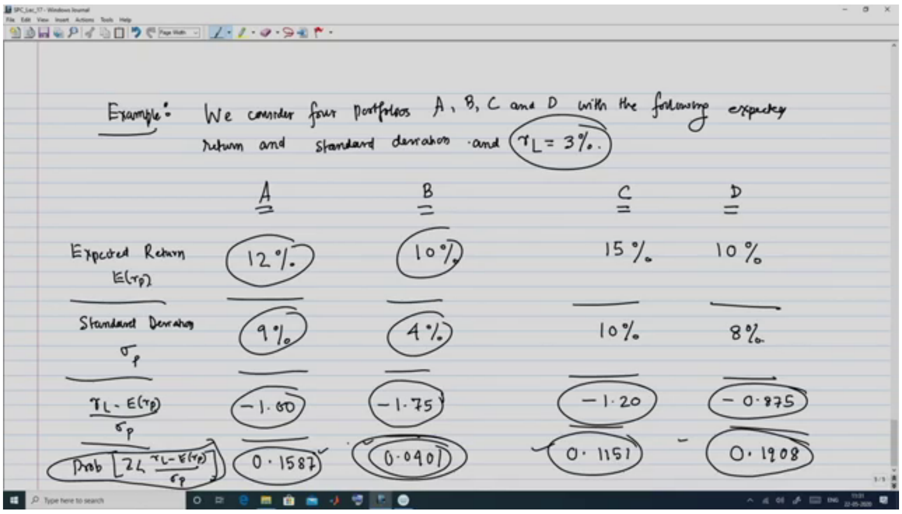

So, now we look at an example, to illustrate the Roy's safety first criterion. So, we consider four portfolios, which will designate as A, B, C and D; with the following expected return and standard deviation, with and along with the safety threshold r_L being 3 percent. So; that means, that you want to choose a portfolio which gives at least the 3 percent return. So, let us have these portfolios A, B, C and D. So, first of all let us have the expected return denoted by $E(r_P)$. So, for portfolio say, A say it is 12 percent, for portfolio B it is 10 percent, for portfolio C it is 15 percent and for portfolio D it is 10 percent ok.

So, next we have is the standard deviation σ_P. So, for portfolio A this is 9 percent, portfolio B is 4 percent, portfolio C it is 10 percent, and portfolio D it is 8 percent. Then, what is going to be our $\frac{r_L - E(r_p)}{\sigma_p}$?

I remember r_L is 3 percent. So, in the first case it is going to be 3 percent minus 12 percent over 9 percent. So, this becomes minus 1.00. In the second case is going to be 3 percent minus 10 percent over 4 percent. So, this is going to be minus 1.75. Likewise, we will have minus 1.20 and the last case we will have minus 0.875.

So, now what is going to be the probability? So, will now calculate, what is the probability of Z being less than $\frac{r_L - E(r_p)}{\sigma_p}$. So, what is the probability that Z is less than minus 1? This probability from the table turns out to be 0.1587.

The probability that Z less than minus 1.75; obviously, this is going to be minus 0.0401. As you can see this is smaller than the first one, because this minus 1.75 is to the left of minus 1.00. Then for minus 1.20, you will expect that you lie between this and this and it turns out that this is 0.1151. And, for the last case you will have this to be 0.1908 ok.

So, now, you remember that the Roy safety criterion basically means that, what is going to be the minimum of this probability. And, the minimum for this probability is going to be this one.

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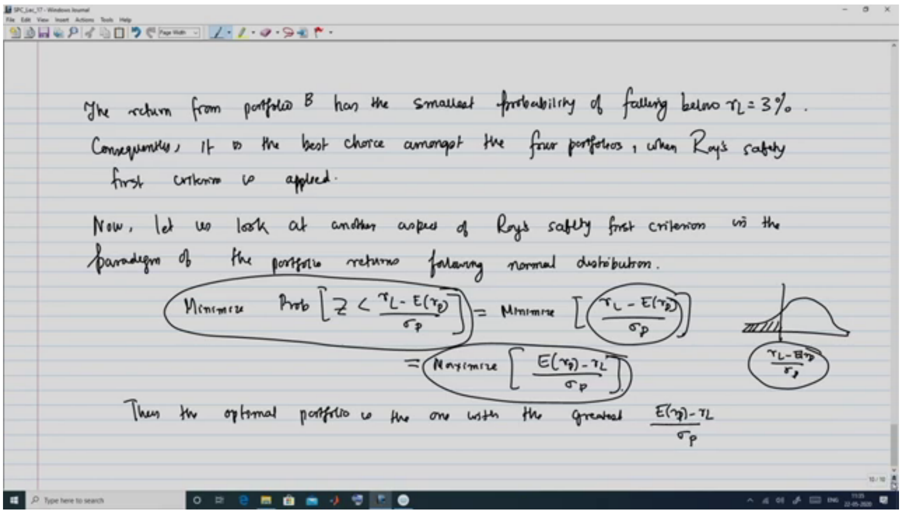

So, accordingly I can write that the return from portfolio B has the smallest probability of falling below r L equal to 3 percent. Consequently, it is the best choice amongst the four portfolios, when Roy's safety graph first criterion is applied ok.

So, now let us look at another aspect of Roy's safety first criterion, in the paradigm of the portfolio returns following normal distribution. So, accordingly we will have a minimize the probability. So, I start of from where I left of earlier that the it is the for the problem reduces to minimization of probability of Z P being less than r L minus E r P over sigma P, on the assumption that r P formed follows a normal distribution.

And, this is equal to minimize $\frac{r_L - E(r_p)}{\sigma_p}$. Remember that here we had this $\frac{r_L - E(r_p)}{\sigma_p}$ and we wanted to minimize the area under the curve. So, the minimization of the area under the curve which is shaded, that amounts to minimization of this quantity. Now, the minimization of this quantity is the same as maximization of the negative of the that is $\frac{r_L - E(r_p)}{\sigma_p}$.

So, the Roy safety first criterion then means that it in case of the returns of the portfolio being normally distributed be reduces to minimization of the probability here, which is then equivalent to maximization of the expected return of the portfolio, that is the excess return over your a safety threshold divided by σ_P.

So, this means that so, thus the optimal portfolio is the one with the greatest $\frac{r_L - E(r_p)}{\sigma_p}$ ok.

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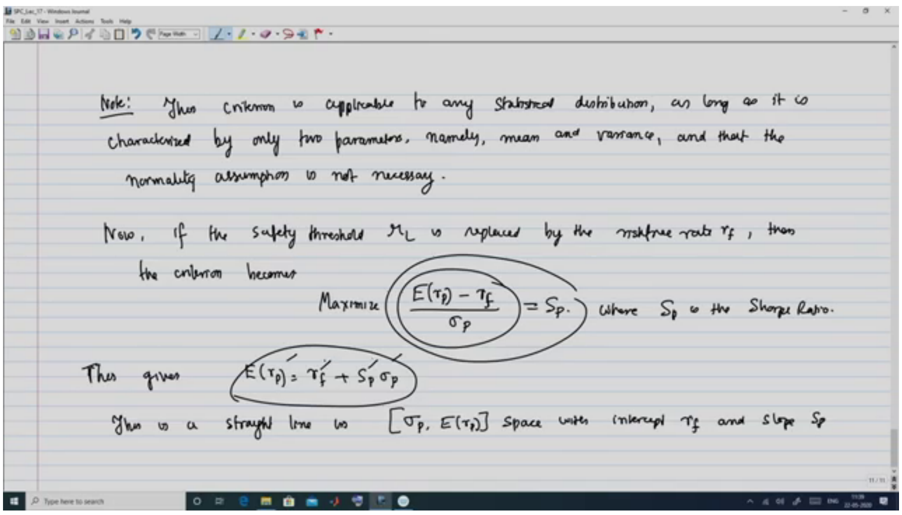

So, I just want to make a note here that this kind of criterion is applicable to any statistical distribution as long as it is characterized by only two parameters, namely mean and variance. And, that the normality assumption is not necessary ok.

So, I want to conclude with one last observation considering a particular case of the safety threshold r_L being replaced by the risk free rate r_f, then the criterion becomes

$$\text{Maximize } \frac{E(r_p) - r_f}{\sigma_p} = S_p$$

Here S_P is the sharp ratio. So, this means that the maximization or the Roy's criterion essentially then reduces in case of the safety threshold or r_L being equal to the same as the risk free rate it reduces to the problem of maximization of the sharp ratio.

So, accordingly what you can write is that,

$$E(r_P) = r_f + S_P \sigma_P$$

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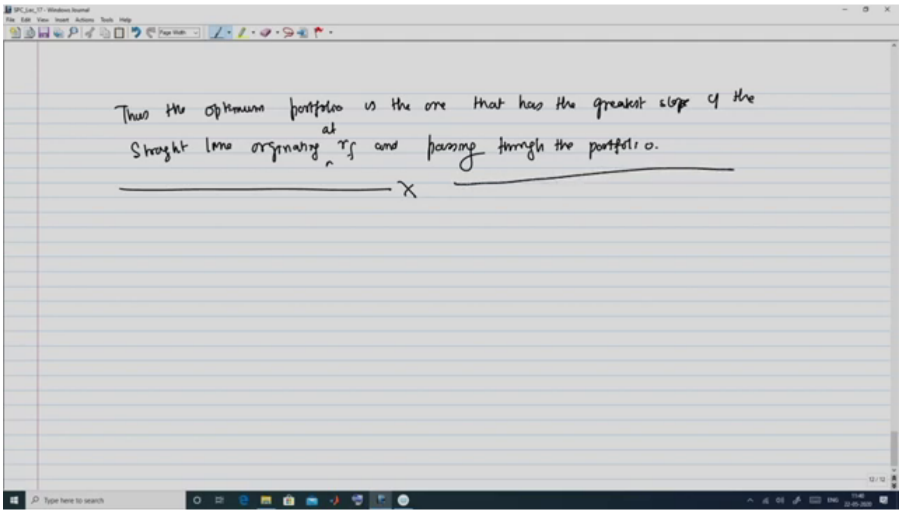

Thus, the optimum portfolio using the Roy safety first criterion is the one, that has the greatest slope of the straight line originating at r_f and passing through the portfolio alright. So, this brings us to the end of this class.

So, just to do a recap we are slowly moving towards our detailed analysis of the non-mean variance framework. And, we began with a motivation of why we need a non-mean variance framework. And, then we looked at the first of this non mean variance criterion that is the geometric mean return.

And, then we looked at how the maximization of the expected utility of the terminal wealth in case of the utility function be a log utility function. Amounts to the maximization of the GMR that is a geometric mean return. And, then we started talking about the safety first criterion which is driven by the motivation of minimizing the probability or the percentage chance, that the return of your portfolio will fall below a predefined safety threshold, which you denoted by r_L.

And, we will talk about several such safety first criterion. And, in today's class we began with the discussion on the Roy safety first criterion, which seeks to essentially minimize the probability of the return of your portfolio falling below the certain threshold.

And, through an illustration of an example of where the returns are normally distributed, we saw how this is accomplished and then we looked at a graphical interpretation of this. And, then we concluded by connecting this Roy's safety first criterion to the sharp ratio that we have already seen as a criteria for portfolio performance evaluation.

So, in the next class we will continue our discussion and look at the other safety first criteria's that have been developed over the period of time. So, this concludes our lecture for today. Thank you for watching.