Mathematical Portfolio Theory
Module 04: Non-Mean-Variance Portfolio Theory
Lecture 16: Portfolio theory with utility functions

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Hello viewers, welcome to this next lecture on the NPTEL MOOC course on Mathematical Portfolio Theory. You will recall that in the previous week we talked mostly about utility theory and we talked about what are the characteristic of utility functions, particularly in the context of the risk attitude of investors. And then we talked about maximization of expected utility in terms of his definition.

And we talked about the two important concepts, extending the basic definition of utility, namely absolute risk aversion, and the relative risk aversion. And you looked at some common examples of utility functions in the context of the preceding discussion.

On the properties that risk neutral or risk averse investor who is also rational investor and we tabulated those properties. And then we were planning on moving on to the next discussion on using utility theory in the context of portfolio optimization. So, that is the topic that we are going to start off in today's lecture.

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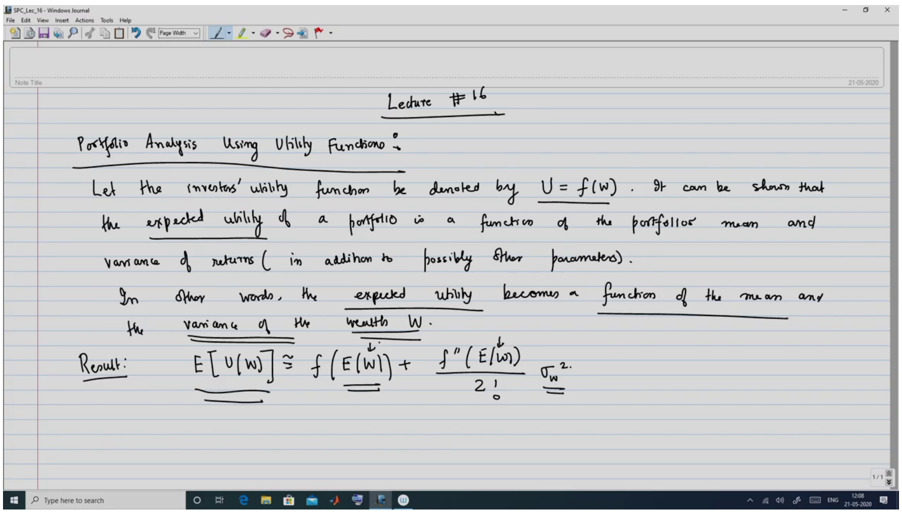

So, as I said we will talk about portfolio analysis using utility functions. So, when you talk about utility functions, this is something investor specific and so, we say that we let the investors utility function be denoted by some $U = f(W)$.

Now, it can be shown that, the expected utility of a portfolio. So, this is the utility function. So, consequently you can calculate the expected utility of a portfolio and it can be shown that it is a function of the portfolios mean and variance of returns. Of course, in addition to possibly other parameters.

So, in other words the expected utility becomes a function of the mean and the variance of the wealth W. So, let me be more specific with this result which gives a quantitative formulation of this. So, the expected utility of the wealth:

$$E[U(W)] = f(E(W)) + \frac{1}{2!} f''(E(W))\sigma_w^2$$

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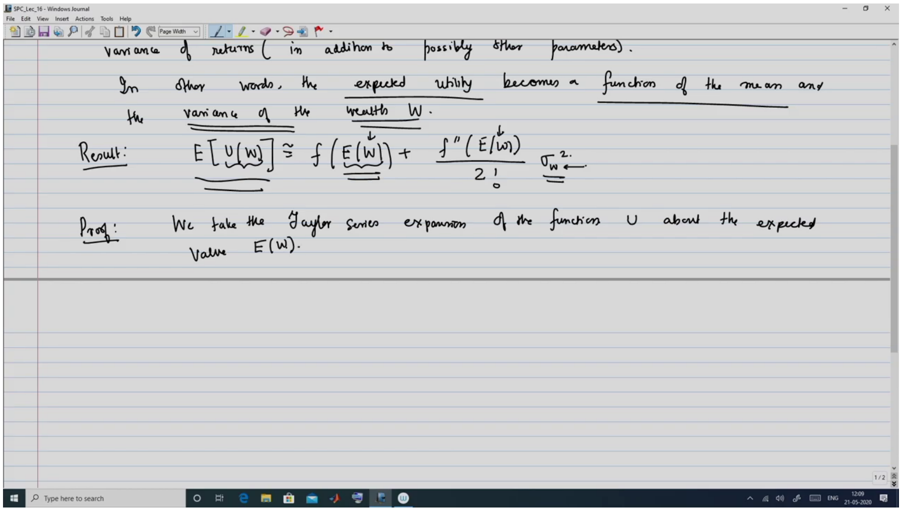

So, now, by looking at the form of the result one can immediately guess that this has been derived I making use of the Taylor series. So, accordingly we take the Taylor series expansion of the function U. So that means, we take the Taylor series expansion of this $U(W)$ about, now since I have the term $E(W)$. So, this indicates to me that I will take that Taylor series expansion about the expected value $E(W)$.

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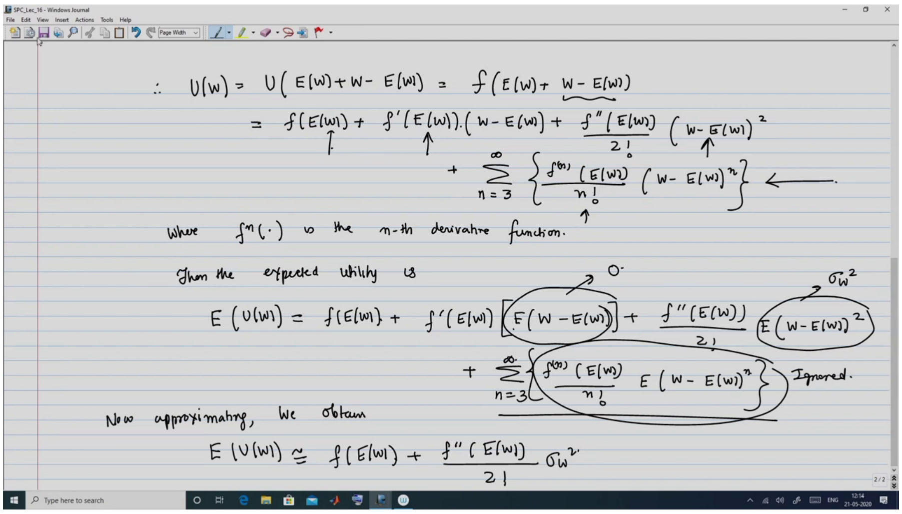

So, the derivation of the results are shown above.

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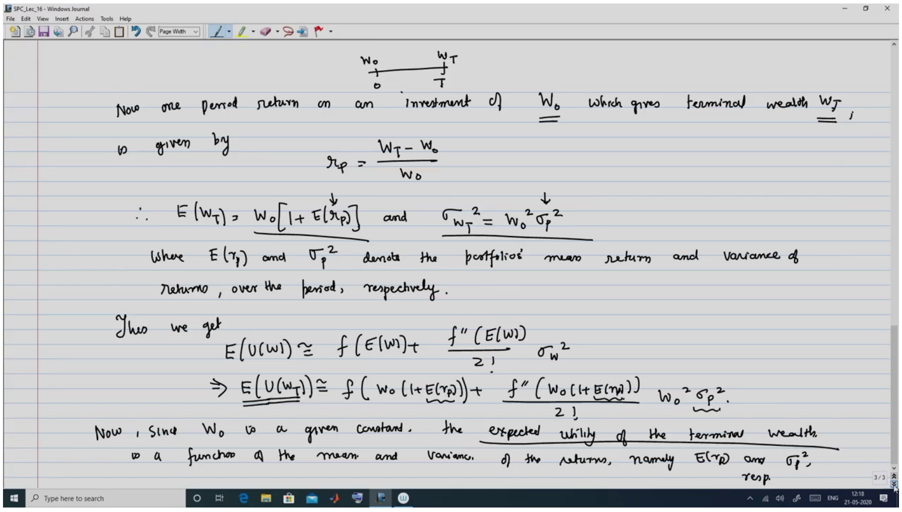

Now we recall that the one period return on an investment of W_0. So, W_0 which gives terminal wealth W_T so; that means, you have a time window 0 to T you start off with a wealth W_0 and your terminal wealth is W_T. So, W_0 is deterministic and W_T is random this is given by the following relation. So, this will be

$$r_p = \frac{W_T - W_0}{W_0}$$

So,

$$E(W_T) = W_0[1 + E(R_p)], \sigma_{W_T^2} = W_0^2 \sigma_p^2$$

Now, we can easily derive the expression for $E[U(W_T)]$.
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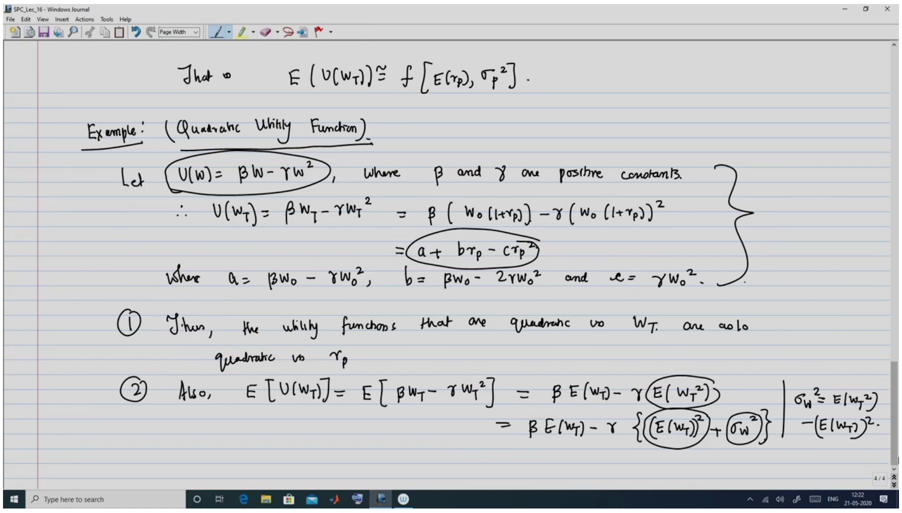

So, that is

$$E[U(W_T)] = f[E(r_p), \sigma_p^2]$$

So, let us now look at an example to elucidate it in a better manner. So, for this we look at a quadratic utility function.

So, let

$$U(W_T) = \beta W_T - \gamma W_T^2,$$

and this can be written as

$$U(W_T) = a + br_p - cr_p^2$$

And secondly, also expected utility of the terminal wealth what is this going to be? This is shown in the figure.

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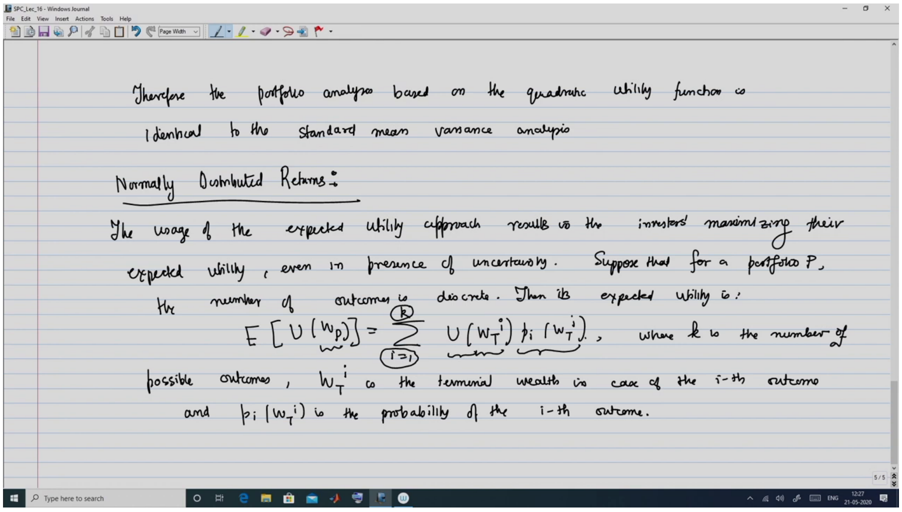

So, therefore, the portfolio analysis based on the quadratic utility function is identical to the standard mean-variance analysis. So, this means basically for example, if I am trying to maximize the expected utility, then this would be the same as minimizing the sigma W square, for example. So, this is one place you can reconcile this with the mean variance framework.

So, next we come to a normally distributed returns. So, the usage of the expected utility approach results in the investors maximizing their expected utility, even in presence of uncertainty.

So, suppose that for a portfolio P, the number of outcomes is discrete. Then its expected utility is which will denote by

$$E[U(W_p)] = \sum_{i=1}^{k} U(W_T^i) P_i(W_T^i)$$

So, $U(W_p)$ takes k possible different values as a random variable which will denote as $U(W_T^i)$ with the corresponding probability be given by P_i. So, this is something like summation random variable into the corresponding probability to get the expectation and remember that this is for the discrete case.

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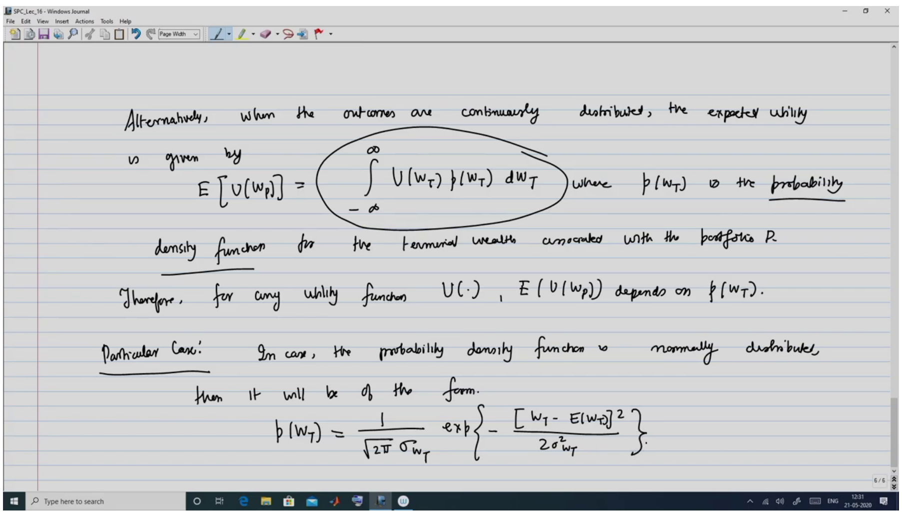

Now, alternatively. So, we will take a cue from here the way the expected utility is defined here and alternatively, when the outcomes are continuously distributed, the expected utility is given by

$$E(U(W_p)) = \int_{-\infty}^{+\infty} U(W_T)P(W_T)dW_T$$

Where, I have introduce this new notation $P(W_T)$ is the probability density function for the terminal wealth associated with the portfolio P.

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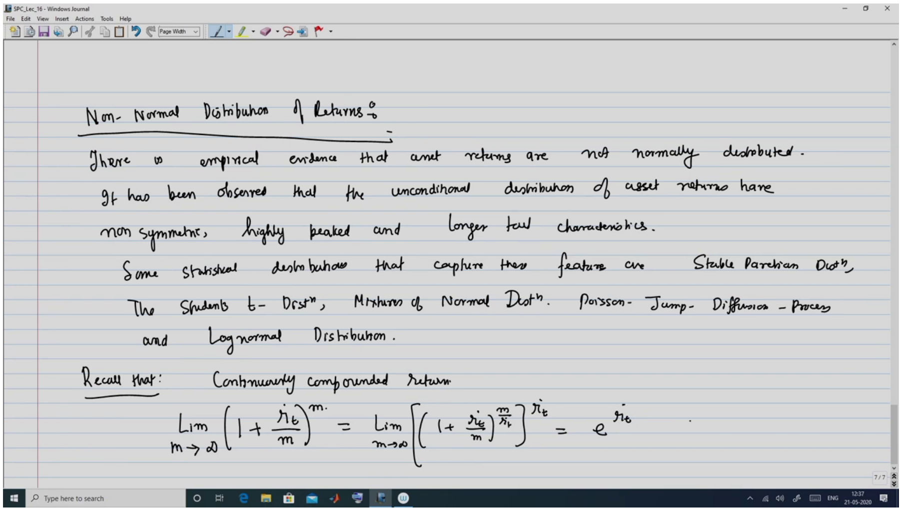

Now, let us move on to non normal distribution of returns. So, let me first begin with a motivation for this. So, I state that there is empirical evidence; that means, data based evidence that asset returns are not normally distributed. It has been observed that the unconditional distribution of asset returns have non symmetric, highly peaked and longer tail characteristics.

So, there is a basic assumption that you know the returns of an asset are normally distributed. So, which means that they are going to be following as normal random variable, and then because a normal distribution is symmetric, one would assume that this using the normal distribution to look at asset return would be consistent with the actual distribution of returns.

However, in practice where you look at the returns and you look at, if you plot the histogram and then smoothen it out, then you notice that there are certain characteristics that are exhibited by the actual empirical data that are not consistent with the normal distribution. And I have identified three of them here one of them it is non symmetric; that means, it is skewed on one side, one of them is highly peaked so; that means, that the peak of the bell shaped normal distribution is much more in case of the actual.

So, in case of the actual empirical distribution so; that means, that the peak will be higher if you plot the actual distribution as compared to the normal distribution with the same parameters and it has longer tail characteristics.

So, compared to the thickness of the tails of the two distributions on both sides in case of normal distribution, in reality the distribution of returns will exhibit a tails which are thicker than what you would get in case of a normal distribution, ok. So, next what you will do is that, we will enumerate some of the distributions which are not normal, but which are closer to reality in terms of fitting what is going to be this non symmetric and highly peaked and longer tail characteristic that are exhibited by the returns. So, accordingly some statistical distribution that capture these features are stable paretian distribution, the students t distribution, mixtures of normal distribution, Poisson-Jump - Diffusion-Process and finally, log normal distribution and we are going to discuss this log normal distribution in a little more detailed, ok.

So, just to do a recap let us recall that for continuously compounded return what did we have? So, we had that if the rate of interest rate of return is say $\dot{r}_t$ and if this if the interest payments are happening m times in a year, then the compounding factor is going to be 1 plus r t dot over m raised to m.

And if you want this to happen in a the compounding a happening in a continuous manner, you would recall that we had let m tends to infinity and this gives $e^{\dot{r}_t}$

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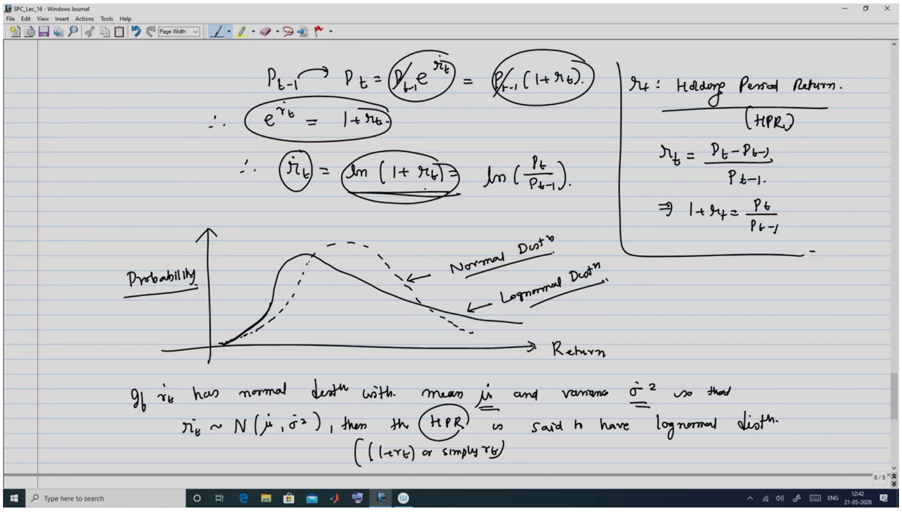

So, if I were to plot the return against the probability of that return based on historical data, if it was normal then the data would fit this normal distribution curve, but in reality, it is going to be fit something like this. So, this is the log normal distribution. So, if I say that the asset returns or fitting a log normal distribution, it is going to look something like this as compared to the normal distribution.

So, now if $\dot{r}_t$ has normal distribution with mean $\dot{\mu}$ and variance $\dot{\sigma}^2$ so that r t dot follows normal distribution with mean me mu dot and variance sigma dot square, then the HPR that is the holding period return r_t, this is said to have log normal distribution alright. So, its a HPR which is $(1+r_t)$ alright.

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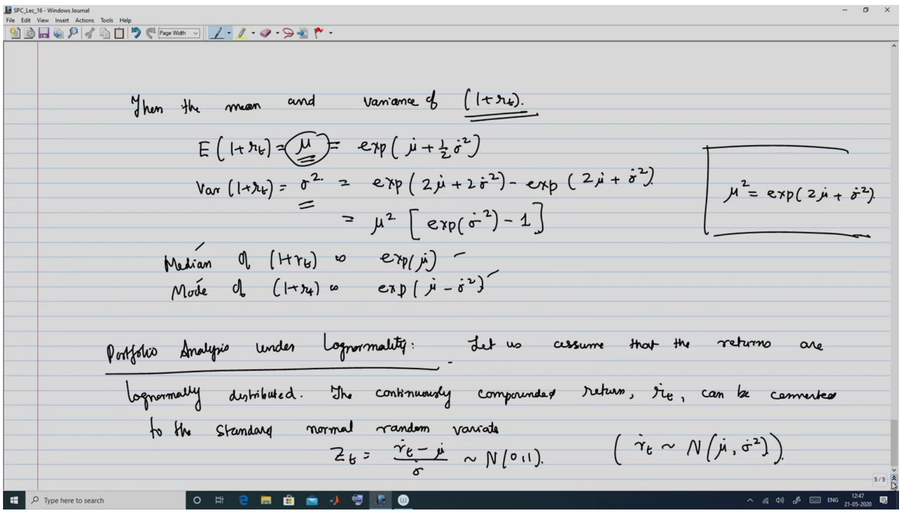

So, the expectation, variance, median and mode are given.

So, now that we have talked about the properties of log normality in the context of recognizing the fact that the historical returns data based on the empirical analysis results in the historical data being more amenable and more realistic in terms of fitting the distribution to a log normal distribution as compared to a normal distribution.

So, now that we have talked about the log normality of returns let us now move on to the discussion on log normality in the context of portfolio analysis. So, therefore, we start this topic of portfolio analysis under log normality. So, for this let us assume that the returns are log normally distributed.

Now, the continuously compounded return r t dot can be converted to the standard normal random variate. What is this? This is

$$Z_t = \frac{\dot{r}_t - \dot{\mu}}{\sigma} \sim N(0,1)$$

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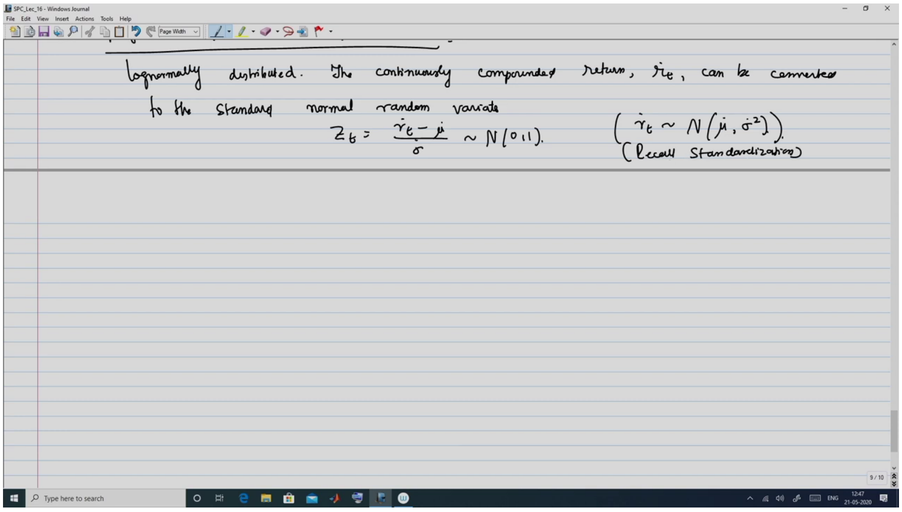

So, I have just done the transformation we have already or standardization. So, I just recall the concept of standardization, when you had introduce aspects of probability theory.

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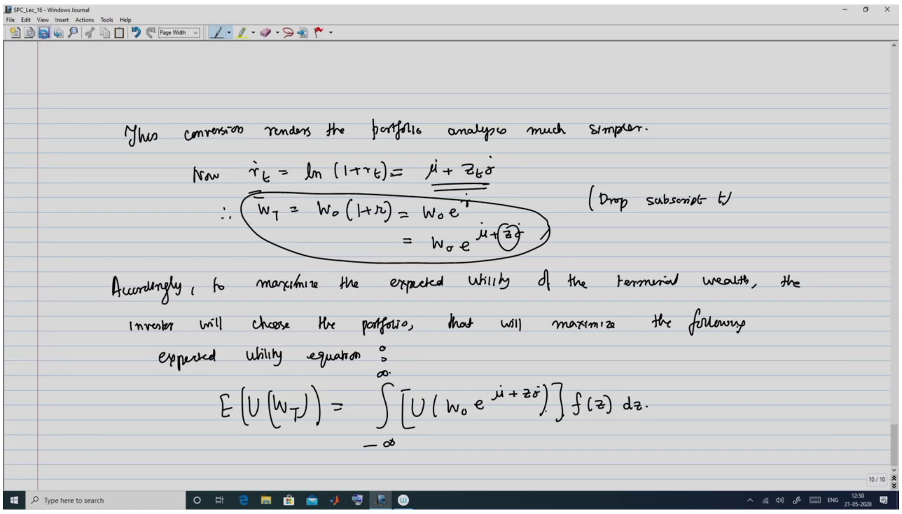

So, this conversion or standardization renders the portfolio analysis much simpler. So, it makes the analysis much simpler. Now,

$$\dot{r}_t = \dot{\mu} + Z_t\dot{\sigma}$$

So,

$$W_T = W_0 e^{\dot{\mu} + z\dot{\sigma}}$$

So, accordingly to maximize the expected utility of the terminal wealth the investor will choose the portfolio that will maximize the following expected utility equation.

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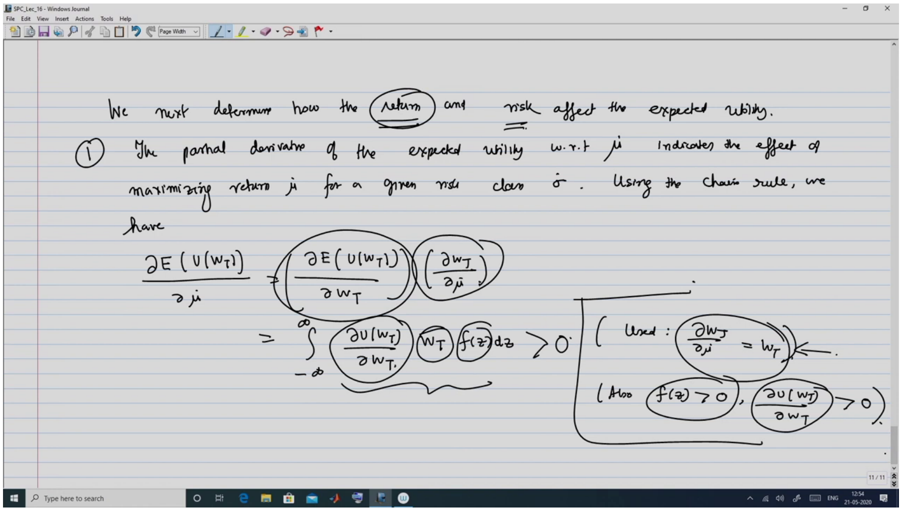

So, we next determine how the return and risk affect the expected utility alright. So, we look at these two points in terms of sensitivity to return and sensitivity to risk. So, accordingly we will start off with this form of the expected utility and take the partial derivative of this with respect to mu dot and sigma dot square.

So, the partial derivatives of the expected utility with respect to the return that is mu dot, this indicates the effect of maximizing return mu dot for a given risk plus sigma dot. So, accordingly using the chain rule we have. So, I will take the partial derivative of the expected utility of the terminal wealth with respect to mu dot.

Now, the chain rule will give me the partial derivative of the expected utility of terminal wealth with respect to the terminal wealth multiplied by the terminal wealth partial derivative with respect to mu dot.

Now this is going to be given by. So, first of all let me look at this term. So, I will take this partial derivative with respect to W_T. So, I will take the partial derivative of this with respect to W_T under integration sign. So, this will give me

$$\int_{-\infty}^{+\infty} \frac{\partial U(W_T)}{\partial W_T} W_T f(z) dz$$

So, accordingly this term is positive, this term obviously, is positive and $f(z)$ is always positive. So, this means that since all the terms are positive. So, this is always going to be greater than 0 alright.

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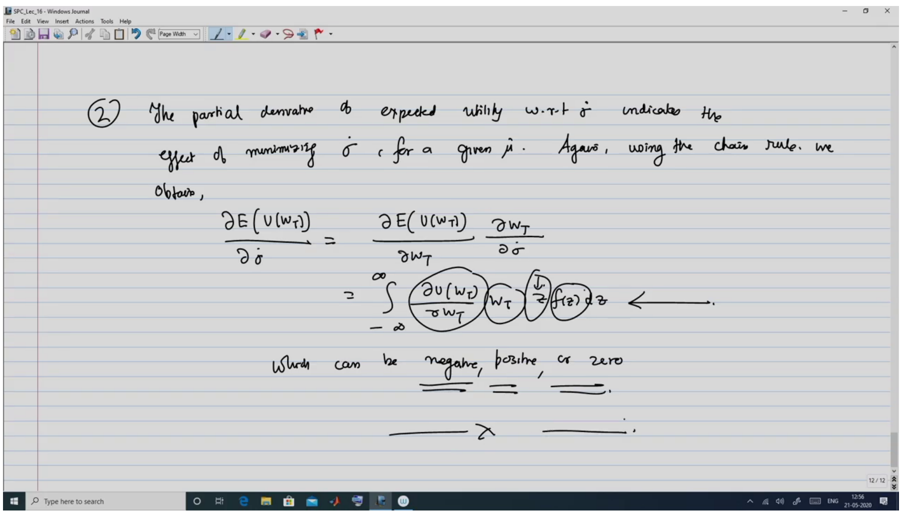

So, now, we come to the next point. So, the second point is we take the partial derivative now with respect to the sigma dot. So, the partial derivative of expected utility with respect to sigma dot indicates the effect of minimizing sigma dot for a given mu dot.

So, again using the chain rule, we obtain, the partial derivative of the expected utility of terminal wealth with respect to sigma dot. This again as before is going to be a partial derivative of the utility of the expected utility with respect to the wealth and then we take the partial derivative of the wealth with respect to sigma dot.

And this turns out to be

$$\int_{-\infty}^{+\infty} \frac{\partial U(W_T)}{\partial W_T} W_T z f(z) dz$$

So, this is the same as before except that now we will have this additional z here and this can be so, which can be negative, positive, or zero right.

So, remember that this is positive, this is positive and this is positive, but we cannot you know depending on what your z is this entire term here can either be negative positive or zero ok. So, this brings us to the end of this lecture. So, just to do a recap what we did today is that we looked at the portfolio theory in the paradigm of utility functions.

And in particular what we did was we looked at an example of a quadratic function quadratic utility function and then we talked about the expected utility and its maximization and we looked at a particular example where we motivated why a log normal distribution is a better fit to the through the return distribution of any particular asset during the holding period return.

And we looked at the sensitivity of the expected utility of that in terms of the expected return and variance of a continuously compounded convention being used. So, in the next class we will continuing our discussion on a portfolio theory in the context of utility functions and then we will move on to looking at certain criteria's analogous to the mean, variance criteria's, but extended in this case to the non mean variance framework.

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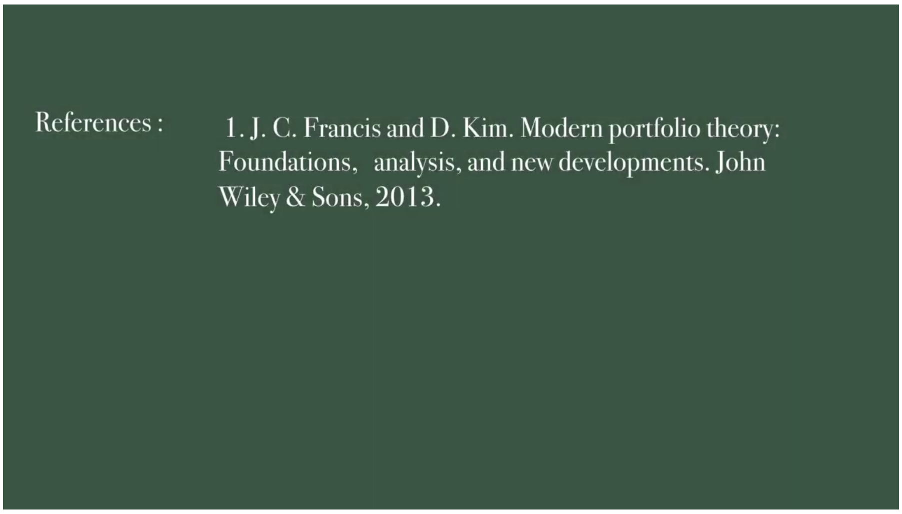

Thank you for watching.