Mathematical Portfolio Theory
Module 04: Non-Mean-Variance Portfolio Theory
Lecture 15: Absolute Risk Aversion and Relative Risk Aversion

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Module – 04 Non-Mean-Variance Portfolio Theory Lecture – 03 Absolute Risk Aversion and Relative Risk Aversion

Hello viewers welcome to this next lecture on the MOOC course on Mathematical Portfolio Theory. So, call that in the previous two classes we focused ourselves on the utility the expected utility and some of the properties in terms of risk attitude of investors namely investors who are risk averse, who are risk loving and who are risk neutral. And we quantified them in terms of the utility as a function of the wealth level. And we specified the fact that $U'(W) > 0$ in all cases and the $U''(W)$ that is the second derivative of U will change depending on the risk attitude of the particular investor. So, in today's class we will delve into a little more detail on this aspect and we will introduce and discuss about two concepts namely the Absolute Risk Aversion and the Relative Risk Aversion.

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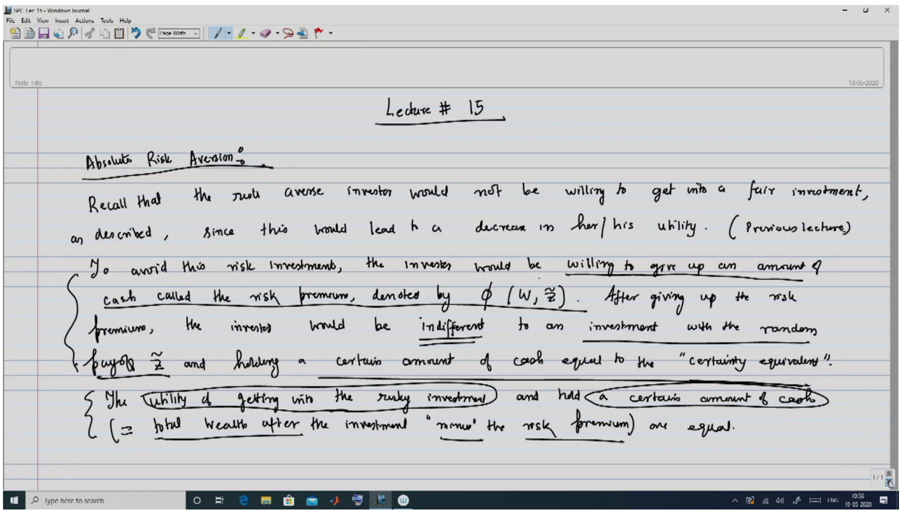

So, accordingly we start this lecture with the concept of Absolute Risk Aversion, ok. So, we begin by recalling that the risk averse investor would not be willing to get into a fair investment, recall we had looked at that investment where you could gain or lose 300 on an investment of 1000 as we had described earlier, since and the reason this happens is that it would lead to a decrease in the investors utility. So, for this, I can refer to the previous lecture.

Now, in order to avoid this risky investment. So, what I am going to state is an interpretation of certainty equivalent. So, in this case the investor would be willing to give up an amount of cash called the risk premium, denoted by phi. So, you have already introduced this notation $\phi(W, \tilde{Z})$. Now, after giving up the risk premium the investor would be indifferent to an investment with the random $\tilde{Z}$ and holding a certain amount of cash equal to the "certainty equivalent" alright.

So, just to elaborate this so, what I am saying is that we have already seen the example of the risk averse investors and we have said that the risk averse investor would not be willing to get into this investment, recall the particular example of the investment was that you start off with a wealth level W_0 and the probability with probability half it could you could have a gain of $\tilde{Z}$ which in the example was 300 or you can have a loss of $\tilde{Z}$ again with the probability of half.

So, the reason for avoiding this investment is that this would lead to the decrease in the utility of the investor. Now in order to avoid this risky investment the investor would be willing to give up an amount of cash called risk premium.

So; that means, that once the investor gives up / is willing to give up this risk premium then when you look at the investment in a random payoff z tilde; that means, this particular investment is holding an amount of cash equivalent to certainty equivalent.

Then after giving up the risk premium the investor would be indifferent to this two choices namely the investment in the fair or this risky proposition and holding the an amount of money same as the certainty equivalent alright.

So, now the utility of getting into the risky investment and holding a certain amount of cash which is equal to total wealth after the investment minus the risk premium they are equal. So, this is just a manifestation of the previous term that I said that it there will be indifferent to an investment in the random payoff z. So, this is equivalent to the statement that utility of getting into a risky investment.

And the second point of certainty equivalent, this is equivalent to a holding a certain amount of cash and remember certain amount of equivalent will be given by total wealth minus risk premium and they are equal. So, this statement is essentially just stating the preceding statement in a slightly different form ok.

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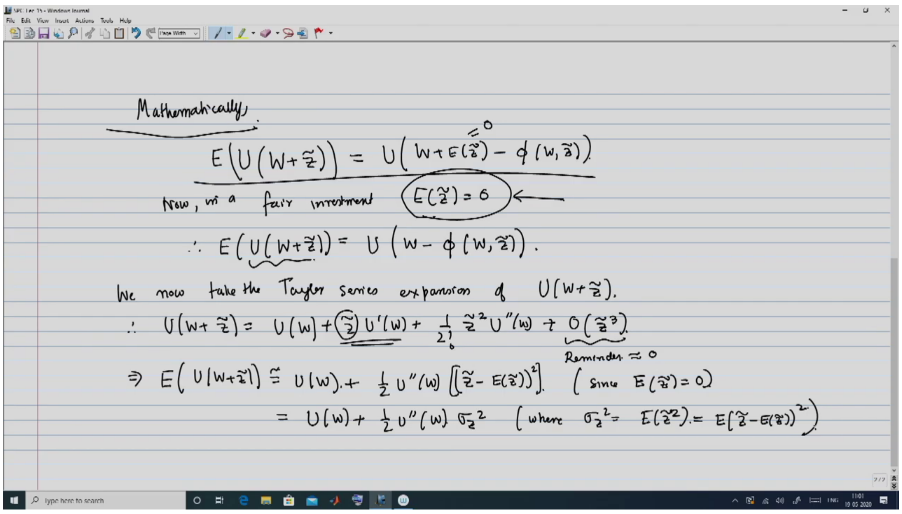

So, mathematically this would mean the following. So, you have the wealth W. So, you start off with a wealth level W and then you get an additional amount of z tilde which could be plus 300 and minus 300 if you consider specific example and you consider utility of this.

Now since this is a random variable so, I have to calculate the expected utility of this and this must be:

$$E(U(W+\tilde{Z})) = U(W+E(\tilde{Z})) - \phi(W,\tilde{Z})$$

So, this is the mathematical manifestation of my previous statement given here.

Now, in a fair investment what we will have, we will have

$$E(\tilde{Z}) = 0$$

So, therefore, this line I can write

$$E(U(W+\tilde{Z})) = U(W-\phi(W,\tilde{Z}))$$

We now take the Taylor series expansion of $U(W+\tilde{Z})$ that is this component.
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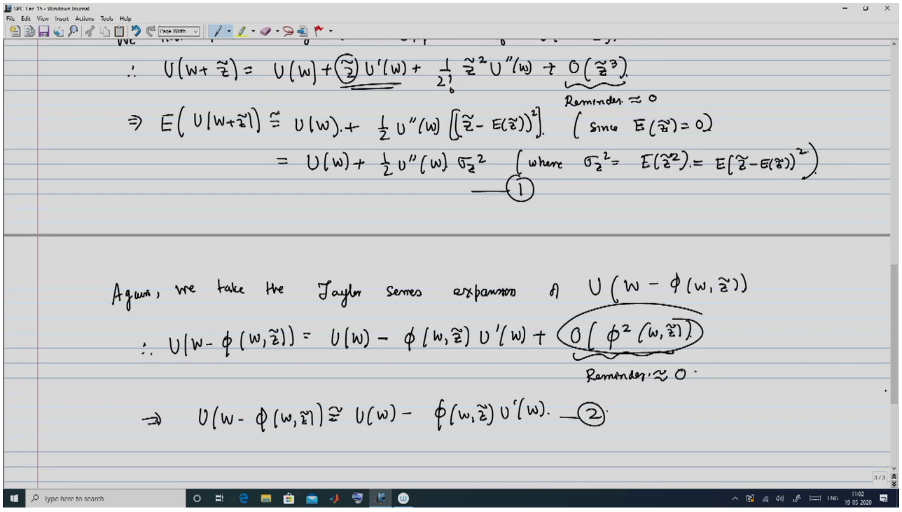

See the derivations for the required result.
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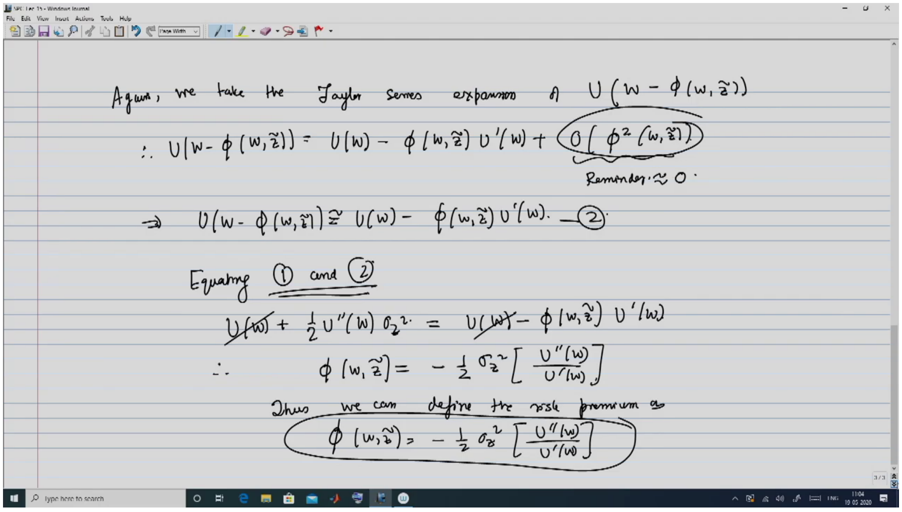

So, finally it si proved that:

$$\phi(W,\tilde{Z}) = -\frac{1}{2}\sigma_z^2[\frac{U''(W)}{U'(W)}]$$

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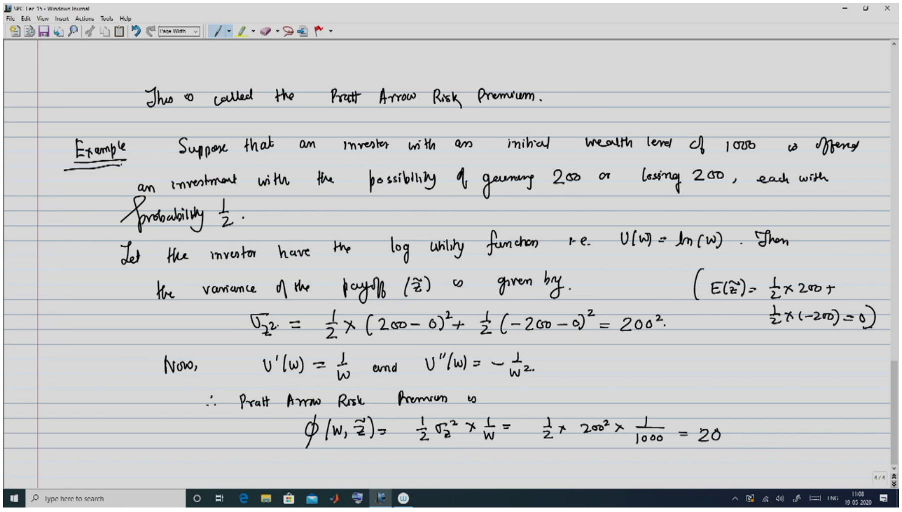

So, this risk premium is called the Pratt so, this is a very classical concept. So, it is the Pratt Arrow Risk Premium ok. So, let us now look at an example. So, suppose that. So, it is the same situation as the example considered in the previous lecture.

So, we consider an investor with an initial wealth level of 1000 who is offered an investment with the probability, actually I should say with the possibility of gaining 200 or losing 200; that means, it can go up to 1200 or come down to 800, each with probability, with probability half, ok.

Now, if I want to discuss the Pratt arrow risk premium. So, I have to talk about the utility of the investor. So, if suppose that the investor has the log utility function. So, that is $U(W) = \ln(W)$ then the variance of the payoff what is the payoff that is z tilde is given by sigma z square. So, remember here expectation which is half into 200 plus half into minus 200 this is equal to 0, i.e., $E(\tilde{Z}) = 0$

So, σ_z^2 is going to be half into the random variable 200 minus 0 square plus half into the random variable minus 200 minus 0 square. So, this turns out to be equal to just 200 square. Now you see that

$$U'(W) = \frac{1}{W},\ U''(W) = -\frac{1}{W^2}$$

So, therefore

$$\phi(W, \tilde{Z}) = 20$$

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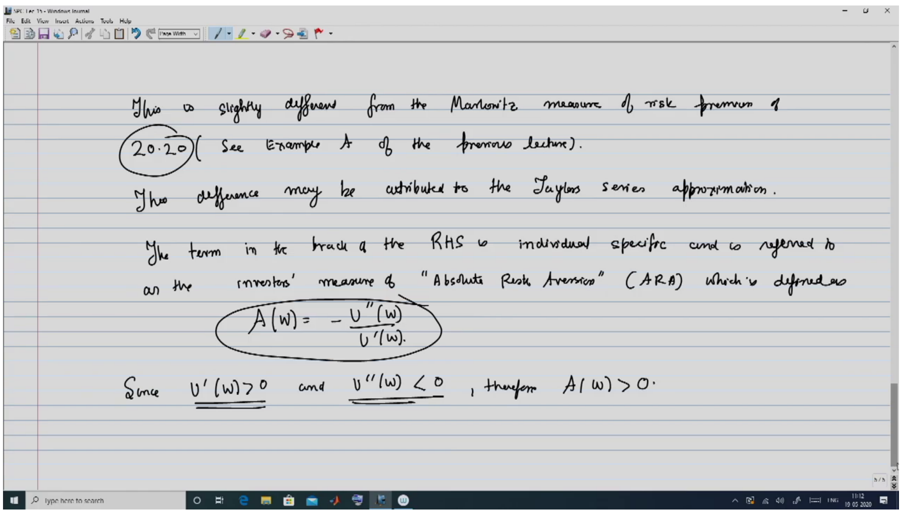

Now, this risk premium calculated this way, this is slightly different from the Markowitz measure; that means, the example A that you did in the previous lecture of risk premium of 20.20. So, you please see example A of the previous lecture.

Now, this difference of 20 and 20.20 may be attributed to the errors that keeps in because of Taylor series approximation. So, this is some sort of a truncation error ok. Now the term so, we refer back to the Pratt arrow risk premium.

So, the term in the bracket of the RHS is individual specific and is referred to as the investors measure of "Absolute Risk Aversion" or ARA which is defined as

$$A(W) = -\frac{U''(W)}{U'(W)}$$

So, just recall that we had these two terms here this is going to be half sigma square term and this term which involves the utility. So, this term here is investor specific and this term is independent of the investor since it pertains to the specific in risky investment that is being offered to the investor ok. So, now, you have defined what the absolute risk aversion is ok.

Since $U'(W) > 0$ and $U''(W) < 0$. So, I am talking from the point of view of a risk averse investor. So, therefore, we have

$$A(W) > 0.$$

So, now, I will make a couple of observations.

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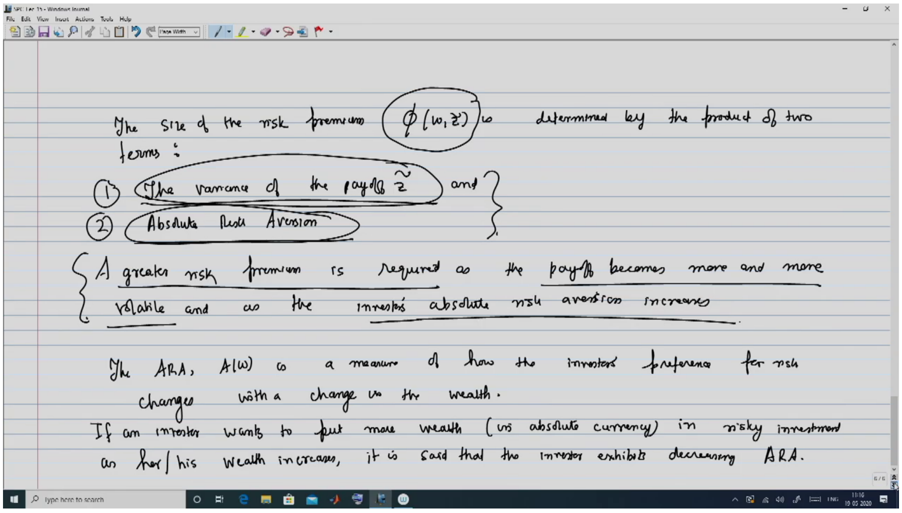

And it is the following the size of the risk premium $\phi(W, \tilde{Z})$ is as I have already pointed out determined by the product of two terms, the first one is that the variance of the $\tilde{Z}$ and so; that means, it is the sigma z square term and the absolute risk aversion ok.

So, a greater risk premium is required as the payoff becomes more and more volatile; that means, as σ_z^2 becomes higher and higher a greater risk premium is required and as the investors absolute risk aversion increases.

So, this statement he follows immediately from the fact that the risk premium phi W tilde is can be decomposed into two quantities, namely, the variance of the payoff z tilde which is denoted by sigma z square and the absolute risk aversion.

So, this risk premium has to be higher under two circumstances the first circumstances is when this variance increases. So, this means that the payoff becomes more and more volatile. So, resulting in sigma z square increasing and consequently the risk premium increases.

The other scenario is that if the risk absolute risk aversion increases all right which is this case then also it will result in an immediate increase of risk premium. So, the increase in risk premium typically can then be attributed these two factors which is why we have identified these two factors here.

So, the absolute risk aversion A of W is a measure of how the investors preference for risk changes with a change in the wealth. So, if an investor wants to put more wealth in absolute currency or sometimes it is called the absolute dollar amount in risky investment as her or his wealth increases it is said that the investor exhibits decreasing absolute risk aversion.

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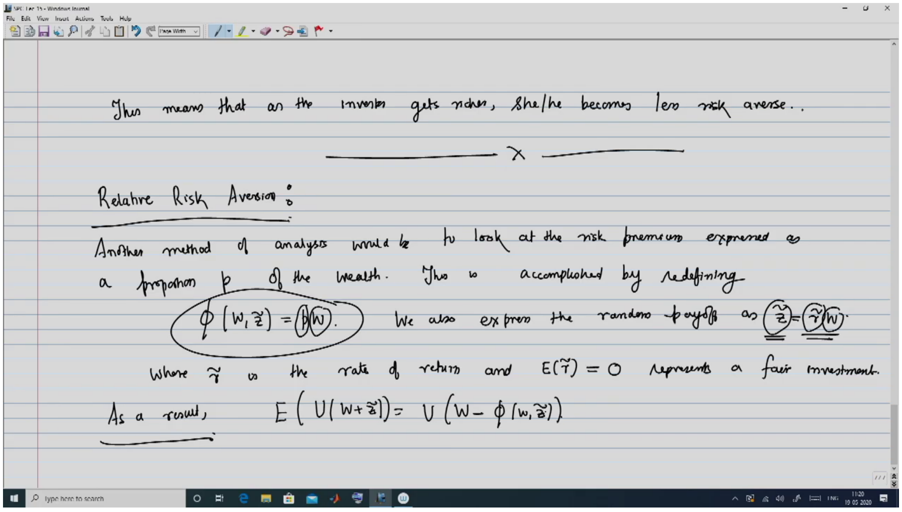

So, this means that as the investor gets richer she or he becomes less risk averse ok. So, this concludes our discussion on absolute risk aversion. And now we move on to the next concept called the Relative Risk Aversion. So, we start off by observing that this is another method of analysis would be to look at the risk premium expressed as a proportion p of the wealth. So, this is accomplished by redefining

$$\phi(W,\tilde{Z}) = P(W)$$

So, the reason why you are doing this basically we want to see the risk premium as a proportion p of the wealth level. So, we also accordingly by the same motivation express the random payoff as what is the random payoff? It was $\tilde{Z} = \tilde{r}W$.

where, $\tilde{r}$ is the rate of return and $E(\tilde{r}) = 0$. As a result,

$$E(U(W+\tilde{Z})) = U(W-\phi(W,\tilde{Z}))$$

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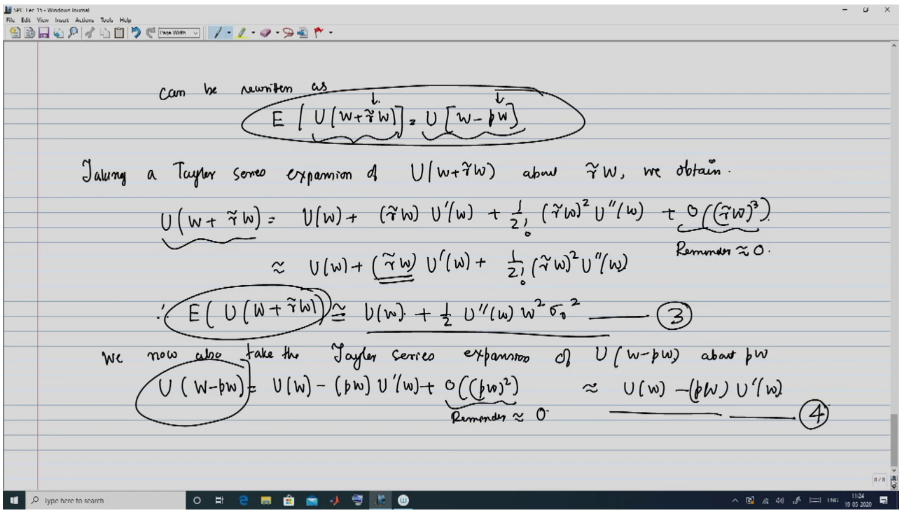

And what we are going to do is we are going to rewrite this as this can be rewritten as

$$E[U(W+\tilde{r}W)] = U[W-PW]$$

The results are described above.

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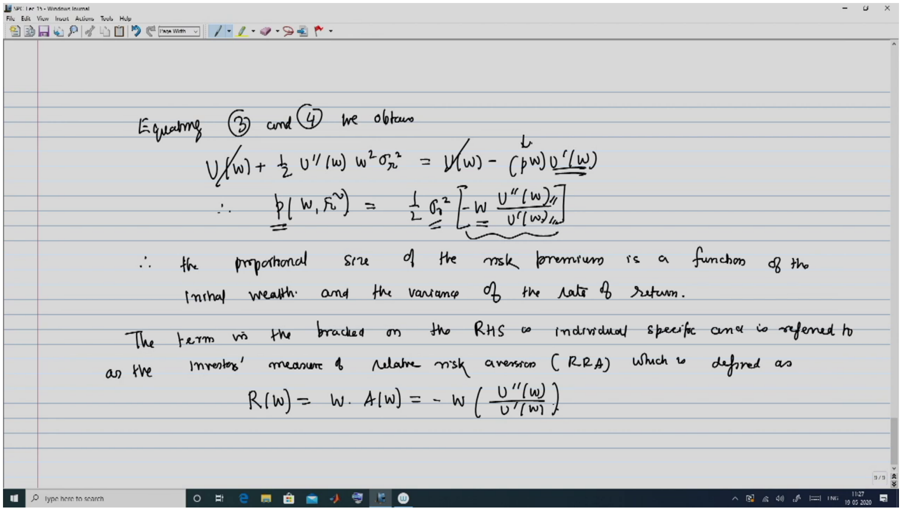

So, equating 3 and 4 we obtain

$$U(W)+\frac{1}{2}U''(W)W^2\sigma_r^2 = U(W)-(PW)U'(W)$$

$$P(W,\tilde{r}) = \frac{1}{2}\sigma_z^2[-w\frac{U''(W)}{U'(W)}]$$

Thus,

$$R(W) = WA(W) = -W\frac{U''(W)}{U'(W)}$$

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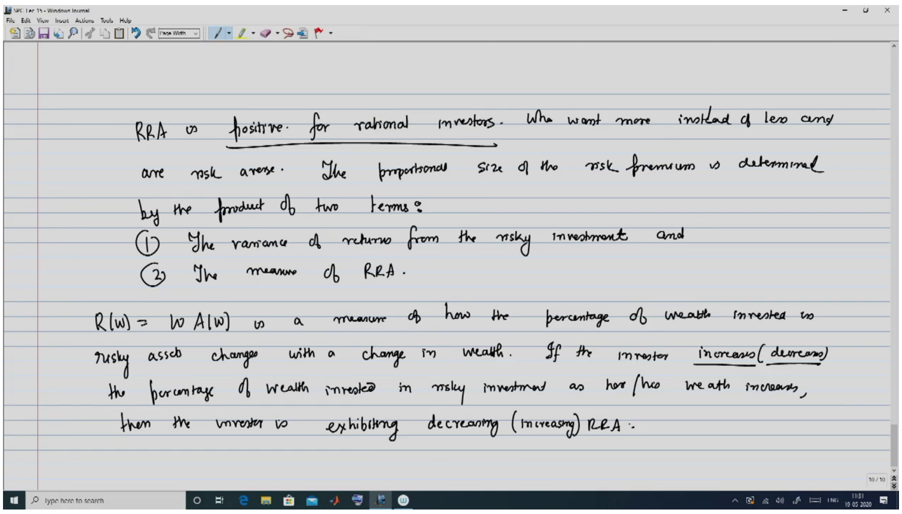

Now, this relative risk aversion RRA is going to be positive for a rational investors because $U''(W) < 0$ and $U'(W) > 0$. So, for rational investors so, accordingly this is positive for rational investors who want more instead of less and are risk averse.

So, we can say that the proportional size of the risk premium; that means, p is determined by the product of two terms. So, the first one is the σ_r^2. So, it this is the variance of returns from the risky investment and secondly, the measure of the relative risk aversion ok.

So, the final punch line on this. So, I will say that $R(W) = WA(W)$ is a measure of how the percentage of wealth invested in risky assets changes with a change in wealth.

So, if the investor increases or equivalently decreases the percentage of wealth invested or in the risky investment as her or his wealth increases then the investor is exhibiting. So, in the case of increases the investor is exhibiting decreasing RRA and in case the investor decreases the percentage of wealth in risky investment then it means that the investor is increasing their relative risk aversion.

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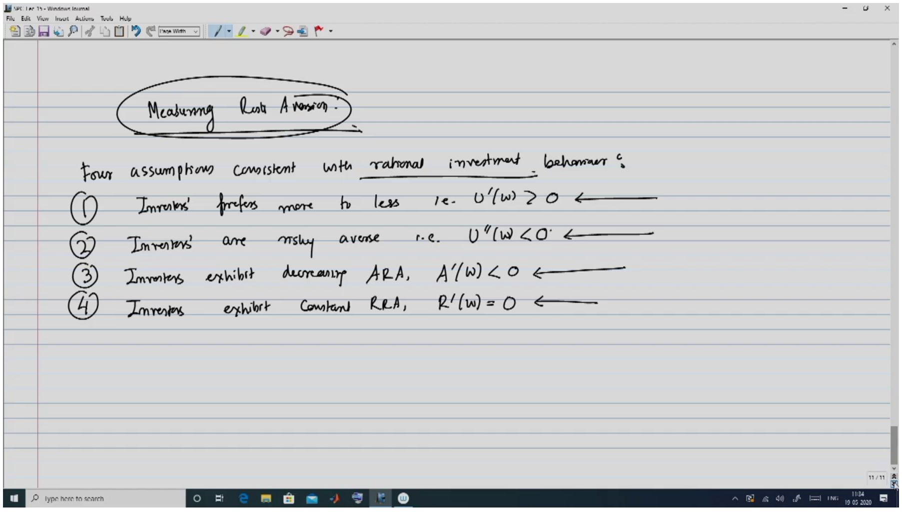

So, we come to the last topic of this lecture this is on a measuring risk aversion. So, I will just state some assumptions related to risk aversion. So, four assumptions consistent with rational investment behavior:

- Prefers more to less: $U'(W) > 0$
- Risk averse: $U''(W) < 0$
- Exhibit decrease ARA, $A'(W) < 0$
- Exhibit constant RRA, $R'(W) = 0$

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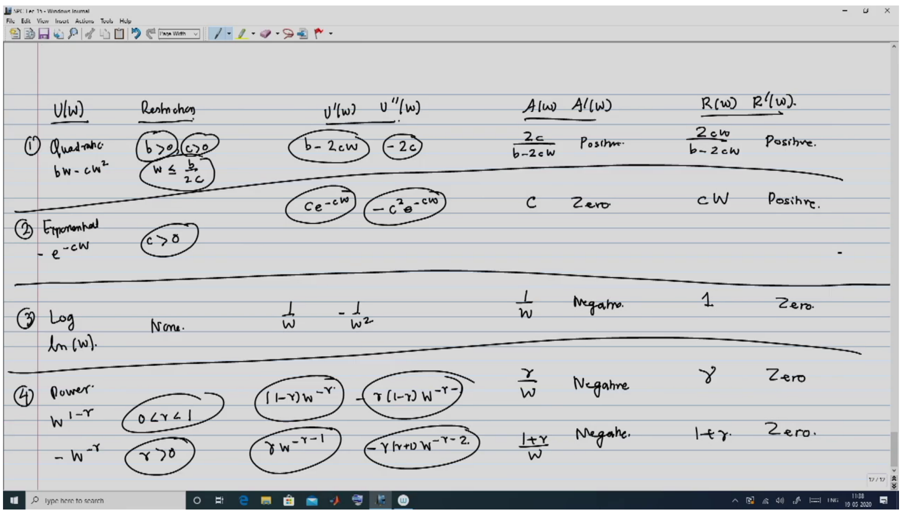

So, now what you do is we look at a few examples of utilities. So, let us look at some utility functions $U(W)$ along with what is the restriction. What is going to be the $U'(W), U''(W)$, what is $A(W), A'(W), R(W), R'(E)$? So, this will be some sort of a ready reckoner which you might need to refer to again and again ok.

Four types of utility function and their characteristics are given in the above table.

So, this concludes our preliminary discussions on utility theory and from the next class onwards we will start looking at the usage of utility theory in the paradigm of portfolio analysis.

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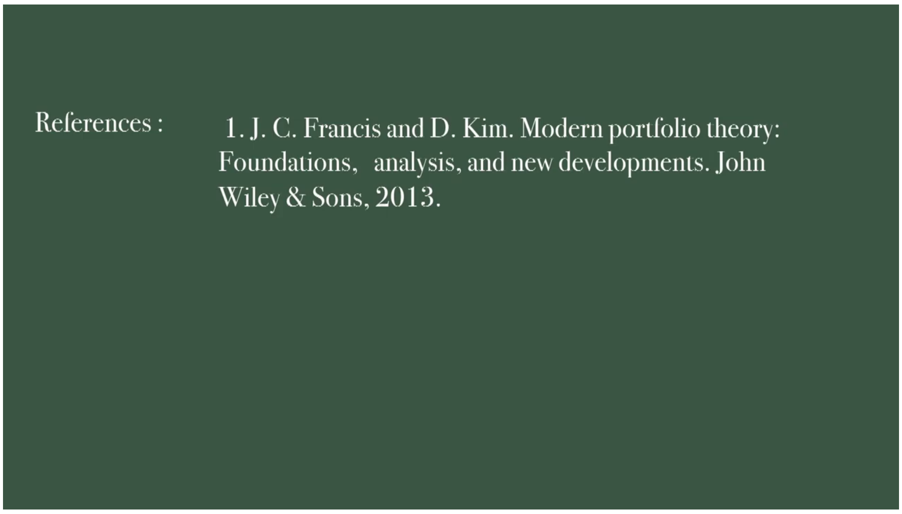

Thank you for watching.