Mathematical Portfolio Theory
Module 04: Non-Mean-Variance Portfolio Theory
Lecture 14: Risk preferences of investors

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Hello viewers, welcome to this next lecture on the MOOC course on Mathematical Portfolio Theory. You would recall that in the last class we started talking about utility theory and we looked at the economic interpretation of the utility functions and how we can actually use the utility theory in terms of assigning utils or numerical values amongst various investment alternatives.

And then we concluded by looking at an example highlighting the fact that when you talk about utility of a wealth level, given their fact that the utility of a wealth level is a random variable we need to calculate what is the expected utility. And, then we are driven by the maximization of the expected utility amongst different alternatives in terms of investments. In today's class I will elaborate a little more on a notion of the risk appetite of individual investors and we will draw motivation from the fact that the last time we talked about three categories of investors in terms of the risk appetite namely risk averse, risk loving and risk neutral. So, let us begin this lecture with a prelude to that discussion and I will then discuss the characteristics of all these three types of investors from the point of view of the nature of their risk attitude.

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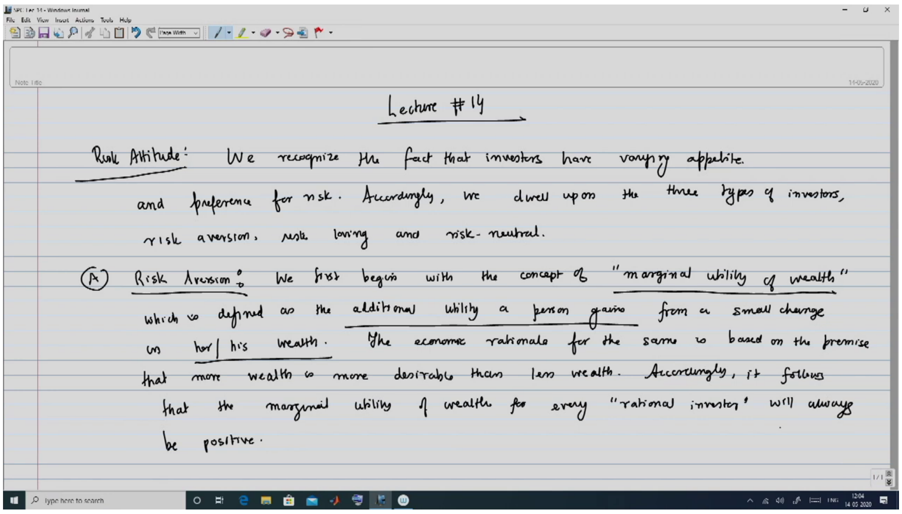

So, we start off with as I mentioned will talk about risk attitude. So, the motivation for this is that we need to recognize the fact that investors have varying appetite and preference for risk. And, accordingly we dwell upon the three types of investors and we have already introduced them in the previous class namely the investors who have a risk aversion, who are risk loving and who are risk neutral.

So, let us start off first with the investor who have the attitude of risk aversion. So, we first begin with the concept of what is known as the "marginal utility of wealth." So, this is a fairly well established economic concept which is defined as the additional utility a person gains from a small change in her or his wealth.

So, the economic rationale for the same is based on the premise that more wealth is more desirable than less wealth which is sort of an obvious statement. And accordingly, it follows that the marginal utility of wealth for every "rational investor" will always be positive.

So, just to sort of look at the crux of the matter we say that this concept of marginal utility of wealth is defined as the additional utility a person gains from a small change in his or her wealth. So; that means, given a certain wealth level say W_1 the corresponding utility is going to be $U(W_1)$. So, if it is changed to if the wealth increases to W_2 then the utility; obviously, will change to $U(W_2)$.

And, then essentially the additional utility; that means, the difference between $U(W_2) - U(W_1)$ as a result of the worth level changing from W_1 to W_2 this is the basis of the concept of marginal utility of wealth. And, accordingly because of the obvious attitude of any investor that more wealth means they essentially get an additional utility.

So; obviously, from that point of view the marginal utility of any rational investor will obviously, always end up being positive.

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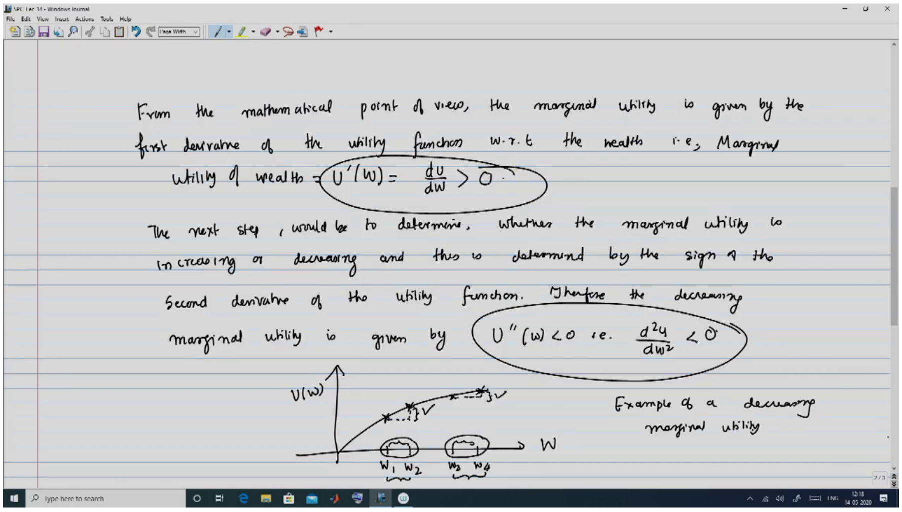

From the mathematical point of view so, since this is a mathematical portfolio theory course so, will have to look at the mathematical perspective of this. So, accordingly the marginal utility is given by the first derivative of the utility function with respect to the wealth that is, a marginal utility of wealth it is given by

$$U'(W) > 0$$

And so, another way of looking at it is that U as a function of W you expect that for a rational investor this is going to be an increasing function. And, that is the reason why the marginal utility which is defined by the additional increase it is essentially the rate of change of the utility vis a vis the wealth level.

So, that is the reason why the definition of marginal utility in order to capture that relative change in the utility corresponding to a unit change in the wealth level as the marginal utility. And, since this is an increasing function so; obviously, the marginal utility defined by U prime of W that is going to be positive.

The next step would be to determine whether the marginal utility... So, we now look at the nature of the marginal utility itself. So, you have to determine whether the marginal utility is increasing or decreasing, and this is determined by the sign of the second derivative of the utility function. So, therefore, the decreasing marginal utility, so it turns out that this is actually decreasing.

So, the decreasing marginal utility is given by

$$U''(W) < 0$$

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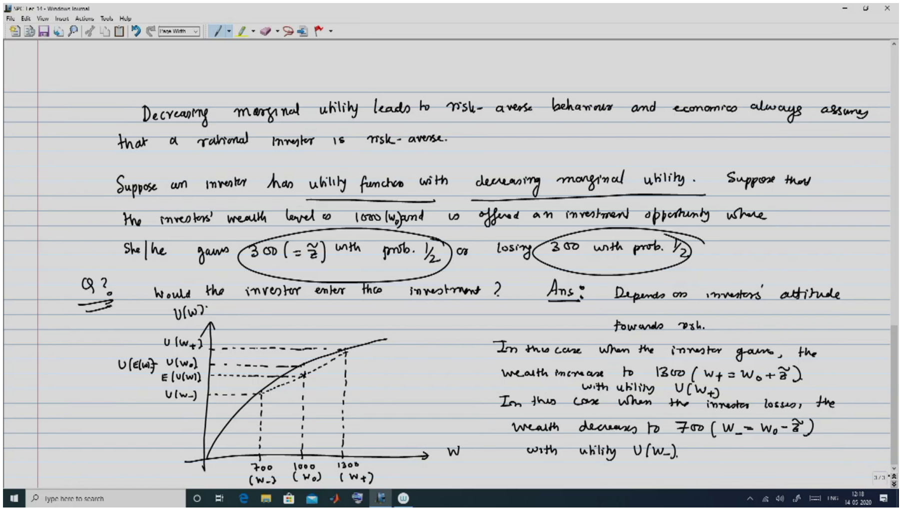

So, in summary what we can write is that decreasing a marginal utility leads to risk averse behavior and economics always assumes that a rational investor is risk averse ok. Now, we would make a sort of a comparative study of the three types of investors in terms of the risk appetite through a specific example which will then also at the same time reconcile with a general setup.

So, accordingly let us begin with the case of an investor. So, suppose an investor has a utility function with decreasing marginal utility. So, it is basically $U(W)$ will have $U'(W) > 0$ and $U''(W) < 0$. Now, suppose that the investor's wealth level is 1000 and is offered an investment opportunity where the investor can gain an amount of 300. So, let me just identify it to some $\tilde{Z}$ with probability half or losing 300 with probability half. And the question now this investor is risk averse as because I have said it is a decreasing marginal utility.

So, the question is that should the or would the so, we need to see it from the point of view of the investor that whether he or she would be interested to enter this investment opportunity? And the answer to this is that it depends on investor's attitude towards risks. So, I am saying that it depends on investor attitude towards risk because I am going to use the same example to look at all the three cases ok.

So, let us look at graphically let us see what this situation looks like. So, this is my wealth and this is my $U(W)$, now since I have said that the investor's utility function are decreasing marginal utility. So, the graph will look something like this and let us identify the initial amount of 1000 with the notation W naught and the gain can be 300 which means that either it can go up to 1300. And, let us identify with W subscript plus and it can go down to 700 let us identify this with W minus ok.

So; that means, the corresponding utility of W minus will be here, the corresponding utility for W_0 is here and this is nothing, but utility of expected wealth before the investment and for W plus the utility is going to be U of W plus. Now once you have made the investment remember that it can either go down to 700 with probability half or it can go to 1300 with probability half. So, the expected utility will be somewhere here. So, this means that this is going to be the expected utility. So, this is going to be the expected $U(W)$. So, in this case when the investor gains the wealth increases to 1300 which I will denote as $W_+ = W_0 + \tilde{Z}$. So, identify 1000 as W naught so, please make a note of it. Now in the now in this case so, the other alternative is in this case when the investor loses then the wealth decreases to 700 and you will identify this with $W_- = W_0 - \tilde{Z}$. And in the first case the utility is $U(W_+)$ and in the second case the utility is $U(W_-)$ minus ok.

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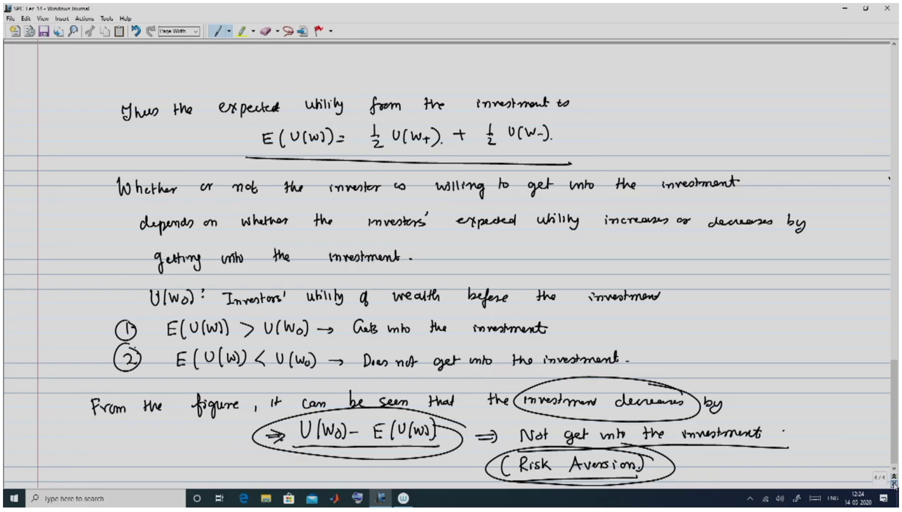

So, thus the expected utility so,

$$E[U(W)] = \frac{1}{2}U(W_+) + \frac{1}{2}U(W_-)$$

So, it is obvious that the investor has an amount of 1000 or W naught and they are, obviously, going to get into the investment only if they see that they have the possibility of the expected utility being more than the utility of the amount that they are currently holding and that is what is going to drive home their decision ok. So, let us now introduce a few notations. So, remember that W naught was the initial wealth level. So, this is $U(W_0)$ is going to be the investors' utility of wealth before the investment now there are two possibilities. If $E(U(W)) > U(W_0)$ and $E(U(W)) < U(W_0)$

So, remember just going back to the graph you observe carefully here this is the wealth level that the investor would have without making the investment and this is the wealth level that they would they are expected to have if they invest. And, you observe that this wealth level after investment is less than this wealth level.

And the difference between these two, that means,

$$U(W_0) - E(U(W))$$

this is going to be the investment decrease, as a result of the investment or the value decrease of the utility. And, then that is the reason why the investor is not going to get into the investment because the investor is risk averse. And, this has happened because the graph is a concave graph that is the reason why we have this quantity ending up being negative.

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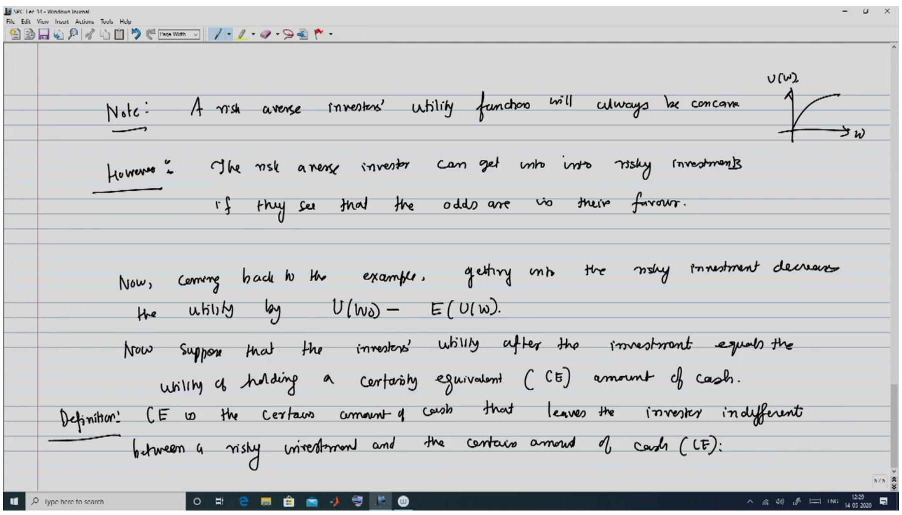

Now, note that a risk averse investors utility function will always be concave right. So, this means that it is going to look something like this. Now; however, the risk averse investor can get into risky investments if they see that the odds are in their favor. So; that means, if the expected utility is going to be more than the utility of their current wealth level then of course, the risk averse investors are likely to get into the investment.

So, risk aversion does not mean that the investor will not take any risk it just simply means that they were willing to take the risk only if they see that their expected utility which is a manifestation of their likelihood of making you know a gain is more than the whatever is their initial investment. So, U of U W if it ends up being greater than the or U of W naught then; obviously, they will get into a risky investment is just that they are going to be their this need adverse nature will ensure that they will have this criteria set up in order to determine whether they want to get into the risky investment or not ok.

So, now we come back to the example. So, for the point of view of the investor getting into the risky investment decreases the utility by

$$U(W_0) - E(U(W))$$

ok. So, now, suppose that the investor's utility after the investment equals so the utility of holding a certainty equivalent which will abbreviate with CE amount of cash. So, the definition for certainty equivalent is the certain amount of cash that leaves the investor indifferent between a risky investment and the certain amount of cash which is the certainty equivalent.

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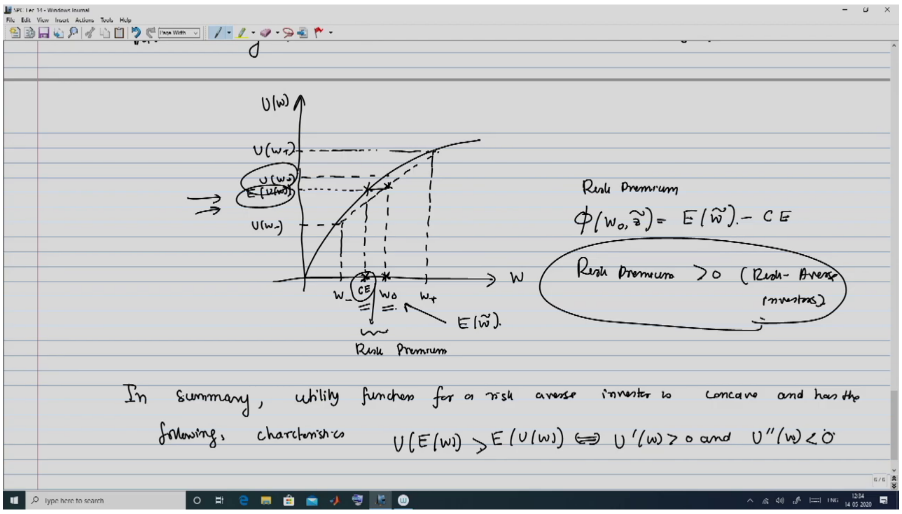

So, we look at this revisit the graphical interpretation of this. So, you have W and so, I just want to add one more thing.

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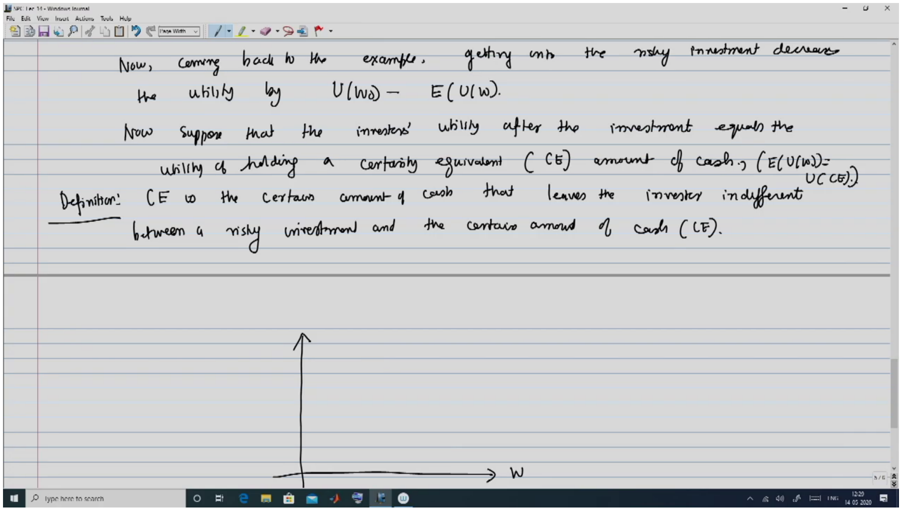

So, here this means that you will have

$$E(U(W)) = U(CE)$$

because so, this gives the definition of certainty equivalent. So, in this case so, let us look at the graph again.

Now, please understand that the expected utility of W. So, what you could have is that this expected utility of W if it is equivalent to the utility of a certain wealth level here then this wealth level is called certainty equivalent. So, just to explain this, so, what we have is that once we have made the investment then the corresponding expected utility.

So, in the previous example it was given by half into U of W plus and half of U of W minus this gives the expected utility to be this value here. Now, we extrapolate this utility value of utility to hit the utility curve and then I drop down to calculate the corresponding value on the wealth axis W which gives me this utility. So, these certain this amount of money is what you call as the certainty equivalent and it is different from W_0 in this case.

So, accordingly what happens is that. So, there is a difference between W naught which is the original amount and the certainty equivalent or the amount that gives the same utility as the expected utility. So, this difference that we have here this is known as the risk premium and the risk premium this is given

$$\phi(W_0, \tilde{Z}) = E(\tilde{W}) - CE$$

So, in this case the risk premium so, here W_0 is what we write as $E(\tilde{W})$ so, the risk premium is nothing, but this is going to be always greater than 0 for risk averse investors. So, in summary and this is the summary for the risk averse investor.

The utility function for a risk averse investor is concave and has the following characteristics that U utility of the expected wealth is going to be so; that means,

$$U(E(W)) > E(U(W)) \iff U'(W) > 0 \text{ and} U''(W) < 0$$

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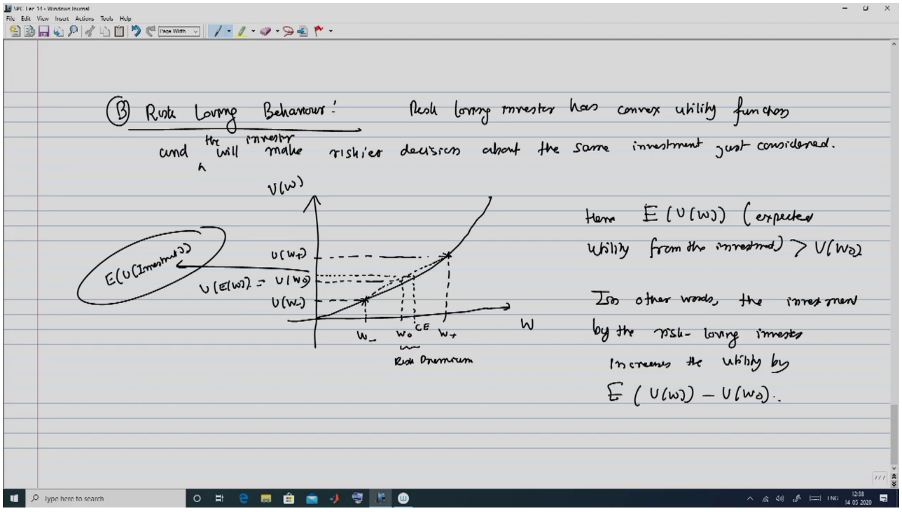

We next come to what is known as the risk loving behavior which is the opposite of the risk averse behavior. So, in this case the risk loving investor typically has convex utility function and will make so, and the risk loving investor will make riskier decisions about the same investment that we just considered.

So, let us look at this graphically again. So, this is U of W and it is a convex function now this is W minus. So, correspondingly I will have U of W minus this is W naught. So, this is going to be U of W naught which is nothing, but the utility of the expected wealth as before and this is W plus. So, this is going to be the utility of W plus. Now what is the expected utility? So, expected utility will be basically given by the line joining these two points.

So, it is basically half of utility of W minus plus half of utility of W plus and then so, this is the corresponding value that we have here. And so, what you will have now is this value is going to be expected utility of the investment. And once I extend this here then this point here is going to be my certainty equivalent and this is going to be the risk premium.

So, here you observe that the expected utility from the investment so, that is here E of U of W this is. So, this is the expected utility from the investment this is greater than U of W naught. So, in other words the investment by the risk loving investor increases the utility by

$$E[U(W)] - U(W_0)$$

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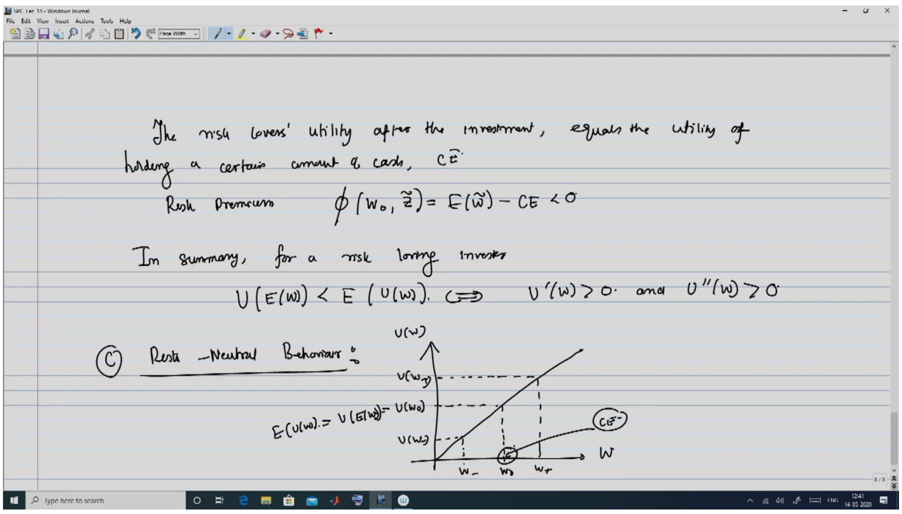

So, now we start off with the notion of certainty equivalent. So, the risk lovers so, centrality equivalent in this case will be defined in a similar way. So, the risk lovers utility after the investment equals the utility or utility of a holding a certain amount of cash, cash namely certainty equivalent.

So; that means, in this case the risk premium which again is denoted by

$$\phi(W_0, \tilde{Z}) = E(\tilde{W}) - CE$$

In summary,

$$U(E(W)) < E(U(W)) \iff U'(W) > 0, U''(W) > 0$$

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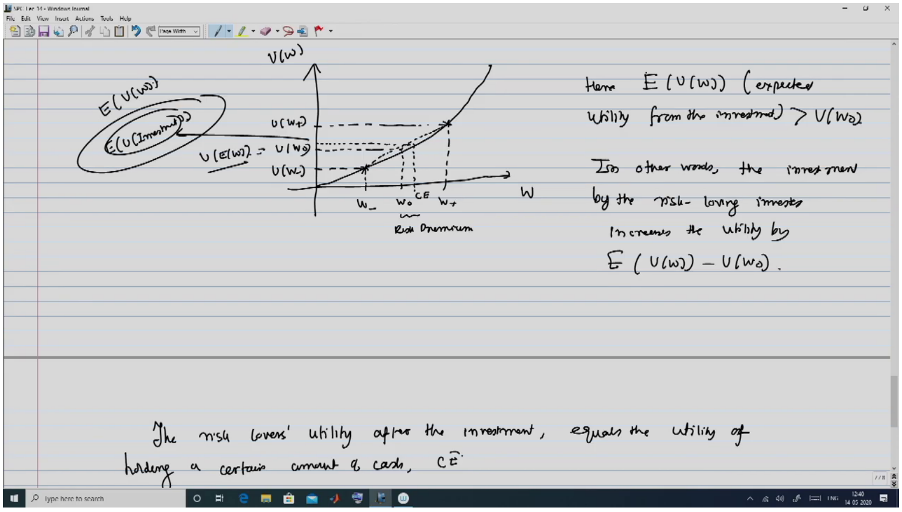

So, finally, we come to the last category of investor namely the risk neutral behavior. And we will consider the following graph. So, again we look at W and it is going to be a straight line. So, we have W minus. So, this is going to be U of W minus then we have W plus. So, I have here U of W plus and then we have W naught here.

So, this is going to be U of W naught which is nothing, but the utility of expected wealth as and which is nothing, but. So, this is actually the utility of the wealth level utility of the expected wealth and this is the same as expected utility of wealth and so the certainty equivalent is going to lie at W naught.

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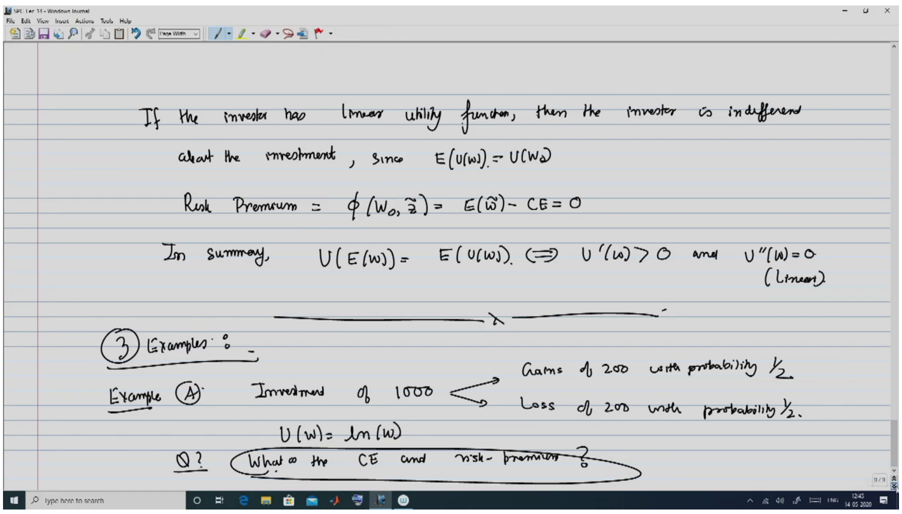

So, this means that if the investor has a linear utility function then the investor is indifferent about the investment and the reason why the investors indifferent about the investment is because we have U of W naught utility of initial wealth is the same as the expected utility of the wealth after the investment.

So, in this case the risk premium is equal to

$$\phi(W_0, \tilde{Z}) = E(\tilde{W}) - CE = 0$$

So, in summary

$$U(E(W)) = E(U(W)) \iff U'(W) > 0, U''(W) = 0$$

And in each of the cases; obviously, the U prime of W is positive, but in case of the risk averse we have U double prime less than 0 and in case of risk loving we have U double prime greater than 0 and in case of the risk total investor we have U double prime being equal to 0 alright. So, now, we will look at the three different examples. So, each example pertains to the 3 kinds of risk in risk appetite.

So, let us look at example A for the risk neutral investor. So, you consider an investment of 1000 and this investment is such that you could have a gain of 200 with probability half and there could be a loss of 200 with probability half and here suppose the utility function is the log utility. So, log utility as you can see it is a it is a concave function and the question is what is the C E and what is the risk premium? Ok.

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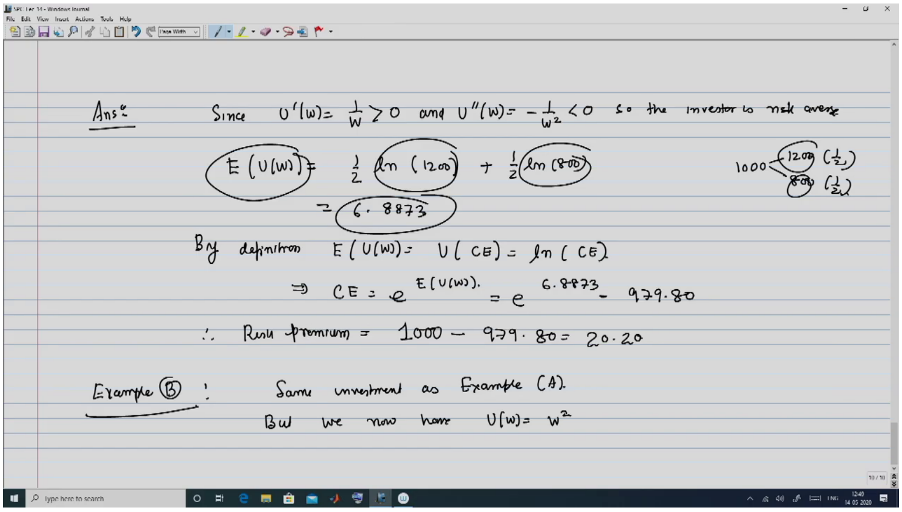

So, let us try to answer this question, now since U prime of W is 1 over W remember wealth is always positive. So, this U prime of W is greater than 0 and U double prime of W is equal to minus 1 over W square which is less than 0. So, it follows that. So, the investor is risk averse. Now what is going to be the expected utility of this?

The expected utility is going to be the utility so, what is the amount of money then you get you can basically go from 1000 to either 1200 with probability half or you can come down to 800 with probability half. So, the respective utilities are going to be natural log of 1200 and the natural log of 800 with the respective probabilities being half.

So, it is the utility into the respective probability C added up. So, that gives us the expected utility and this turns out to be equal to 6.8873. Now, my question was to find out what is the certainty equivalent and henceforth figure out what is the risk premium. So, by definition of certainty equivalent $CE = e^{E(U(W))}$

Now, let us look at an example pertaining to the risk loving investor. So, it is the same framework. So, the same investment as example A, but we now have the utility function to be W square.

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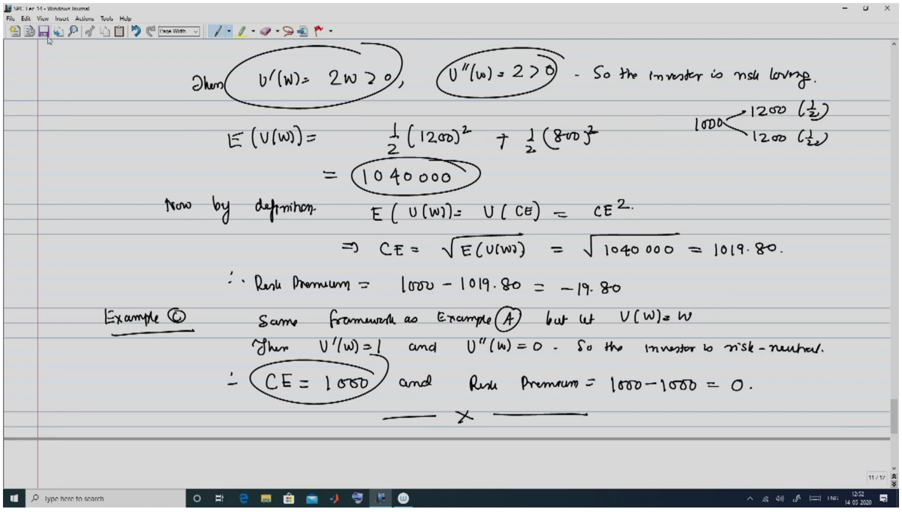

So, you observe that U prime of W is equal to twice W which is greater than 0 and U double prime of W is equal to 2 which is greater than 0. So, the investors are from this two we can conclude that the investor is risk loving. Now what is going to be the so, remember again with a one investment of 1000 it can go up to either 1200 probability half or down to 1200 with probability half.

So,

$$CE = \sqrt{E(U(W))}$$

And so, by definition the risk premium is the original amount 1000 minus certainty equivalent 1019.80 this is minus 19. 80. So, to conclude our discussion I look at the last example. So, in this case you will have the same framework as example A again, but in this case I will let the utility function to be linear function. So, then $U'(W) = 1$ and $U''(W) = 0$. So, the investor is risk neutral.

So therefore, what is going to be the certainty equivalent? The certainty equivalent; obviously, is going to be the same as 0 original amount 1000. And, consequently the risk premium is going to be equal to 1000 which is the original amount minus certainty equivalent which is given here and then this being adds up being to be 0. So, this brings us to the end of this lecture just to summarize what we have done again.

So, we have extended upon the concept of utility functions that we had discussed in the previous class and we looked at two key characteristics for each of the three different kinds of investors in terms of risk appetite. In terms of risk appetite namely risk neutral sorry risk averse, risk loving and risk neutral and identified them in terms of the nature of the graph namely concave, convex and linear and in terms of the marginal utility and whether the marginal utility is diminishing increasing or simply equal to 0.

And in order to illustrate this point, this three different nature of investors we consider an identical example, but with different utility functions defined in a way that is consistent with the concavity, convexity and linearity property of the three kinds of investors. And, each of the cases we derived what is going to be their certainty equivalent and what is going to be their risk premium.

So, this brings us to the end of this discussion and from the next class we will continue more discussion on this in terms of certain parameters that are used in case of capturing the risk appetite. And, then we will move on to talking about utility theory in the paradigm of portfolios.

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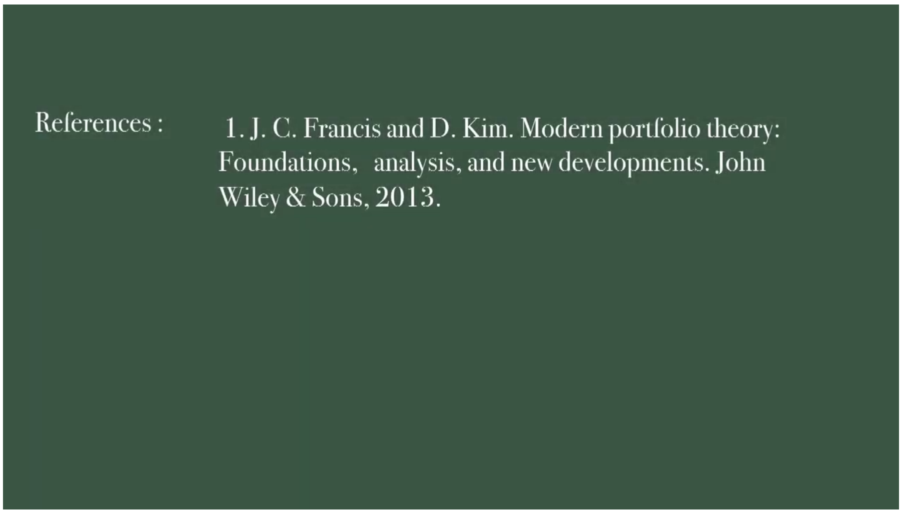

Thank you for watching.