Mathematical Portfolio Theory
Module 04: Non-Mean-Variance Portfolio Theory
Lecture 13: Utility functions and expected utility

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Hello viewers, welcome to this next lecture on the MOOC course on Mathematical Portfolio Theory. In this lecture, we will now start a new component of the course a namely the Non-Mean Variance Portfolio Theory which is going to address some of the issues that are unaddressed as far as the Markowitz framework is concerned along with some of the shortcomings.

Now, in this broader framework of what I am calling as the non-mean variance portfolio theory, I am also going to include and begin with what is known as the utility theory and a utility functions.

So, what is the utility theory all about? Utility theory is an important concept in economics which quantifies the amount of pleasure or utility that one derives from one's wealth level.

And for this purpose what we will do is that we will look at the actions of the utility functions and look at the non-mean variance framework essentially driven by the goal of maximizing the expected utility of investors. And subsequently this will be used when you talk about a multistep portfolio allocation namely dynamic programming as well as when you talk about the Hamilton-Jacobi Bellman equation.

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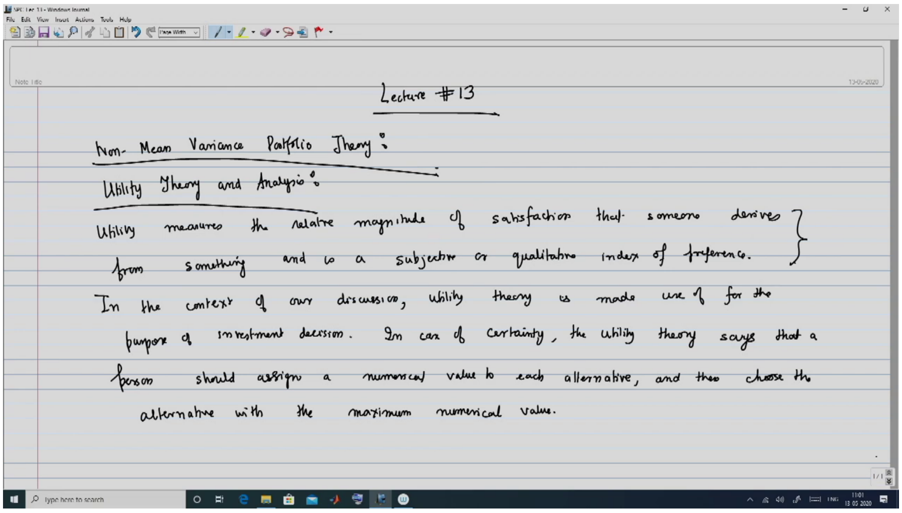

So, accordingly we begin this lecture as I said that this topic is going to be on non-mean variance portfolio theory. So, the first topic that you look at is what is known as the utility theory and consequent analysis.

So, a utility measures or is an indicator of the relative magnitude of satisfaction that someone derives from something and is a subjective or qualitative index of preference. Now, this is a very general statement. So, on a more localized level, we say that in the context of our discussion on portfolio theory, utility theory is made use of for the purpose of investment decisions, and will discuss extensively on how this is done. So, in case of certainty, the utility theory says that a person should assign a numerical value to each alternative, so in this case, it is going to be each alternative portfolios, and then choose the alternative with the maximum numerical value.

So, what are the utility theory says is that we will use utility theory in order to tag or assign a number for all the alternatives that are available to us. So, in this case, in the context of portfolio theory will essentially look at different alternative portfolios that are available for investment. And to each of them, we will use the utility theory to assign a number which is a qualitative indicator of the amount of satisfaction that we can derive by holding on to that portfolio.

And accordingly it also enables us to identify which one of them is preferred over the other in terms of investment and from the perspective of our utility as an individual investor.

And note that this utility is a concept that is subjective and can vary from investor to investor depending on their individual preference for risk. And this is another topic that we will start off towards the end of this lecture ok.

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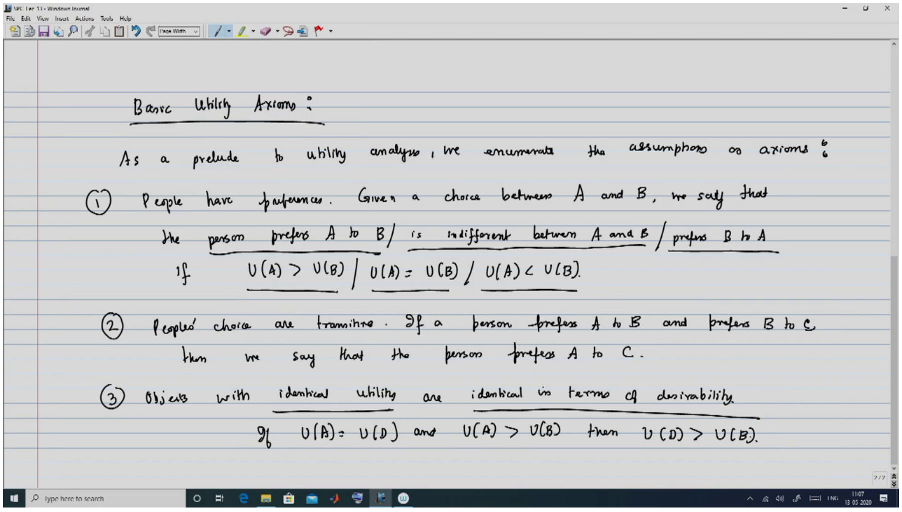

So, now we begin with the basic utility axioms. So, this is going to be a fairly elaborate discussion. So, I say that as a prelude or as a preliminary discussion the utility analysis, we enumerate below the assumptions or axioms, right.

1. Given a choice between two alternatives say A and B, we say that the person prefers the alternative A to B is indifferent between A and B, and prefers B to A if respectively we have

$$U(A) > U(B); U(A) = U(B); U(A) < U(B)$$

2. if a person prefers A to B, and prefers B to C, then we say that the person prefers A to C. So, this means that if we have three alternatives available A, B and C, and if an individual prefers A over B and also B over C, then this implies that the person prefers A over C. So, this is consistent with our typical definition of what is transitivity.

3. alternatives have identical utility, then it implies that they are identical in terms of desirability or preference from the point of view of the person concerned. So, this means that

$$U(A) = U(D) \text{ and } U(A) > U(B) \implies U(D) > U(B)$$

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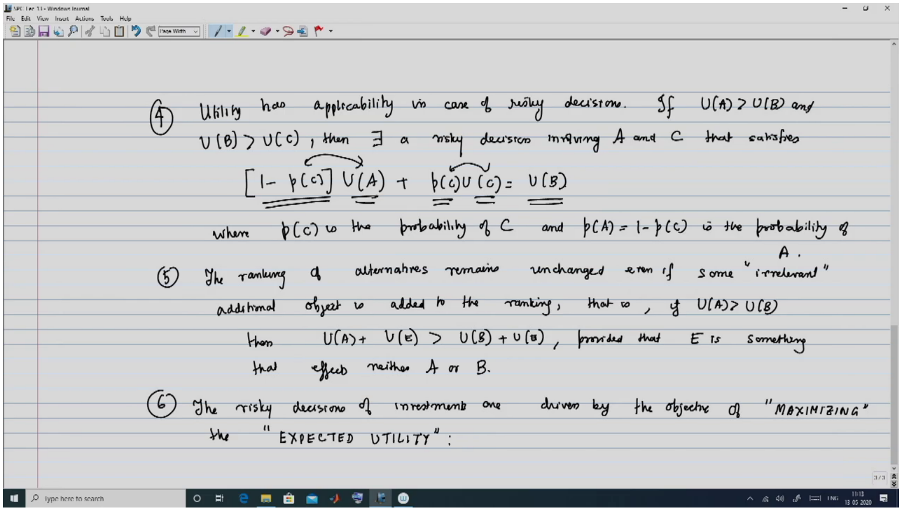

4. Utility has applicability, that means, it can be used in case of risky decisions. If $U(A) > U(B)$ and $U(B).U(C)$, then $\exists$, A, C that satisfies

$$[1 - P(C)]U(A) + P(C)U(C) = U(B)$$

where $P(C)$ is the probability of C and $P(A) = 1 - P(C)$.

5. The 5th axiom is that the ranking of alternatives remains unchanged even if some irrelevant additional object is added to the ranking.

$$U(A) > U(B) \implies U(A) + U(E) > U(B) + U(E)$$

provided that E is something that effects neither A or B, ok.

6. And now we come to the 6th axiom, and this is the really important axiom. So, this says that the risky decisions of investments are driven by the objective of maximizing the expected utility.

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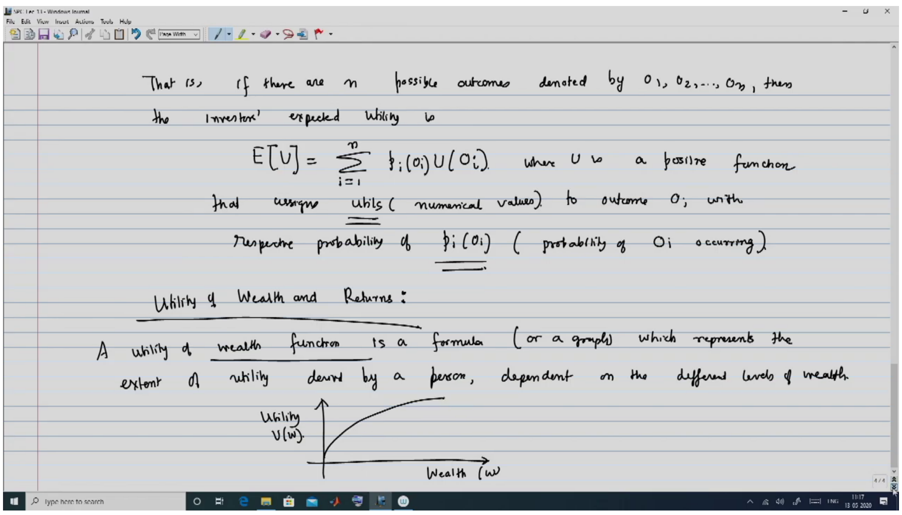

For n possible outcomes denoted as $O_1, O_2, ..., O_n$, the expected utility is given by

$$E(U) = \sum_{i=1}^{n} P_i(O_i)U(O_i)$$

So, the next thing that we come to is the utility of wealth. So, we will now start getting a little more specific in terms of precisely stating what this function U could be, what are its characteristics and how we can make use of them. And finally, we will also look at how we can actually calculate the expected utility in the paradigm of an investment or (Refer Time: 18:33) several alternative investment.

So, accordingly we start off with the topic of utility of wealth and consequently a course of returns. So, a utility of wealth function is a formula or a graph which represents the extent of utility or satisfaction derived by a person dependent, so that means, this wealth function is dependent on the different levels of wealth.

So, graphically a typical utility function in case of risk averse investor will have the wealth on the x-axis and the corresponding utility U w on the y-axis, and it is basically going to look something like this.

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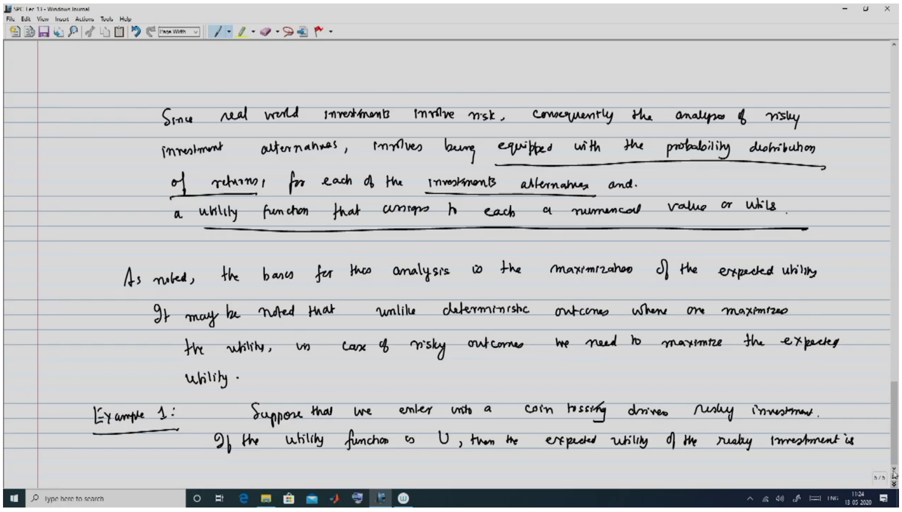

A since investments in real world involve risk, so the consequence of this is that the analysis of a risky investment alternatives involves being equipped with the probability distribution of returns. And this is for each of the investments. So, this is an observation that we have already encountered in slightly different from earlier.

And that is that in case of real or investment and by this I primarily referred to risky investment, the analysis of all these risky investments even in the framework of utility theory requires you to have the information about the probability distribution. Remember that when you are always looking at randomness in any structure in terms of investment, obviously, you will need to have the corresponding underlying probability distribution. So, in order to even analyze this in case of utility theory, you also need to have to be equipped with the probability distribution of what of the returns, because eventually what you are going to look at is we are going to look at the maximization of the expected utility of returns.

And this information about the probability distribution of returns, this has to be made available in case of all the investment or alternatives that are actually being considered to be determined for the best investment in the utility theory framework.

So, therefore, a utility function, so each of the investments alternatives, and a utility function that assigns to each of these alternatives a numerical value or utils, ok. So, this means that you first have to be equipped with the probability distribution of returns. And for each of this alternative investment and also you have to have a utility function that assigns a util to each of them.

So, just to sort of sum this up, it means that when you are doing the analysis you need to be essentially equipped with two things the first is of course the probability distribution of the returns, and consequently once you have that you need to also have a utility function on the basis of which you are going to assign utils or the numerical value to each of those alternative investment ok.

So, now, as noted in the, so as I have already mentioned here that eventually what is our goal is to maximize the expected utility. So, drawing motivation from that, so we say that as already noted the basis for this analysis involving the utils is the maximization of the expected utility. Now, it may be noted that unlike deterministic outcomes such as bonds, where one maximizes the utility in case of risky outcomes such as investment in a portfolio of stocks, we need to maximize not the utility because the return is a random variable, so idea to maximize the expected utility all right.

So, now, let us look at a couple of example to have a better clarity on this topic of expected utility and its consequent maximization. So, first we look at a couple of examples involving the utility, and since it is a random return and it is a risky it is random return resulting from a risky investment, so we will have to talk about not just the utility, but also the expected utility. So, first one I look at a very simple generic example and the next one is going to be a specific example involving multiple assets and multiple investors and for each of those investors, we assign a different utility function depending on their risk preference.

So, first we look at this example 1. So, here suppose that we enter into a coin tossing driven risky investment. Now, if the utility function is U, then the expected utility of the risky investment is given by

$$E(U(\text{coin toss})) = P(\text{head})U(\text{head}) + P(\text{tail})U(\text{tail})$$

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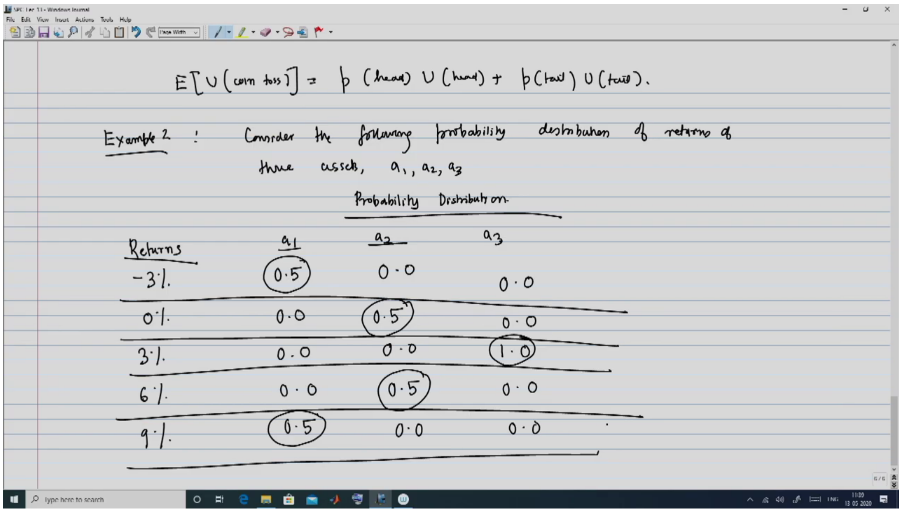

And this is going to be dependent on the coin toss and this is going to be equal to probability of a head multiplied by the utility when a head comes plus probability of a tail into utility of a tail. Let us look at an example 2 which is going to be a little more elaborate example with more numbers that show up, so that we can actually start looking at the concept of utils in practice.

So, this example 2, so what you do is that we consider the following probability distribution of returns of three assets, and I will identify this assets as a1, a2 and a3. So, now, we table it the probability distribution. So, we consider five different possible values of returns, namely, minus 3 percent, 0 percent, 3 percent, 6 percent and 9 percent.

So, for asset a1, we see that we assigned a probability of 0.5 to the return of 3 percent; probability of 0 to 0 percent; probability of 0 to 3 percent; probability of 0 to 6 percent, and the probability of 0.5 to 9 percent. So, this means that the returns can be minus 3 percent with probability 0.5, or 9 percent with probability 0.5. So, this is like a binomial model.

For asset a 2, we have minus 3 percent has the probability of 0; then 0 percent return comes with a probability of 0.5; 3 percent return comes with the probability of 0; 6 percent return comes with a probability of 0.5, and 9 percent return comes with a probability of 0. And finally, so for asset 2, 0 percent returns as a probability of half; and 6 percent return comes with a probability of half.

And in case of the third asset which is a bond, 3 percent return comes at the probability of 3 percent with a probability of 1; and the remaining returns all have a probability of 0.

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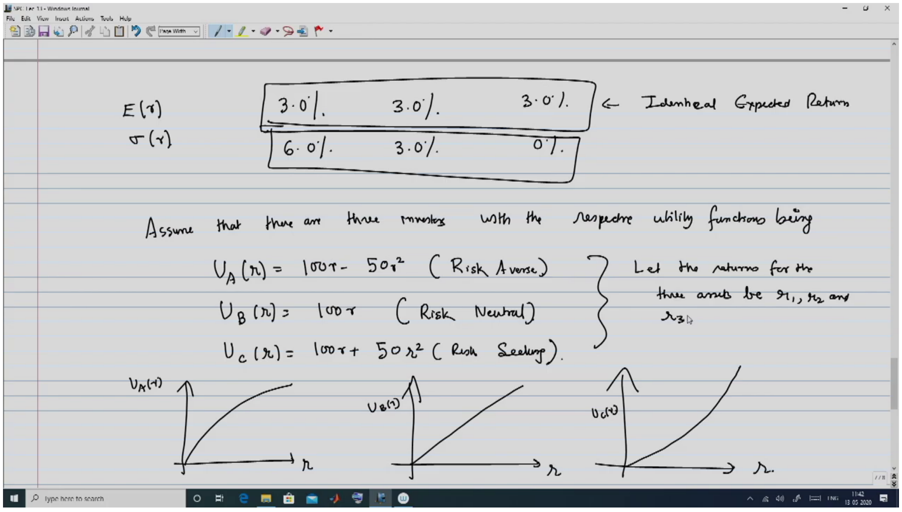

So, you see that. So, from here you can see that the expected return for the first asset is going to be a minus 3 percent into 0.5 plus 9 percent into 0.5. So, this is 3.0 percent. For the second asset, it is 0 percent with probability 0.5 and 6 percent with probability 0.5. So, again the expected return is 3 percent.

And in the third case it is a certain return of 3 percent with probability 1. However, even though they have identical expected returns, they have different risk as given by the standard deviation. So, this turns out to be 6.0 percent in case of the first asset, 3.0 percent in case of the second asset; and since the third asset is a bond, so obviously, the risk associated with this is going to be 0 percent. So, this means that, so here I have identical expected return ok.

So, now, what do you do is we look at, so we assume that there are three investors with the respective utility functions being. So, will give the utility function as

$$U_A(r) = 110r - 50r^2 \text{ Risk averse}$$

$$U_B(r) = 100r \text{ Risk Neutral}$$

$$U_C(r) = 100r + 50r^2 \text{ Risk Seeking}$$

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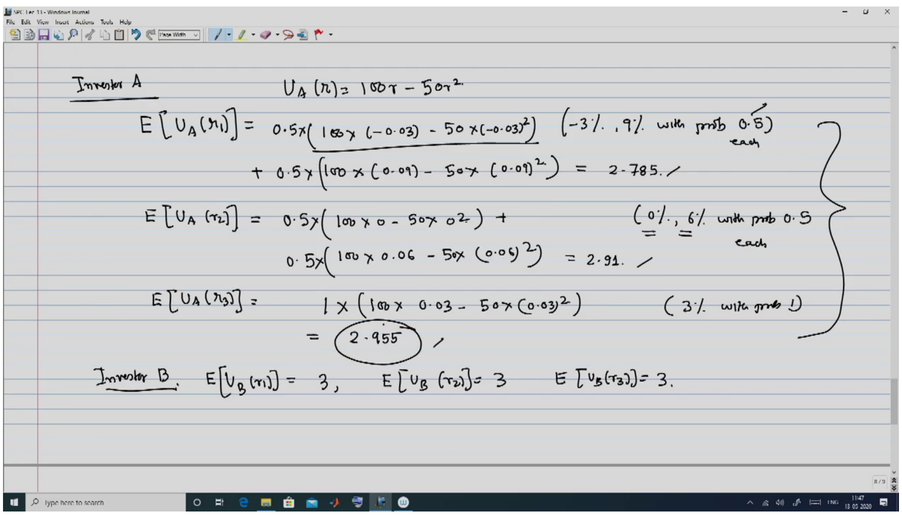

Examples are given above.

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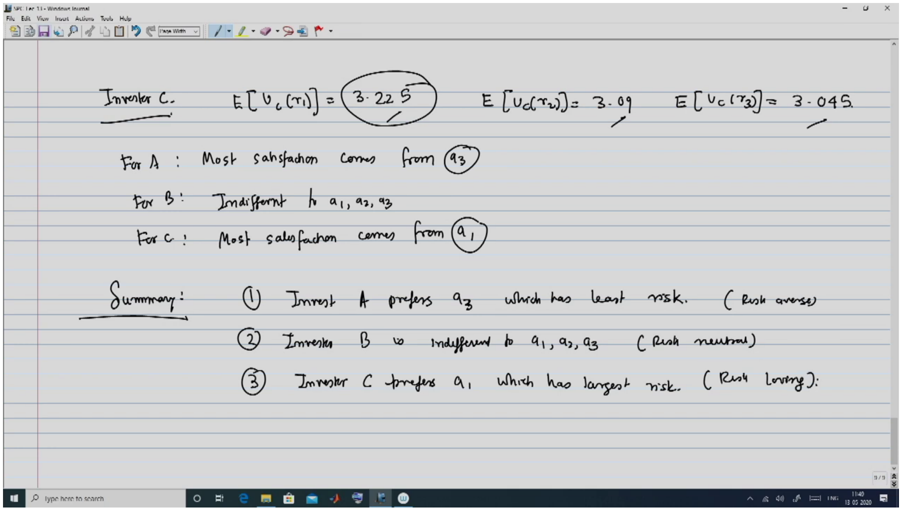

Now, let us just come to the last investor, investor C. So, in case of investor C, we have expected utility for first asset this turns out to be 3.225, expected utility in case of second asset this turns out to be 3.09 and expected utility in case of the third asset this turns out to be equal to 3.045 ok.

So, now, that we have the numbers in place we are now in a position to make an interpretation of this. So, for A you see most satisfaction comes from asset a 3. So, if you observe carefully here asset 1 utility expected utility, asset 2 is expected utility and asset 3 is expected utility, you see that the highest is in case of the third asset. So, for the investor A, most satisfaction will come from a 3.

Now you see in case of investor B all the expected utilities are identical namely equal to 3. So, this means that for B the investor is indifferent to each of the three asset. And for C you see that the highest value, so just 3.225, 3.09, 3.045, so in case of investor C the highest value is 3.225 of the expected utility. So, the most satisfaction comes from a_1.

So, in summary, in now remember that investment a_1 had the highest risk; a_2 had the second highest, and a_3 had the least highest or the least value. So, investor A, prefers a_1 rather a_3 which has least risk, and this is consistent with our earlier statement that investor A is risk averse. Investor B is indifferent to a_1, a_2, a_3.

So, again this is consistent to the utility function for risk neutral and investor C prefers a_1 which has largest risk, and this again is consistent with that investor C is risk loving. So, this is risk averse, risk neutral and risk loving.

So, we come to the last topic or rather just introduction to the next topic that you are going to do and that is going to be on risk attitude. And we will discuss this in the next class. By in terms of risk attitude, what we mean is that there are different investors have different level of risk preferences. So, as we have seen in the example some of this some of the some investor they can be risk averse, some of them can be risk loving, and some of them can be risk neutral.

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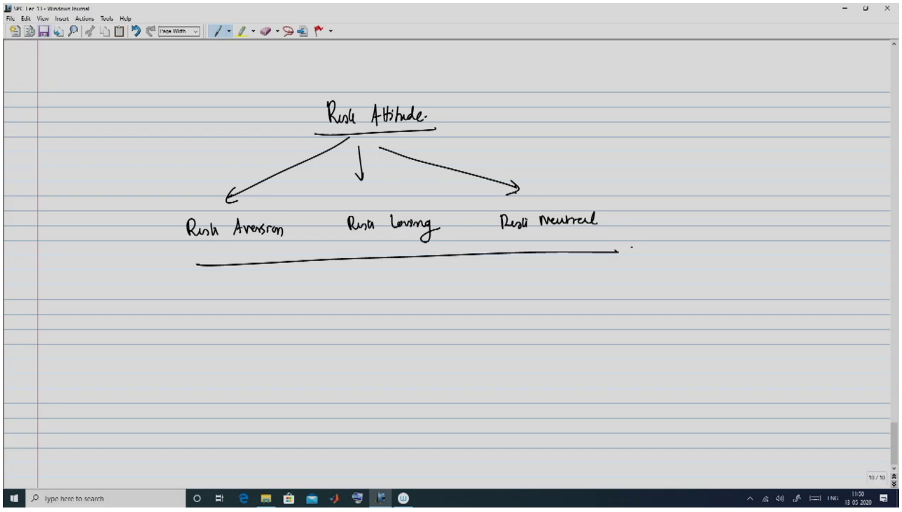

So, what you are going to do is subsequently this concept of risk attitude, we will see this in the paradigm of utility functions, and we will look at the case of risk aversion, risk loving and risk neutral. So, just to sum up what we have discussed today, we introduced the concept of utility functions and which is a quantitative representation of the satisfaction then one derives from any investment strategy that is in the context of our discussion.

And we talked about the fact that utility functions are used to assign a numerical value or utils among the different investment alternatives which in terms helps us to identify our preferences. And we recognize the fact that we need to only calculate what is the utility in case of the returns of any investment being completely deterministic. However, in case we get into risky investment, then the corresponding utils is going to be a random variable and so we need to calculate the expectation of that.

So, irrespective of whether you are looking at just the utility for the deterministic case or the expected utility in case of the random; in case of the random case, we need to the eventual goal from the point of view of the investor is to maximize the utility or the expected utility in the respective cases. And we defined what is going to be the expected utility, and then you looked at two examples one which is a simple example involving a binomial structure.

And then we looked at a (Refer Time: 43:15) tailored example of a three assets and three investors, the three assets were designed in such a way that each of them had their expected utilities to be identical, but they had different levels of risk. And we choose the utility functions for the different investors to reflect their nature or their preferences of being – risk averse, risk neutral and risk loving.

And why this utility functions qualify as a risk averse, risk neutral and risk loving respectively is something that we will look at when we discuss the next topic of risk attitude, and namely characteristics of utility functions for investors who are risk averse, who are risk loving and who are risk neutral respectively. So, this brings us to the end of this lecture.

Thank you for watching.