Mathematical Finance

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Module 3: Modern Portfolio Theory
Lecture 1: Multi Asset Portfolio, Minimum Variance Portfolio, Efficient Frontier & Minimum Variance Line

Hello viewers, welcome to this course on mathematical finance. In todays lecture, we will continue with our discussion on portfolio theory and we will talk a little bit about the two asset portfolio and then we will move on to a multi-asset portfolio and finally we will conclude by talking about what is known as the efficient frontier.

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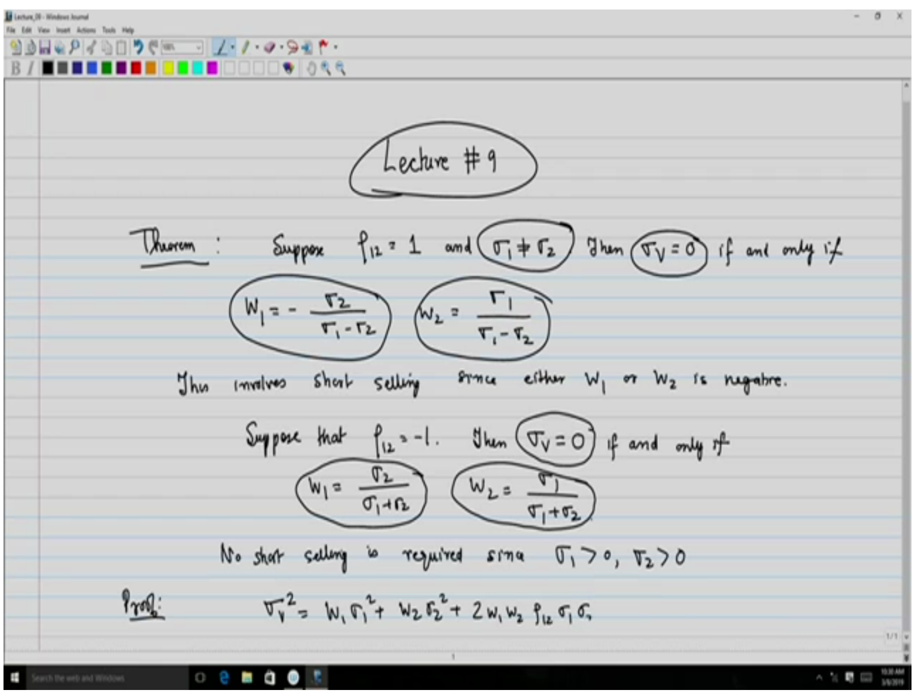

So let us begin the lecture. So we begin the lecture by a theorem related to the previous theorem that we have done. So suppose the correlation coefficient of the two assets in a two asset portfolio is 1. That means, they are perfectly correlated and $\sigma_1 \neq \sigma_2$, then $\sigma_V = 0$ if and only if $W_1 = -\frac{\sigma_2}{\sigma_1 - \sigma_2}$ and $W_2 = \frac{\sigma_1}{\sigma_1 - \sigma_2}$. So essentially, this means that this involves short selling, since either W_1 or W_2 is negative.

So, this means the following that what this results gives you is that under special circumstance when $\rho = 1$ and $\sigma_1 \neq \sigma_2$, a portfolio of two risky assets can end up having a 0 risk that is $\sigma_V = 0$, provided you have made the choice of weights $W_1 = -\frac{\sigma_2}{\sigma_1 - \sigma_2}$ and the 2nd weight will be $W_2 = \frac{\sigma_1}{\sigma_1 - \sigma_2}$. So notice that since $\sigma_1 \neq \sigma_2$, so that means that the denominator is not 0, which is why we needed this condition, $\sigma_1 \neq \sigma_2$ and both σ_1 and σ_2 are positive.

So this means that either W_1 or W_2 depending on which of σ_1 and σ_2 is larger will end up being negative, that means in order to achieve $\sigma_V = 0$, you will need to be involved in short selling. Likewise, suppose that $\rho_{12} = -1$, then $\sigma_V = 0$. Note that in this case, we did not take the condition of $\sigma_1 \neq \sigma_2$. So then you can achieve $\sigma_V = 0$ if and only if $W_1 = \frac{\sigma_2}{\sigma_1+\sigma_2}$ and $W_2 = \frac{\sigma_1}{\sigma_1+\sigma_2}$.

So if it is a perfectly negatively correlated, that is $\rho = -1$, then you can achieve a 0 risk portfolio of these two risky assets provided that $W_1 = \frac{\sigma_2}{\sigma_1+\sigma_2}$ and $W_2 = \frac{\sigma_1}{\sigma_1+\sigma_2}$. Now since both σ_1 and σ_2 are positive. Therefore, in this case your W_1 and W_2 are both positive and hence, $\sigma_V = 0$ can be achieved without resorting to short selling.

Therefore, no short selling is required since $\sigma_1 > 0$ and $\sigma_2 > 0$. So, the proof of this is straightforward and just makes use of the definition of σ_V^2 or the derivation of that. So, σ_V^2 you would recall is $W_1\sigma_1^2 + W_2\sigma_2^2 + 2W_1W_2\rho_{12}\sigma_1\sigma_2$.

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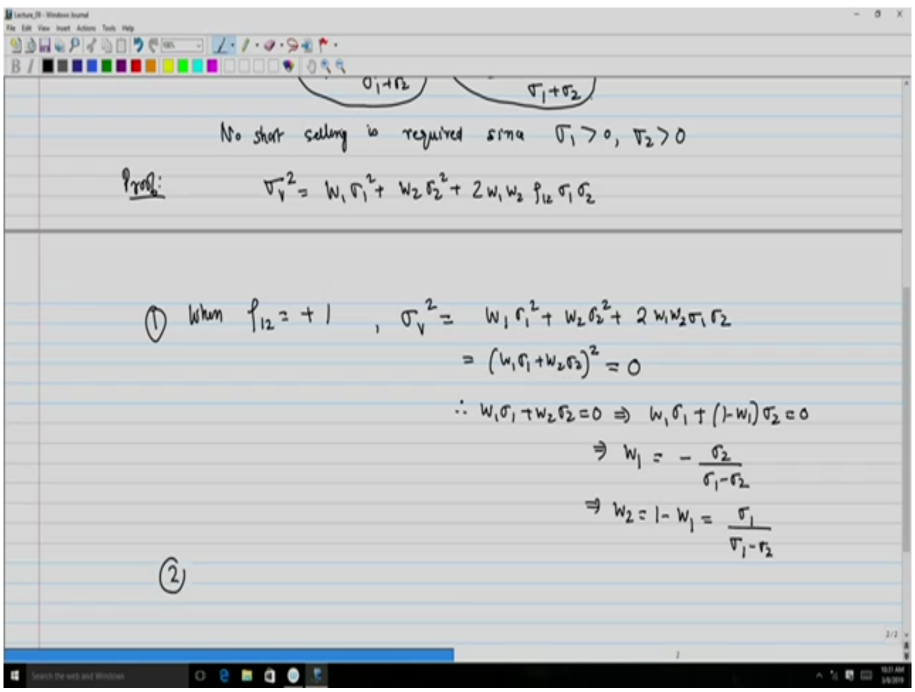

So when $\rho_{12} = +1$, you will get $\sigma_V^2 = W_1\sigma_1^2 + W_2\sigma_2^2 + 2W_1W_2\sigma_1\sigma_2$ and this is $(W_1\sigma_1 + W_2\sigma_2)^2$. And if this has to be equal to 0, then I will need $W_1\sigma_1 + W_2\sigma_2 = 0$ which implies that $W_1\sigma_1$ and I can write W_2 as $1 - W_1$ since some of the weights equal to 1, multiplied by $\sigma_2 = 0$. And I can solve this for W_1 and I obtain $-\frac{\sigma_2}{\sigma_1-\sigma_2}$. And this will give me W_2 is $1 - W_1$ and this is going to be $\frac{\sigma_1}{\sigma_1-\sigma_2}$. So this was the 1st case.

For the 2nd case, when rho is -1, then σ_V^2 is going to be $W_1^2\sigma_1^2 + W_2^2\sigma_2^2 - 2W_1W_2\sigma_1\sigma_2$ and this is $(W_1\sigma_1 - W_2\sigma_2)^2$. And this is going to be equal to 0. Therefore, $W_1\sigma_1 - W_2\sigma_2 = 0$ implies $W_1\sigma_1 - (1 - W_1)\sigma_2 = 0$, which implies that $W_1 = \frac{\sigma_2}{\sigma_1+\sigma_2}$. Therefore, this will imply that $W_2 = 1 - W_1$ which is $\frac{\sigma_1}{\sigma_1+\sigma_2}$. So, this concludes this result.

So, we next move on to another theorem. Now, this theorem is motivated by the fact that your correlation coefficient ρ that you have, lies between -1 and 1, both included. So in this particular theorem, we basically look at a subdivision or dividing this interval -1 to 1 in sub intervals and examine the nature of the portfolio, particularly in the context of its impact on the risk of the portfolio, namely, σ_V^2 in each of the cases. So we will state this theorem without the proof and the proof is left as an exercise.

Suppose that $\sigma_1 \leq \sigma_2$. Then the following holds. Number 1, if $\rho_{12} = 1$, then there is a feasible portfolio V with short selling such that $\sigma_V = 0$ whenever $\sigma_1 < \sigma_2$ and each portfolio V in the feasible set has the same σ_V whenever $\sigma_1 = \sigma_2$.

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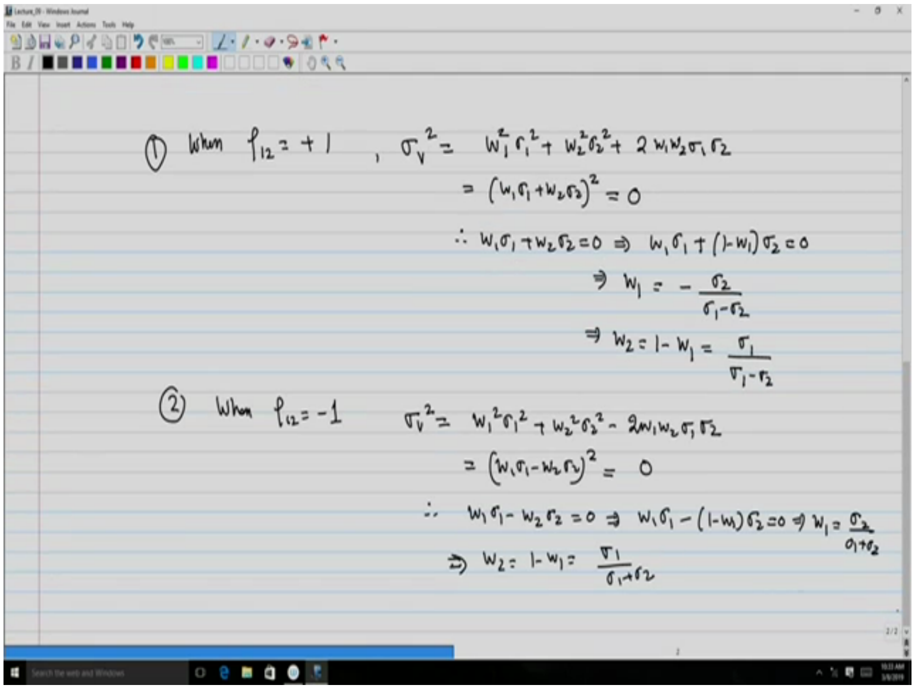

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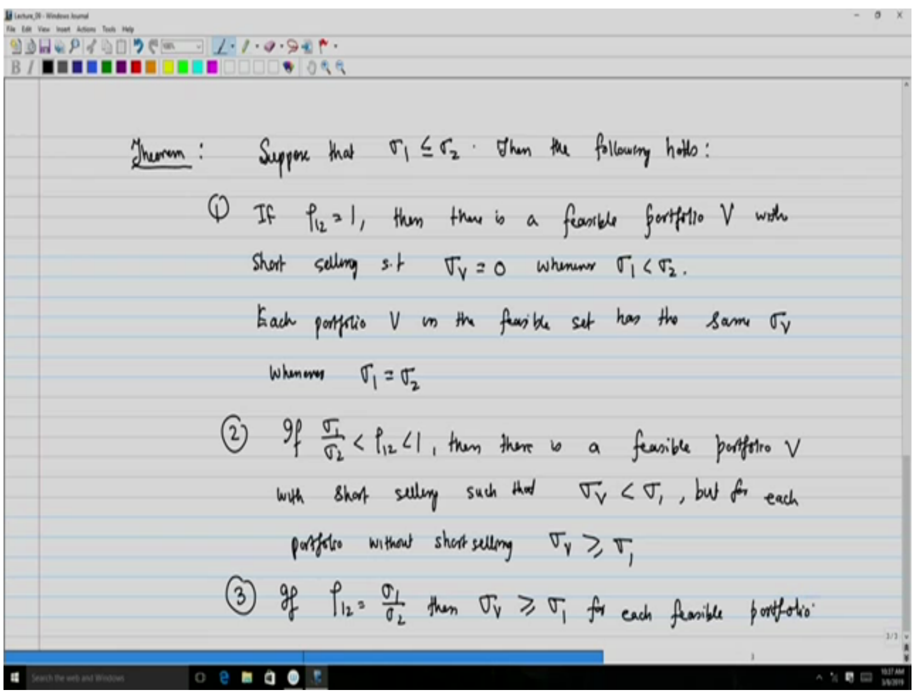

Number 2, if $\frac{\sigma_1}{\sigma_2} < \rho_{12} < 1$, then there is a feasible portfolio, V with short selling such that $\sigma_V < \sigma_1$ but for each portfolio without short selling, results in $\sigma_V \geq \sigma_1$. Number 3, if $\rho_{12} = \frac{\sigma_1}{\sigma_2}$, then $\sigma_V \geq \sigma_1$ for each feasible portfolio V.

Number 4, if $-1 < \rho_{12} < \frac{\sigma_1}{\sigma_2}$, then there is a feasible portfolio V without short selling such that $\sigma_V < \sigma_1$ and finally if $\rho_{12} = -1$, then there is a feasible portfolio V without short selling such that $\sigma_V = 0$. So, notice carefully that for result 1 and result 5, we have essentially made use of the result that we just derived earlier. So notice that in the 1st case, if $\rho = 1$, then it is possible to achieve a risk-free portfolio V such that $\sigma_1 < \sigma_2$.

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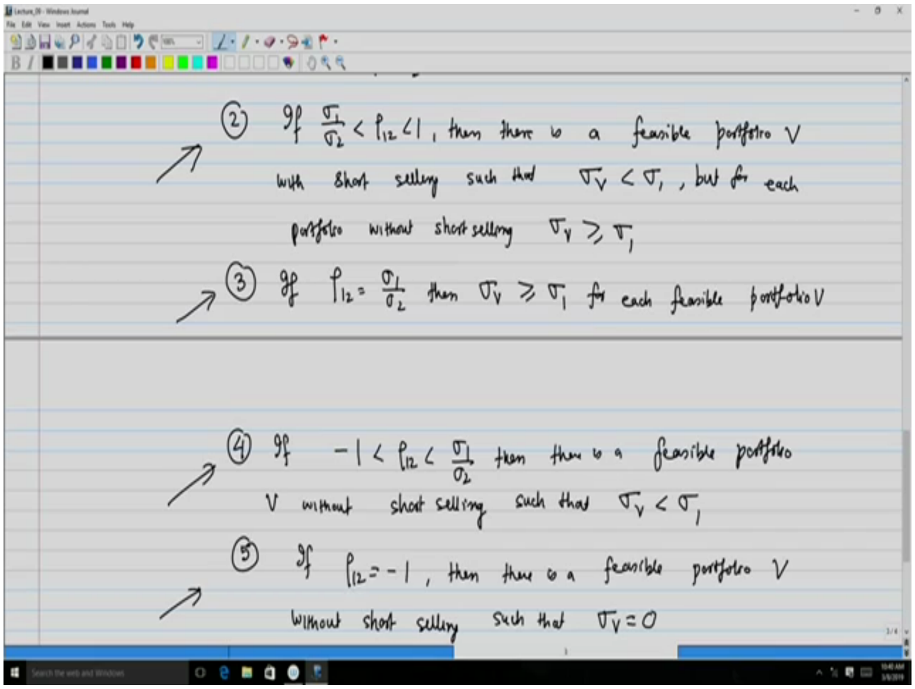

And if $\sigma_1 = \sigma_2$, then basically every portfolio in a feasible set will have the same σ_V and there is no reason to actually choose one over the other. Similarly, for the perfectly negative correlation of rho equal to -1. Again we see that there is a feasible portfolios such that you can have $\sigma_V = 0$. The only difference is that in the 1st case, you can achieve $\sigma_V = 0$, but you will require to do short selling but when $\rho = -1$, you can achieve $\sigma_V = 0$ without resorting to short selling.

Now if your ρ lies between $\frac{\sigma_1}{\sigma_2}$ and strictly less than 1, then you can construct a portfolio such that $\sigma_V < \sigma_1$. So, obviously the assumption was $\sigma_1 \leq \sigma_2$, so we can construct a portfolio such that it is strictly less than the risk. The risk for that portfolio is strictly less than the risk of the individual assets, but in this case, you will require short selling. But in case your rho is lies between -1 and $\frac{\sigma_1}{\sigma_2}$, in that case which is case 4, you can find a feasible portfolio which gives a risk that is less than that of the individual assets without resorting to short selling.

And in case $\rho_{12} = \sigma_1\sigma_2, \sigma_V \geq \sigma_1$. So this is the only case where you pretty much cannot actually achieve a reduction in the portfolio, but in other cases, you can achieve a reduction or even reach 0, sometimes with short selling and in some cases without short selling.

Now, it is time for us to move from two assets to multiple assets. So we start off our discussion about portfolio with several securities or assets. So let us consider a portfolio which is constructed making use of n number of different risky securities or assets with the weight of each asset being Wi is equal to x_i which is the number of units of the i-th asset, multiplied by the current price of the asset i which is $S_i(0)$ divided by the total amount of money available, that is $V(0)$, $i = 1, 2, \ldots, n$.

So, for notational convenience, we introduce the following vectors and matrices. So, first of all, we will introduce $\vec{W}$ of the weights of those assets, namely $(W_1, W_2, \ldots, W_n)$, a vector of all 1s of dimension n which I will denote by a vector $\vec{u}$ and $\vec{m}$ will denote the vector of all the expected returns, $(\mu_1, \mu_2, \ldots, \mu_m)$? So here μ_i is going to be $E(K_i)$ for $i = 1, 2, \ldots, n$. So mu i is going to be the expected return of the ith asset with Ki denoting the random variable for the return of the i-th asset.

So likewise, we also now have, we define a matrix C which is the co-variance matrix, $C_{11}, C_{12}, \ldots, C_{1n}$ and the last row is $C_{n1}, C_{n2}, \ldots, C_{nn}$. So here C_{ij} is the co-variance of the i-th asset and the j-th asset for $i, j = 1, 2, \ldots, n$. So obviously, C_{ii} is going to be σ_i^2 which is the variance of the return of the i-th asset. So now here, the $\vec{W}$ that you have, this is the weight vector. And $\vec{m}$ is the expectation vector and C is the co-variance matrix. Then, the condition that we had, which is $\sum_{i=1}^{n} W_i$, I can rewrite this as

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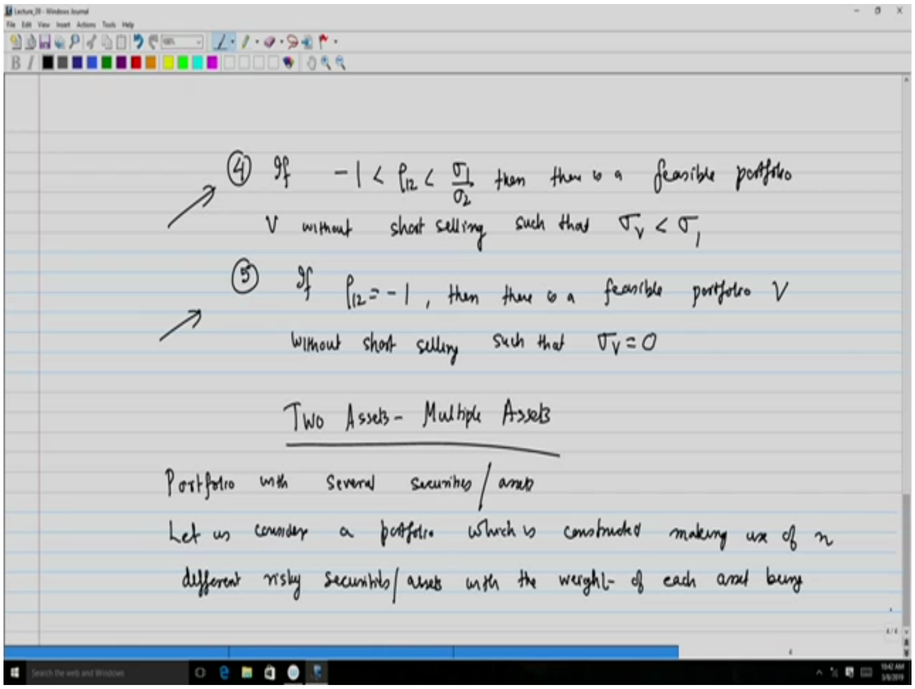

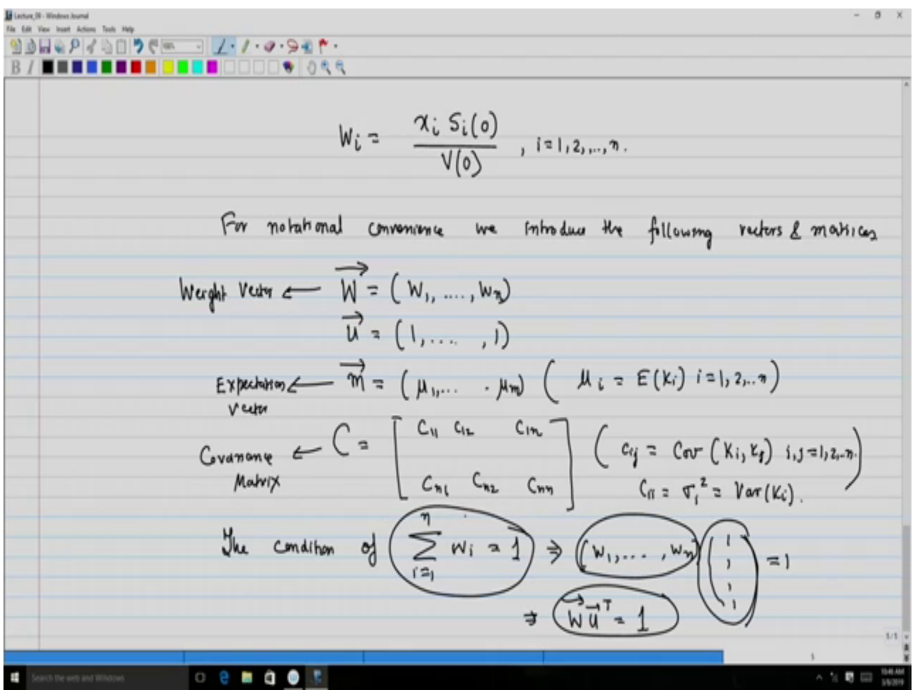

$(W_1, W_2, \ldots, W_n)(1, 1, \ldots, 1)^T$.

So, you do the multiplication here and this will simply become, $\sum_{i=1}^{n} W_i = 1$. So what is this? What can this be written as? So this vector of all these W's, this I have already introduced the notation and that is going to be your $\vec{W}$ and the ones that we have here, this is just going to be $\vec{u}^T$ and this is going to be equal to 1. So this is the vector form for this particular basic constraint that is enforced in case of portfolios that the sum of the weights is equal to 1.

Also assume that determinant of $C \neq 0$. That is C^{-1} exists. Also, please note that the way this C has been defined, this C is obviously also symmetric. In addition to of course our assumption that determinant of $C \neq 0$ to ensure that C^{-1} exists and why C^{-1} needs to exists is something that we see when we are looking at the minimum variance portfolio and the minimum variance line. So now recall that, in case of

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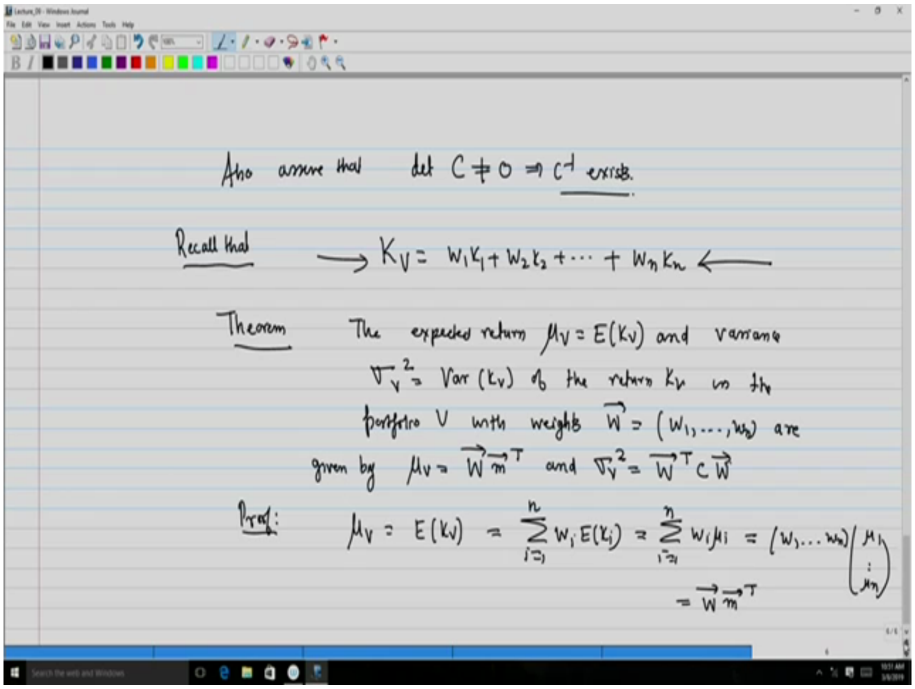

two assets, we had K_V, the return of the portfolio was $W_1K_1 + W_2K_2$. Using the same argument, I can extend this expression for n number of assets which is our current consideration. And this, so I will add this up all the way to W_nK_n.

So now we will use this to state a theorem. The theorem will essentially give the expected return and risk in case of this portfolio of n assets in a manner that is analogous to the expected return and risk that we are seeing in case of two asset portfolio. Therefore, the expected return μ_V which is expected value of K_V and the variance σ_V^2 which is variance of K_V of the return K_V, K_V that we have defined here, in the portfolio V with weights given by this vector $\vec{W}$, that is $W_1, W_2, \ldots, W_n$ are given by $\mu_V = \vec{W}m^T$ and $\sigma_V^2 = \vec{W}^T C\vec{W}$.

So, the proof of this is fairly straightforward. So what is going to be μ_V? μ_V is just $E(K_V)$, so we apply the relation on this and we often summation $W_iE(K_i)$, $i = 1, 2, \ldots, n$ and this is going to be $\sum_{i=1}^{n} W_i\mu_i$, $i = 1, 2, \ldots, n$. And this can be written as $(W_1, W_2, \ldots, W_n)$, multiplied by $(\mu_1, \mu_2, \ldots, \mu_n)^T$. And this is vector $\vec{W}$ and the 2nd vector is $\vec{m}^T$.

Similarly, σ_V^2 is going to be the $Var(K_V)$. So again, we use the variance of a linear combination of random variables from statistics and this turns out to be summation $\sum_{i,j=1}^{n} W_iW_jC_{ij}$ and this if we open up, can be written as $\vec{W}C\vec{W}^T$.

So what we have done so far here is the following that for convenience, in case of multiple number of ourselves, namely n number of assets, we have introduced the weight vector, the unit vector and this expectation vector and the co-variance matrix.

And in terms of this, we have now given our μ_V, that is the expected return of the portfolio of this n assets and σ_V^2 which is the variance of the return of this n number of assets. Now, we are in a position to once we have the vector notation in place, we are in a position to talk about the minimum variance portfolio.

Let us talk about this minimum variance portfolio. What is the minimum variance portfolio? The minimum variance portfolio as the name suggests, is amongst the choices of all the portfolios that you have. It is that portfolio which will result in a minimum variance and equivalently which will result in the minimum risk. So the portfolio with the smallest variance among all feasible portfolios (remember we have talked about what is a feasible portfolio) will be called the minimum variance portfolio or sometimes we

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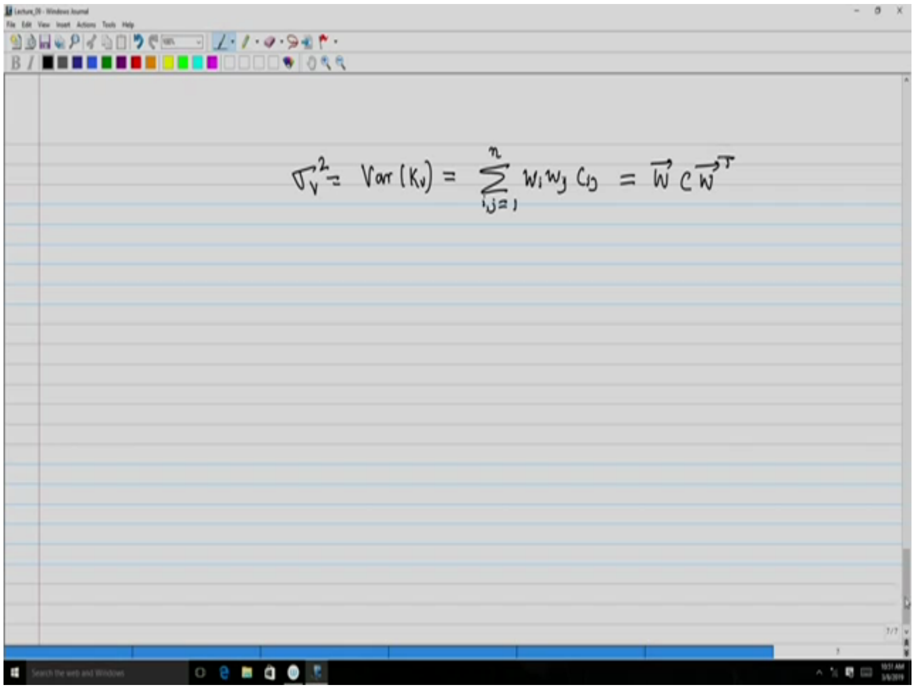

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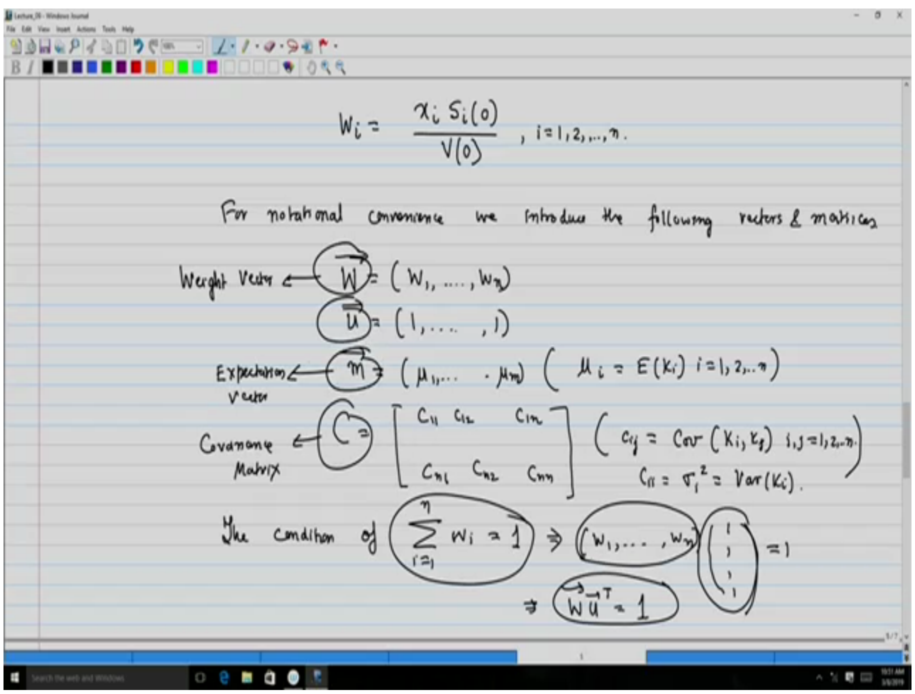

refer to this as MVP. So the question we want to answer next is that how do we determine this MVP? So when I say, how do we determine this MVP, this basically means that we need to determine the combination or the vector $(W_1, W_2, \ldots, W_n)$, so that the consequent σ_V^2 is going to be less than the σ_V^2 for any other, all the other possible combinations of W_1 through W_n.

Therefore, so in order to answer this question, what we need to do is that we need to solve the problem. So I need the minimum variance, so this will require a minimization of σ_V^2 which is same as minimization remember σ_V^2 is given by $\vec{W}C\vec{W}^T$. So this is equivalently minimization of $\vec{W}C\vec{W}^T$ such that we need to impose the constraint, that is $\sum_{i=1}^{n} W_i = 1$. That is $\vec{W}\vec{u}^T = 1$ for all $\vec{W} \in \mathbb{R}^n$.

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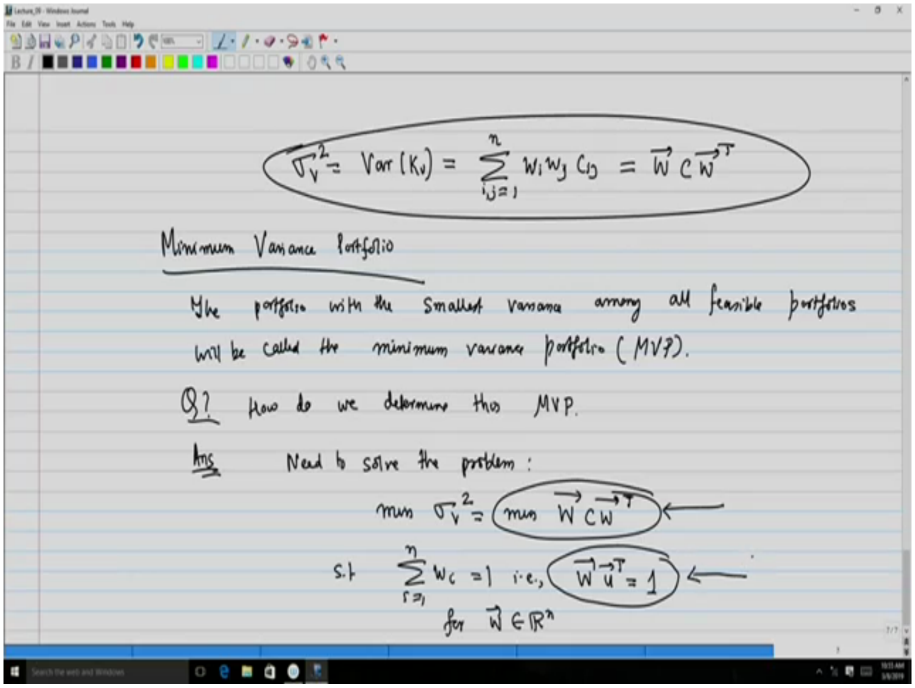

So essentially this means that minimizing the variance in order to attain the minimum variance portfolio, but at the same time being mindful of the condition that the sum of weights must be equal to 1. So this means that this is a constraint optimization problem or more specifically, a constraint minimization problem where this is the minimization that has to be done subject to this particular constraint.

Now, how do we go around solving this? So we have already done this in case of a two asset portfolio but there, we had the convenience that we had just had two weights, W_1 and W_2 and you could parameterize and turn this into a function of a single variable and make use of the derivative test. However, this kind of strategy cannot be readily extended in case of multiple assets.

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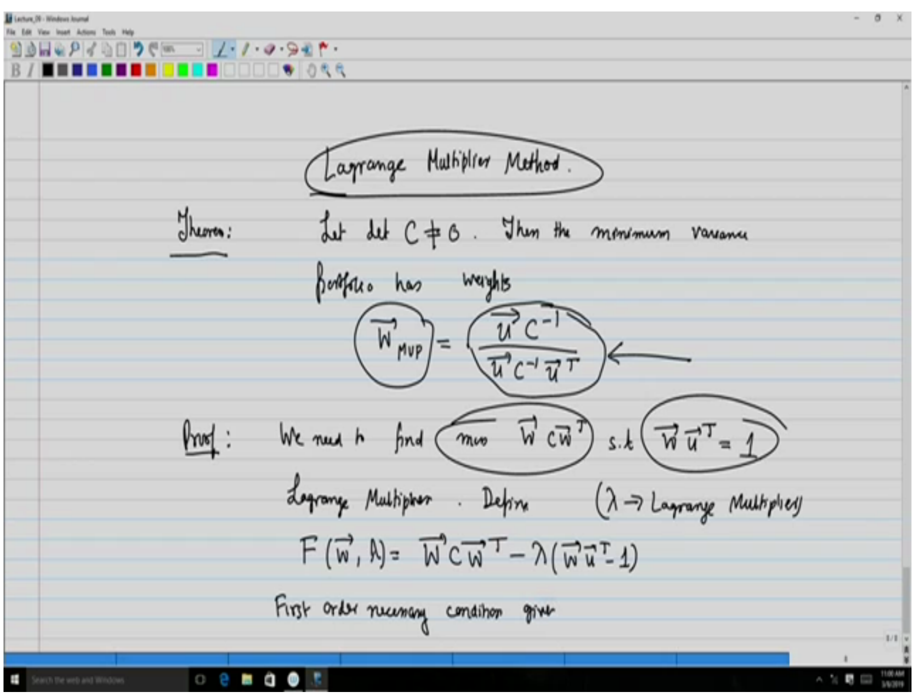

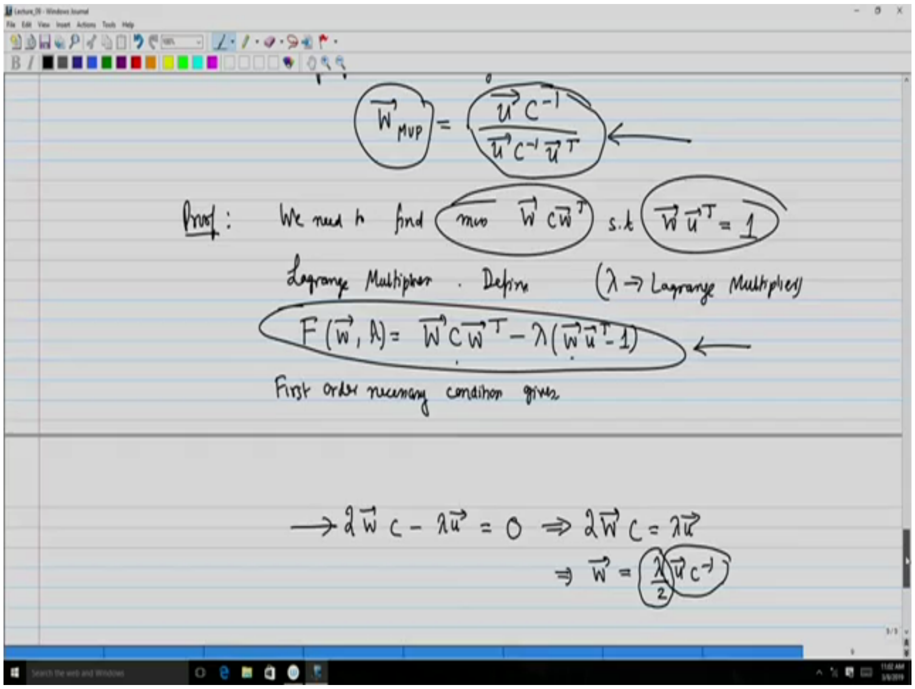

And so, we need to look at some technology of maximizing in case of a multivariable function and then the 1st and the natural choice for this is what is known as the method of Lagrange multiplier. So we will use the method of Lagrange multiplier to determine the weights. So accordingly, we state the theorem. So let $det(C) \neq 0$, so that C^{-1} exists. Then the minimum variance portfolio has weights and remember, this is a vector of dimension $1 \times n$. This weight, $W_{MVP} = \frac{\vec{u}C^{-1}}{\vec{u}C^{-1}\vec{u}^T}$.

So notice that here everything on the right-hand side is a known quantity because once you know what your assets are, you can calculate what your C is going to be and then you can do the necessary operations in order to determine what the $\vec{W}$ is, because u is just a vector of all 1's. So basically once you have done this calculation here, you will get the weights that you need to assign to the different assets and accordingly the resulting portfolio that you get, will be the minimum variance portfolio for amongst all the portfolios that you can construct out of those n number of risky assets.

So the proof for this is the following. The proof for this is drawn from the minimization problem with a constraint that we have already presented. So it means that we need to find the minimal value of $\vec{W}C\vec{W}^T$ such that $\vec{W}\vec{u}^T = 1$. So as I have mentioned, we will make use of the method Lagrange multiplier and accordingly, we define the Lagrangian as $F(W, \lambda)$, where lambda is the Lagrange multiplier. And this is vector $\vec{W}C\vec{W}^T$, which is basically the objective function.

And then minus lambda into this constraint written in the form of $F(X) = 0$, which is $\vec{W}\vec{u}^T - 1$. So we use the necessary condition and what we do is that the 1st order necessary condition gives and the 1st order necessary condition is similar to the 1st derivative test that we have done in case of a single variable function. So take the 1st derivative and set it equal to 0. So in this case, what you do is that you open up this entire F in terms of W_i and W_j and the C_{ij}'s and then you take the derivative of F with respect to $W_1, W_2, \ldots, W_n$ and after that, you reassemble them in a vector form in which case you will end up getting twice $\vec{W}C - \lambda\vec{u}$.

So you can view this as suppose that this was a case where basically it is a one-dimensional case. So we can view this like a quadratic term which upon imposing the 1st order condition becomes $\vec{W}C$ and this as some sort of a linear term, the 2nd term which upon differentiation will basically can get rid of the $\vec{W}$. And the 1st order necessary condition is that this must be equal to 0. So once you have this 0, so this will give you, you can write this as $2\vec{W}C = \lambda\vec{u}$ and you can post multiply this by C^{-1} so that will give you $\vec{W} = \frac{\lambda}{2}\vec{u}C^{-1}$.

So what I have got now is that we have basically got the weight vector in terms of known things, $\vec{u}$ and

C^{-1}. However, the weight is not completely determined because you still have to ascertain what is going to be your $\frac{\lambda}{2}$. So for this purpose, what we will do is that we will substitute it in the constraint. Remember what was the constraint. The constraint was that the sum of the weights is going to be equal to 1. So now how are you going to determine what is going to be your λ?

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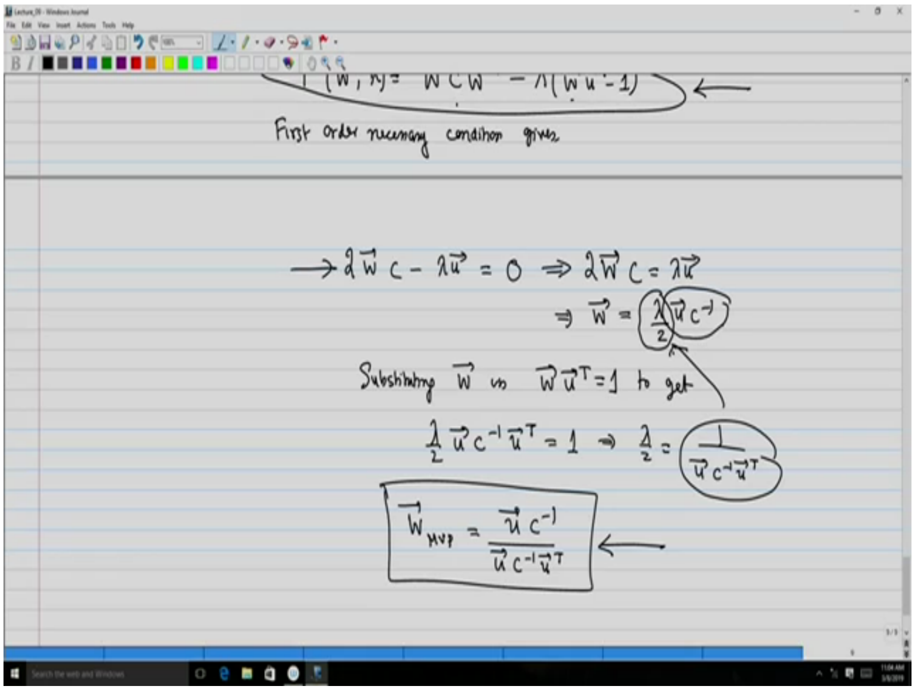

So we substitute $\vec{W}$ in $\vec{W}\vec{u}^T = 1$, what will I get? So what is $\vec{W}$? $\vec{W} = \frac{\lambda}{2}\vec{u}C^{-1}\vec{u}^T = 1$. And this means that $\frac{\lambda}{2} = \frac{1}{\vec{u}C^{-1}\vec{u}^T}$. So then now that we have got our $\frac{\lambda}{2}$ in terms of known quantities, so I can substitute this here to get the weights which in this case obviously is going to be the weights of the minimum variance portfolio and this turns out to be $\frac{\vec{u}C^{-1}}{\vec{u}C^{-1}\vec{u}^T}$.

So this means that once you are able to decide on what is the co-variance matrix, it is fairly straightforward and this involves matrix operations in order to ascertain which is going to be the minimum weight, sorry minimum risk portfolio specified by its weight which is given by this particular formulation. So this essentially was the 1st result that we have done in case of portfolio theory, where we seek to address the question that what is the best way or best strategy in terms of weights when you are going to invest in a portfolio.

So now we come to something what is known as the efficient frontiers. So now in order to talk about efficient frontiers, we need to actually start off with a few definitions. So before that, we give the prelude that given two securities, a rational investor if possible and I will explain why this qualifier if possible has to be included, will choose the security with higher expected return and lower risk or standard deviation. So what I mean by if possible? It means that if as an investor I have choices of two securities and if I am a rational investor and I see that the return on the 1st security is higher than the return on the 2nd security, but the risk of the 1st security is less than the risk of the 2nd security, then obviously I am going to make a choice of the 1st security.

The reason is that I am getting a higher return while being exposed to a lower level of risk. And this typically is unlikely to happen and which is why I have put the qualifier, if possible. However, if we introduce this concept of higher returns and lower risk as a prelude to the following definition and this definition will act as the motivation for what is an efficient frontier. So, a security with expected return μ_1 and standard deviation of return being σ_1 is said to dominate another security with expected return μ_2 and standard deviation σ_2, provided the $\mu_1 \geq \mu_2$, that is the 1st asset will give you a higher expected return than the 2nd asset but $\sigma_1 \leq \sigma_2$.

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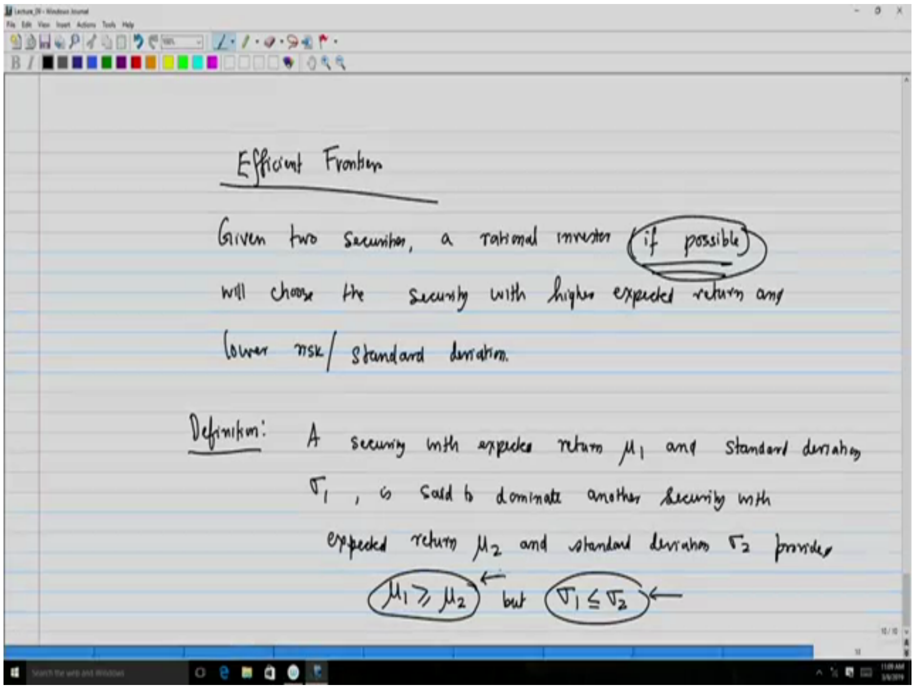

That means the 1st asset will have a lower risk. So if it turns out that your $\mu_1 \geq \mu_2$ and $\sigma_1 \geq \sigma_2$, unlike what I have assumed here, then there is no clear obvious choice but in case of clear dominance and by this, I mean that asset, the 1st asset is actually performing better than the 2nd asset in terms of returns but at the same time, is facing a lower amount of risk than the 2nd asset. Then the choice is very obvious that the 1st asset dominates the 2nd asset. And we must as a rational investor go for the 1st asset.

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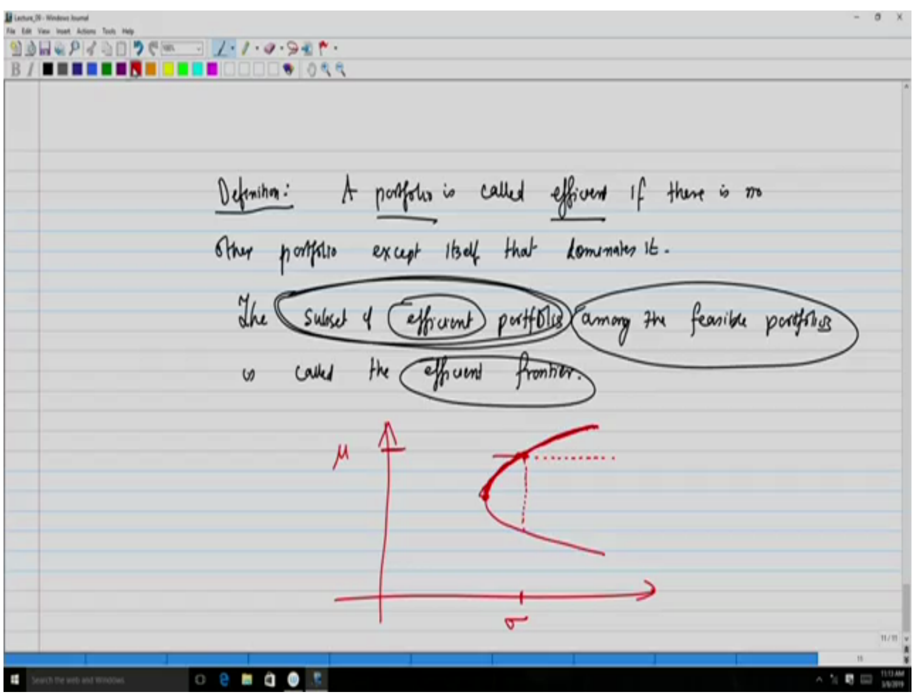

So, now we come to the definition of what is an efficient portfolio before we go ahead and start talking about efficient frontiers. So then this definition says the following. A portfolio is called efficient if there is no other portfolio except itself that dominates it. So this means that a portfolio is said to be efficient if the only portfolio that can dominate it itself. That means that there is no other portfolio which can actually dominate it. So once we define what is an efficient portfolio, we are now in a position to talk about efficient frontier.

So, this subset of efficient portfolios and there could be many such efficient portfolios among the feasible portfolios is called the efficient frontier. So this means that amongst the feasible portfolios which are extremely large in number, there will be a subset of portfolios which are efficient. And this subset of efficient portfolios that we have here, this is what is known as the efficient frontier. So we have to quantify what this efficient frontier is, I mean why we are using the word frontier is that if you go back and look at your mu sigma diagram and we had this hyper bowl, so here what happens is the following that the efficient frontier is this part.

And if you observe very carefully, what does the efficient frontier comprise of? Efficient frontier comprises of all those portfolios where the either the variance is minimized or for a given level of return, I have many different choices of risks or many portfolios at different risks but this point here is going to be the minimum risk portfolio for this given level of return. Likewise, if I fixed the risk level then every portfolio, these are infinitely many portfolios whose risk is same but whose returns are different.

So as a rational investor, I am obviously going to pick the portfolio that has the largest return for that particular given level of risk. So efficient frontier is nothing but either it is a minimum variance portfolio or it is a portfolio where if you fix the return, it will give you the minimum level of risk or it is a portfolio where you fix the level of your risk and it is the portfolio which will give you the highest level of return, which is what you would actually do as a rational investor and that is the reason why the efficient frontier is of great importance from the investors point of view.

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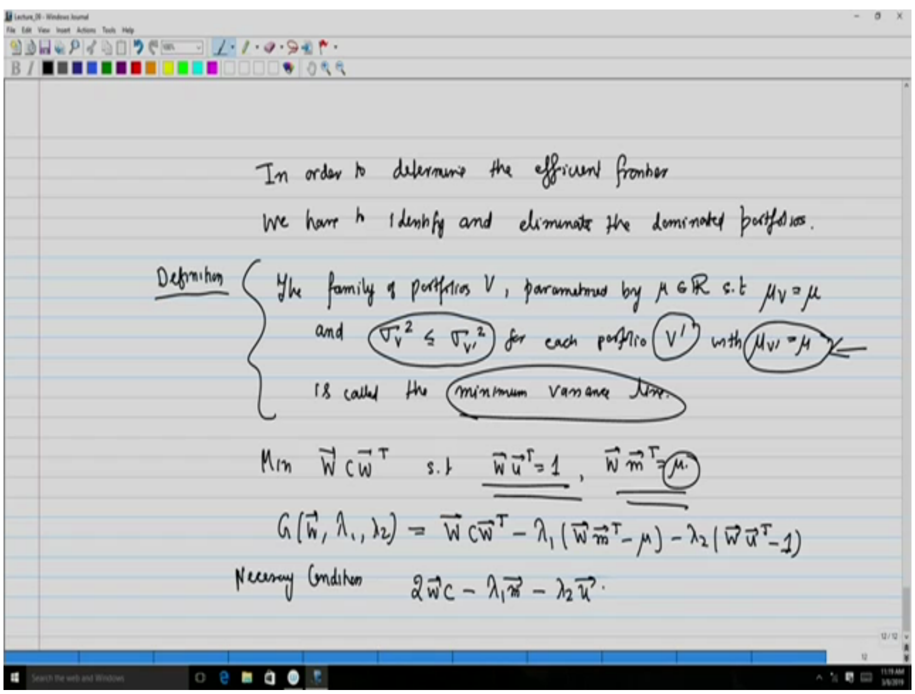

Now, in order to determine the efficient frontier, we have to identify and eliminate the dominated portfolios. So we come to the last result. This is again a very critical result. Just like we had the minimum variance portfolio, we will now talk about what is known as the minimum variance line and so we start off with the definition. So the family of portfolios V parameterized by $\mu \in \mathbb{R}$ such that $\mu_V = \mu$, that means it is a portfolio where you fix our return to be equal to μ and $\sigma_V^2 \leq \sigma_{V'}^2$ for each portfolio V' with $\mu_V' = \mu$ is called the minimum variance line.

So, this means the following that I choose a generic portfolio, V', I fix my return of this V' which I will denote by $\mu_V' = \mu$. Then any portfolio V such that $\mu_V^2 \leq \mu_{V'}^2$ will be identified as something important and the family of all such V is what is known as the minimum variance line. So therefore, I want to so when I am talking about the minimum variance line, I essentially it is a family of portfolios, so it is suggestive that, this statement is suggestive that any such portfolio which satisfies the condition that is specified in the definition, will be a family of portfolios which will be basically a linear function or rather linear in $\vec{W}$.

Accordingly, what we will do is that so in order to compute this, what do we do? We compute the minimization of WCW transpose. So it is just the extension of the previous result of that we have done, where we minimize $\vec{W}C\vec{W}^T$ such that $\vec{W}\vec{u}^T = 1$ and in addition to that, it is the scenario where I will have the return to be fixed. So it is ascertaining, so this definition essentially means that you are trying to ascertain the minimum variance portfolio subject to the sum of the weights being equal to 1, which is sort of the statutory or mandatory constraint. But in addition to that, you are putting in one more constraint, namely, that the sum of the or the expected return is going to be pre-decided at a certain level of mu such that so I have this another constraint which will be $\vec{W}\vec{m}^T = \mu$.

So it means that fix the μ and then decide on the weight of the portfolio. Means for a particular fixed value of μ, there is going to be infinitely many different portfolios with this particular μ and I am trying to find the portfolio with this μ such that that portfolio will have the least variance amongst all these portfolios having a return of μ. So this is what I meant by $\mu V' = \mu$. So again, this is a problem where you were trying to do minimization of a multivariable function.

So accordingly, we define $G(\lambda_1, \lambda_2)$, I need two Lagrange multipliers because I have two constraints. This will be defined as $\vec{W}C\vec{W}^T - \lambda_1\vec{W}\vec{m}^T - \mu - \lambda_2\vec{W}\vec{u}^T - 1$. So from the necessary condition, we get $2\vec{W}C - \lambda_1\vec{m} - \lambda_2\vec{u} = 0$.

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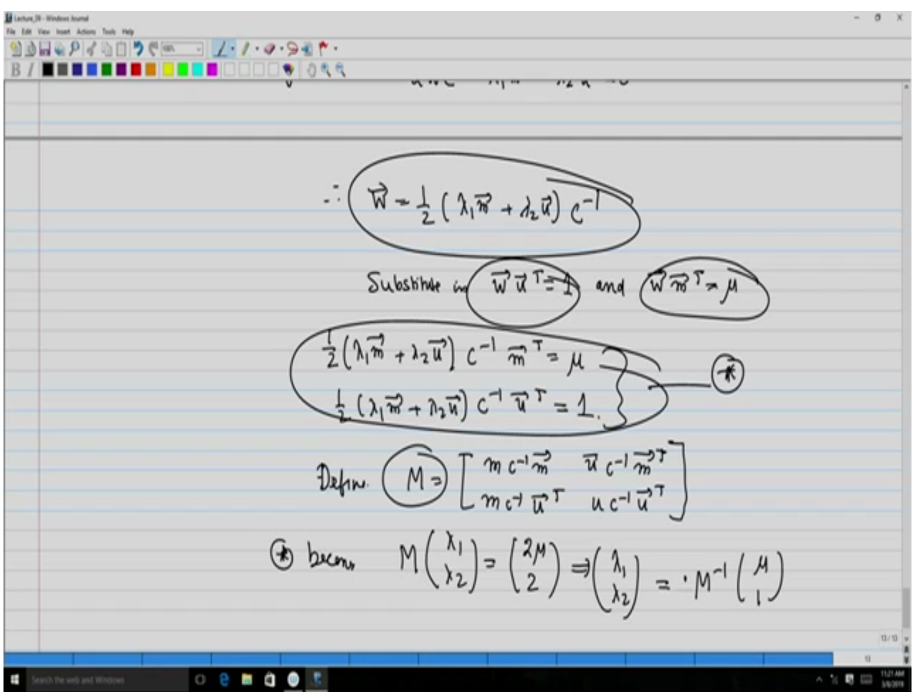

So which will give you your $\vec{W}t = \frac{\lambda_1}{2}\vec{m} + \lambda_2\vec{u}C^{-1}$. So I have got the weights now. And now, it is time for me to substitute this in $\vec{W}\vec{u}^T = 1$ and $\vec{W}\vec{m}^T = \mu$. So what is the consequence of this substitution? As a consequence of the substitution, I will get $\frac{\lambda_1}{2}\vec{m} + \lambda_2\vec{u}C^{-1}m^T = \mu$. And I substitute in this, so I will get $\frac{\lambda_1}{2}\vec{m} + \lambda_2\vec{u}C^{-1}u^T = 1$.

Therefore, now what I can do is that I can define m to be mC inverse m transpose u C inverse m transpose, mC inverse u transpose and u C inverse u transpose. Then if I call this system to be some star, then star becomes m into lambda 1 lambda 2 is equal to twice mu and 1. So I can rewrite this system in terms of this new matrix M and this will give me lambda 1 and lambda 2 to be equal to M inverse (this actually should be

2 here). So this will give me M inverse mu 1 with 2 outside. So I have now discovered what is my lambda 1 and lambda 2. Therefore, what is going to be my weight?

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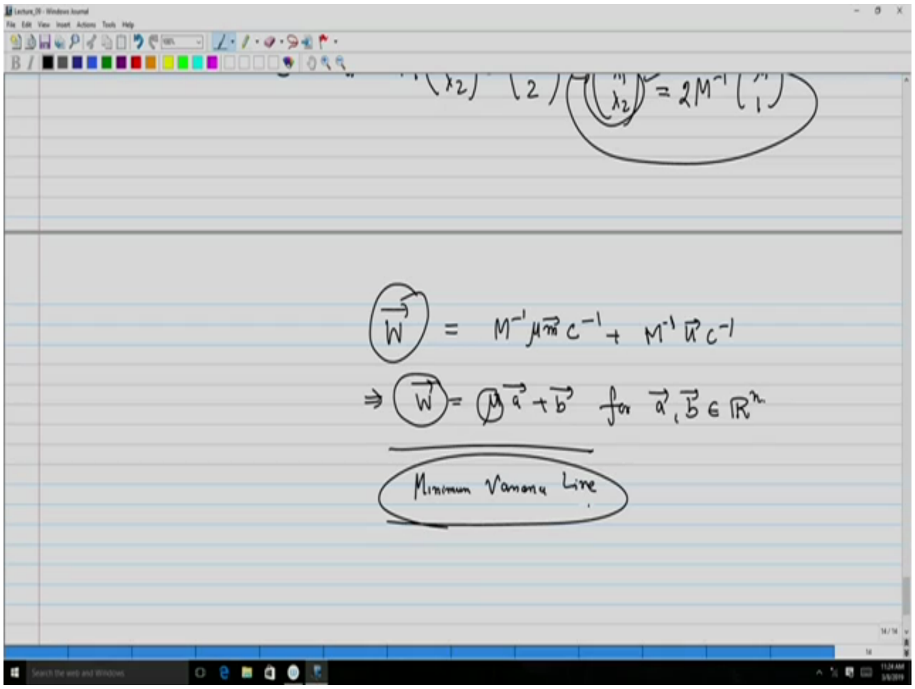

I can rewrite my weight now. So what I have to do is now I have to figure out what is going to be my W. So accordingly, let me recall my question for W here. In this expression for W, I will substitute λ_1 and λ_2 with the values of λ_1 and λ_2 that I have obtained here to obtain the following, to finally get my $W = M^{-1}\mu\vec{m}C^{-1} + M^{-1}\vec{u}C^{-1}$ and this can be written as $W = \mu a + b$, for $a, b \in \mathbb{R}^n$. So this is essentially means that, remember that this is the portfolio, this is not what you see in the (μ, σ) diagram.

This is the portfolio, so any portfolio which will give you the minimum variance at a certain level of return μ is given in this particular form and this is basically any question of a line in terms of μ. So, this is what is known as minimum variance line. As you change the different values of μ, remember that here your a and b are in terms of m as the inverse which does not change. So as you vary your μ, your W is also going to change and you will basically then get a family of portfolios which is what is known as the minimum variance line. So, this concludes more or less our discussion with the two asset portfolio and the efficient frontier. So, subsequently we will talk a little bit more about in terms of the interpretation of efficient frontiers and then we will move onto capital asset pricing model. Thank you for watching.