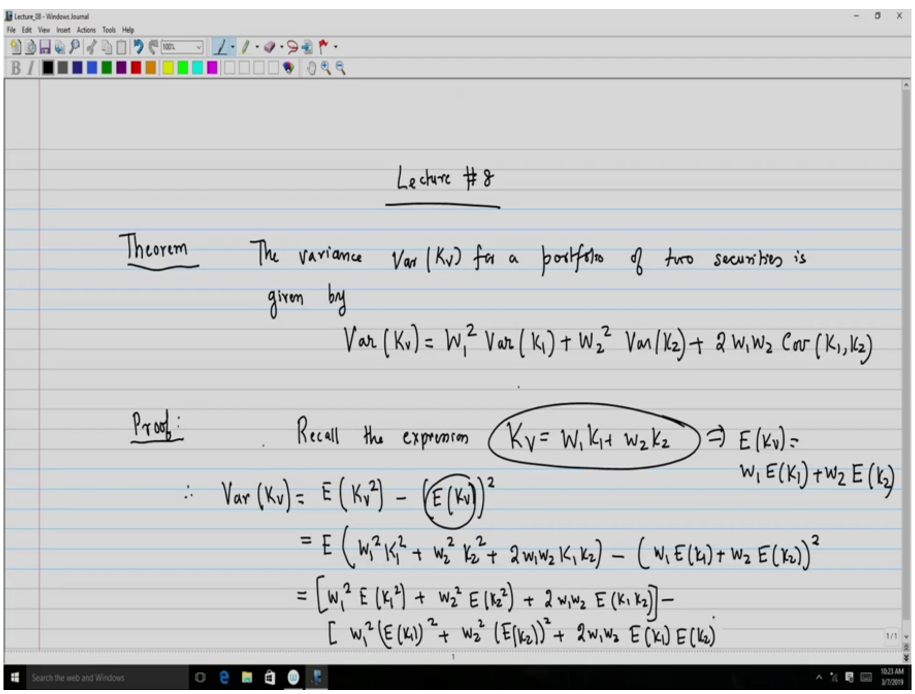
Mathematical Finance

Professor N. Selvaraju
Professor Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

Module 3: Modern Portfolio Theory
Lecture 2: Minimum Variance Portfolio and Feasible Set

Hello viewer, welcome to the 8th Lecture of this course on mathematical finance which is going to be the 2nd Lecture of Module 3.

(Refer Slide Time: 00:42)



In this lecture, we are going to look at some of the results for the two asset portfolio. So remember, in the previous class, what we had done is we had defined what is the return of an asset for a single time period and then we looked at a portfolio which comprised of two assets which are both risky and we looked at what is going to be the return of the portfolio and we had done one result that is the return of the portfolio is going to be equal to the weighted sum of the returns of the two individual assets and then we used it to define what is going to be the expected return of the portfolio of two assets. Now, recall that in the Markovas framework, the two main pillars were returned and risked.

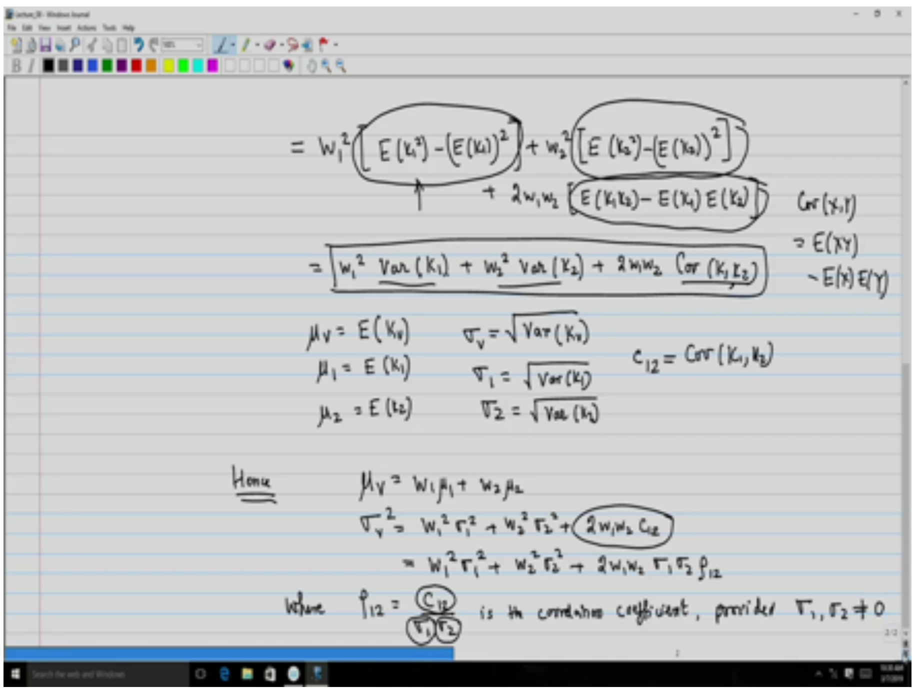
So, we begin todays class with a theorem on the risk that is associated with a two asset portfolio. So state the theorem as follows. So the variance $Var(K_V)$ for a portfolio of two securities or assets is given by $Var(K_V) = W_1^2 Var(K_1) + W_2^2 Var(K_2) + 2W_1W_2 Cov(K_1, K_2)$.

So look at the proof for this. So for the proof, we begin by recalling the expression for the return of this two asset portfolio. So, recall the expression of $K_V = W_1K_1 + W_2K_2$. Therefore $Var(K_V) = E(K_V^2) - [E(K_V)]^2$, this is going to be by definition, so I square this term of K_V. So this gives me $W_1^2K_1^2 + W_2^2K_2^2 + 2W_1W_2K_1K_2$. And for the 2nd case, you remember that we had this result of that $E(K_V)$ or the expected return on the portfolio is going to be W1 into the expected return of the 1st asset + W_2 into expected return of the 2nd asset. So, we substitute the $E(K_V$ here, that is $[W_1E(K_1)+W_2E(K_2)]^2$. So, opening this up, what do we get? We will get, so we make use of the linearity property now.

So this term, $E(W_1^2K_1^2$, this can be written using linearity of expectation as

$$W_1^2E(K_1^2) + W_2^2E(K_2^2) + 2W_1W_2E(K_1K_2) - W_1^2[E(K_1)]^2 - W_2^2[E(K_2)]^2 + 2W_1W_2E(K_1)E(K_2).$$

(Refer Slide Time: 05:49)



So continuing with this, so what we will do is we will combine the terms. So, let us combine the W_1^2 term 1st. So we will get $W_1^2E(K_1^2) - [E(K_1)]^2$. Then we will consider the W_2^2 term. So accordingly we get $W_2^2E(K_2^2) - [E(K_2)]^2$. And then they will take into account the cross terms W_1W_2, so we will get $2W_1W_2E(K_1K_2) - E(K_1)E(K_2)$. So observe carefully, this particular term here is $E(K_1^2) - [E(K_1)]^2$.

Remember that K_1 is a random variable. So by definition of variance, this term that we have here, this term can be replaced with $Var(K_1)$. Likewise, this term that you have here, this is going to be $Var(K_2) + 2W_1W_2$, we have this term. So, remember we had $Cov(XY) = E(XY) - E(X)E(Y)$. So, this term will become $Cov(K_1K_2)$. So, we essentially what we have now is that given the following input that you know what is the return for the 1st and the 2nd asset, and consequently, you know the expected return for the 1st and the 2nd asset and also you know what is the variance or standard deviation which we take as the risk of the returns of both the assets and we also know the information about the co-variances of both of them.

So using these three pieces of information, we can then construct the return or the expected return for the portfolio as a whole and the risk of the portfolio as a whole. So for a wide variety of choices for a wide range of choices for W_1 and W_2, we can examine case-by-case as to what is going to be our expected return and risk and then accordingly make our investment decision. So for the sake of brevity and convenience, we just briefly introduce a simplified notation for some of the input parameters that we are going to use in the course of our discussion on this two asset portfolio.

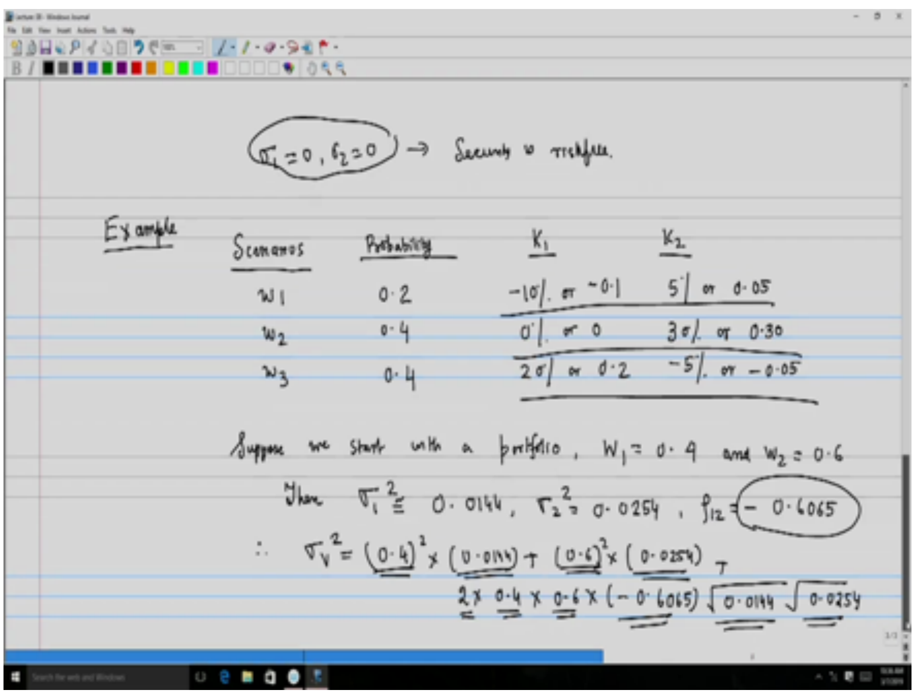
So, we now introduce the notation μ_V. So we will denote μ for the expected return. So accordingly we will define μ_V or identify μ_V with $E(K_V)$. μ_1 will be given as $E(K_1)$ and $E(K_2)$ will be denoted as μ_2. Likewise, the risk which is a standard deviation of K_V in case of the portfolio, this will be denoted by σ_V and $\sigma_1 = \sqrt{Var(K_1)}$ and finally $\sigma_2 = \sqrt{Var(K_2)}$. And there is one final notation.

So, we essentially have defined this term, this term. Finally, one more term is left, that is $Cov(K_1, K_2)$ and we will denote this by c_{12}, so there is the co-variance between the 1st and the 2nd assets return. Hence, the two results that we have obtained for the expected return and the risk for this two asset portfolio, can now be written in the simplified notation that we have just introduced as $\mu_V = W_1\mu_1 + W_2\mu_2$ and $\sigma_V^2 = W_1^2\sigma_1^2 + W_2^2\sigma_2^2 + 2W_1W_2c_{12}$. Now, this particular term, can be rewritten in another form giving of the following expression, $W_1^2\sigma_1^2 + W_2\sigma_2^2 + 2W_1W_2\sigma_1\sigma_2\rho_{12}$, where $\rho_{12} = \frac{c_{12}}{\sigma_1 sigma_2}$.

Remember that this is nothing but the correlation coefficients which is the co-variance between the two random variables divided by the standard deviation of the 1st random variable multiplied by the standard deviation of the 2nd random variable. So here, this is the co-relation coefficient, provided the denominator is not 0, so that means provided both of σ_1 and σ_2, these are not equal to 0.

So, in case of $\sigma_1\sigma_2 = 0$, then ρ becomes undefined. However, from a financial point of view, if we have either $\sigma_1 = 0$ or $\sigma_2 = 0$ or both of them 0, in each of those case, it implies that the asset is no longer risky because of 0 risk measure which in this case is standard deviation and hence, the portfolio no longer remains a portfolio of two risky assets. But instead, it becomes a risk-free asset.

(Refer Slide Time: 12:49)



So that means that if your $\sigma_1 = 0$ or $\sigma_2 = 0$, this essentially becomes that the security is risk-free. Now, let us look at a motivating example. So let us consider three scenarios as before but not necessarily recession, stagnation and boom that we have defined previously. So, just we identified three economic scenarios, ω_1, ω_2 and ω_3 and the corresponding probability of each of those scenarios are 0.2, 0.4 and 0.4, respectively. So we consider two assets with returns K_1 and K_2.

Now the return K_1 in the 1st scenario is -10% or -0.1. In the 2nd case, it is 0% or 0 and the 3rd case is 20% or 0.2. Identically, in case of the 2nd asset, the returns are 5% or 0.05, 30% or 0.30 and -5% or -0.05. So, now what we want to do is that we want to construct a portfolio in particular, a two asset portfolio out of these two given assets with the specified returns under three different scenarios. Accordingly, what we do is that suppose we start with a portfolio and remember for a portfolio we must specify what the weights are.

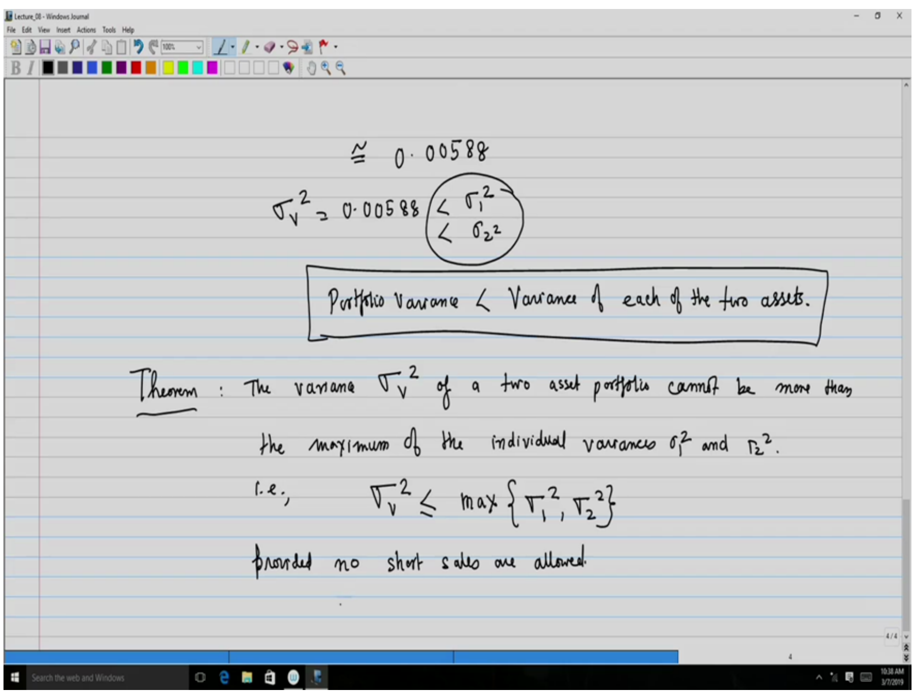
So, we start with a portfolio with weight $W_1 = 0.4$, that means 40% of the wealth is allocated to the 1st asset and W_2 naturally in that case is going to be 0.60 or 0 or 60% being allocated to the 2nd asset. Then you can compute the variance of returns of the 1st asset using the basic definition of variance and this is approximately 0.0144 and for the 2nd asset, this is 0.0254. And the co-relation between them $\rho - 12$ is -0.6065. So they essentially, they are negatively correlated, that also should be obvious considering the fact that in the 1st scenario, ω_1 when the 1st asset is incurring losses on negative returns, you get a positive return in case of the 2nd asset.

In the 2nd scenario when the return is 0 for the 1st asset, you get an extremely high return for the 2nd asset and the 3rd scenario when you have return that is fairly high in case of 1st asset you observe that the return in case of the 2nd asset becomes a negative quantity, that means it is incurring losses. So at a 1st glance when you compare these two returns, these two returns and finally these two returns, under ω_3 you observe that their pattern of returns is behaving in a somewhat opposite direction which is suggestive that your correlation coefficient is likely to be negative which would actually turns out to be -0.6065.

Therefore, we can use the formula for

$$\sigma_V^2 = 0.4^2 \times 0.0144 + 0.6^2 \times 0.0254 + 2 \times 0.4 \times 0.6 \times (-0.6065)\sqrt{0.0144}\sqrt{0.0254}.$$

(Refer Slide Time: 18:17)

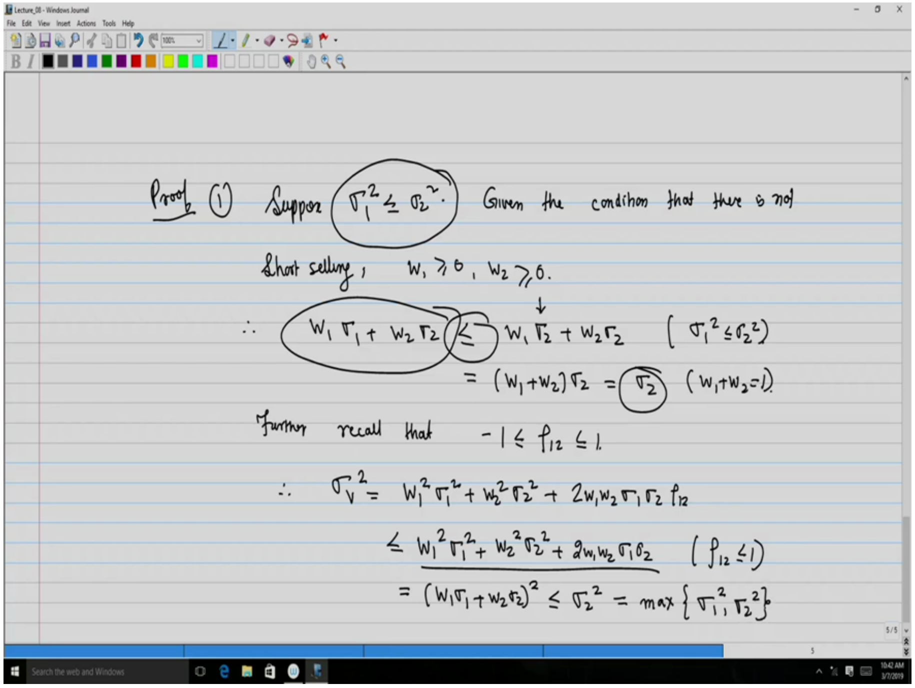


So, this turns out to be equal to approximately 0.00588 and you observe that this $\sigma_V^2 = 0.00588 < \sigma_1^2 < \sigma_2^2$. So what this suggests is that you have a portfolio variance is less than variance of each of the two assets. And this example illustrates the benefit of actually making an investment in a portfolio rather than just investing in one particular asset. So this example motivates the following theorem.

The variance σ_V^2 of a two asset portfolio cannot be more than the maximum of the individual variances, σ_1^2 and σ_2^2. That is, σ_V^2 which is the variance of the portfolio must be less than or equal to the larger of σ_1^2 and σ_2^2, provided no short sales are allowed. So, let us look at the proof of this.

So, for the proof, we can consider two cases. Suppose $\sigma_1^2 \leq \sigma_2^2$. Now given the condition that there is no short selling, remember that a short selling is a scenario where we borrow the stock from an owner and sell it off in the market under the assumption that the stock price will go down and subsequently we can

(Refer Slide Time: 20:55)



purchase it for a price that is lower than the one that we received by selling of the stock and return it to the owner of the stock and this act of selling without having the ownership is known as short selling.

And so this weight in this case is denoted by a negative number. So accordingly, given the condition that there is no short selling, so obviously the weights cannot be negative and so we will have $W_1 \geq 0$ and $W_2 \geq 0$. So therefore, $W_1\sigma_1 + W_2\sigma_2 \leq W_1\sigma_2 + W_2\sigma_2$. Why have I made this statement? Here I have made use of the fact that $\sigma_1^2 \leq \sigma_2^2$. So I have made a change to this particular term.

Now this can be written as $(W_1 + W_2)\sigma_2$ and remember that $W1 + W2 = 1$. Further, we recall that the correlation coefficient from basic probability we get correlation coefficient lies between 1 and -1. Therefore, $\sigma_V^2 = W_1^2\sigma_1^2 + W_2^2\sigma_2^2 + 2W_1W_2\sigma_1\sigma_2\rho_{12} \leq W_1^2\sigma_1^2 + W_2^2\sigma_2^2 + W_1W_2\sigma_1\sigma_2$, since $\rho_{12} \leq 1$. Now this term or this expression now has become a perfect square.

So, I can write this as $(W_1\sigma_1 + W_2\sigma_2)^2$. And here you remember, $W_1\sigma_1 + W_2\sigma_2 \leq \sigma_2$. Therefore, I can write that this is less than or equal to σ_2^2. Remember that I had assumed that $\sigma_1^2 \leq \sigma_2^2$. Therefore, σ_2^2 is nothing but the maximum of σ_1^2 and σ_2^2.

So, for the case 2, if we assume that suppose that $\sigma_2^2 \leq \sigma_1^2$. Remember in the 1st case we chose $\sigma_1^2 \leq \sigma_2^2$. So, in this case, we will make this particular assumption and the proof is analogous or similar. Now, going back to the statement of the theorem, we have said that this result holds provided no short sales are allowed. So naturally, I am curious about what will happen if there is actual short selling. So the question is that what happens if there is short selling.

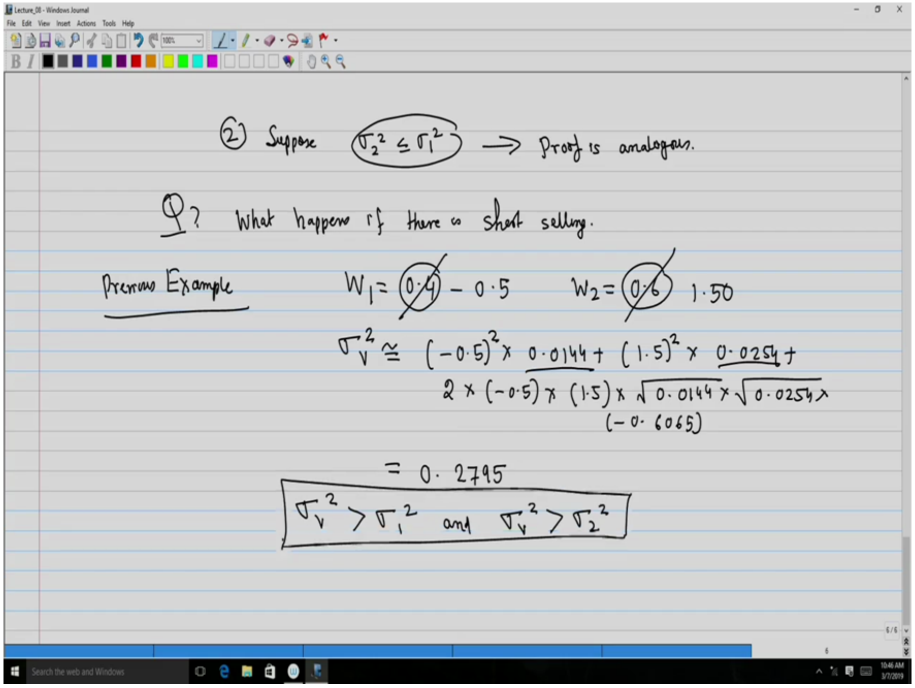
So again, we consider the previous example, but the only thing that changes now is instead of $W_1 = 0.4$, which is just -0.5, that means you are short selling the 1st asset and $W_2 = 0.6$, we change this to 1.50, so that $W_1 + W_2 = 1$. So, in this case, your

$$\sigma_V^2 \approx (-0.5)^2 \times 0.0144 + (1.5)^2 \times 0.0254 + 2(-0.5) \times 1.5\sqrt{0.0144} \times \sqrt{0.0254}(-0.6065).$$

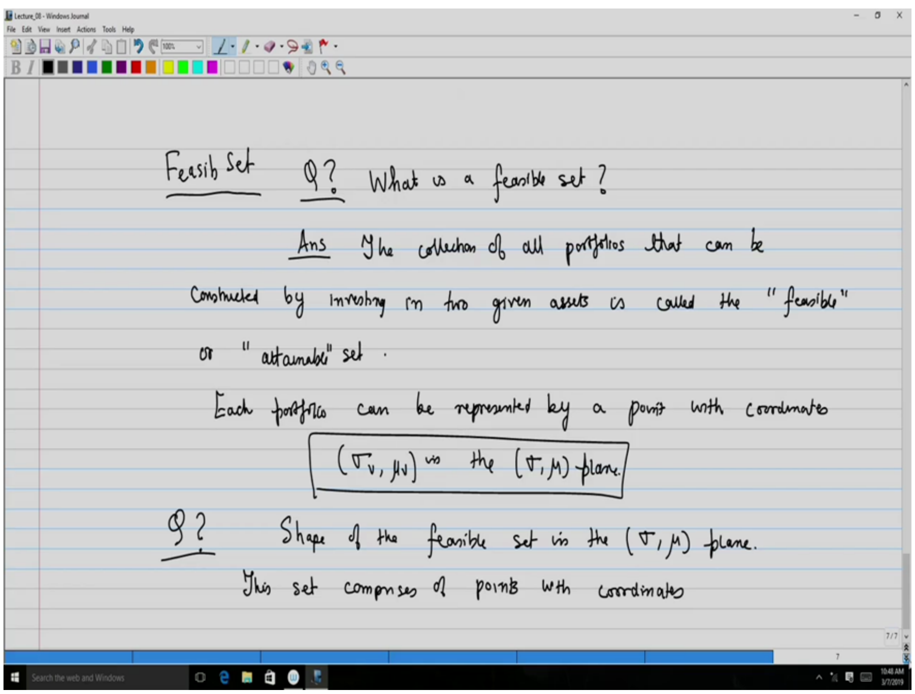
And now you see, in this case, this $\sigma_V^2 = 0.2795$. And interestingly in this case, $\sigma_V^2 > \sigma_1^2$ which is this and $\sigma_V^2 > \sigma_2^2$. So, in this case, as a result of short selling, what has happened is that the overall risk of the portfolio has increased and has reached a point where it is actually larger than the risk that are there for each of those assets individually.

We next come to something which is known as the feasible set. So, as the term feasible suggests, this means what is the possible set and that is what we are going to address. So what is a feasible set? And the

(Refer Slide Time: 24:51)



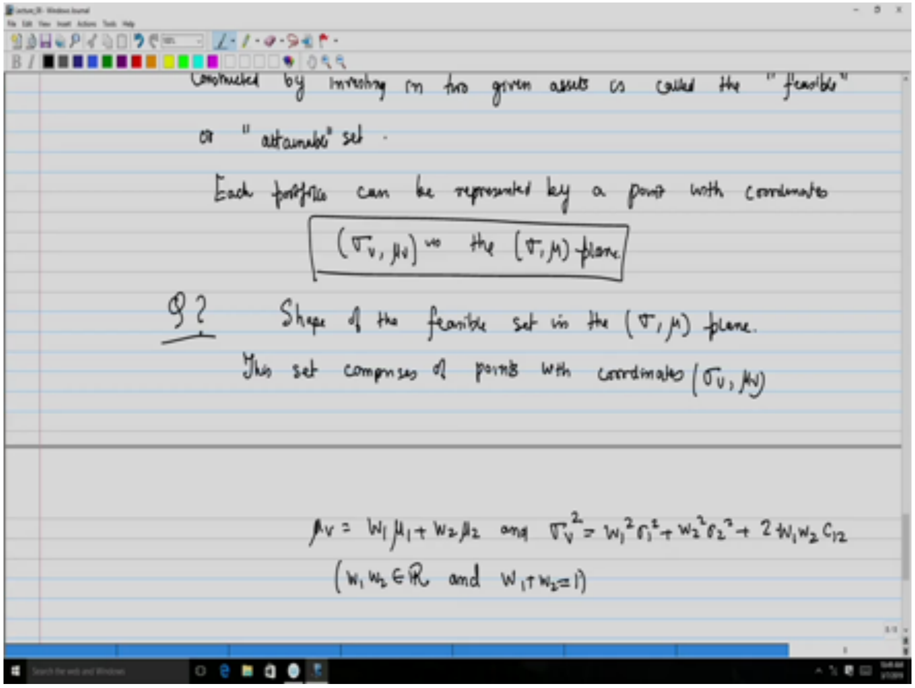
(Refer Slide Time: 28:00)



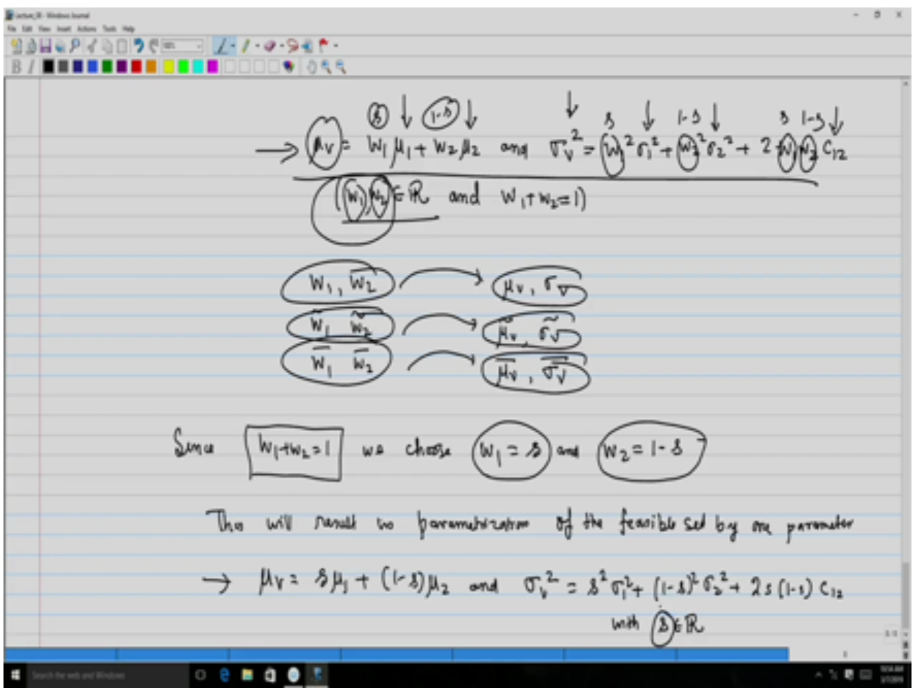
answer is, the collection of all portfolios that can be constructed by investing in two given assets is called the feasible or attainable set. So each portfolio can be represented by a point with coordinates (σ_V, μ_V) in the (σ, μ) plane. So we will explain this even in more details graphically.

So this immediately brings us to another question is what is going to be the shape of the feasible set in the (σ, μ) plane. So to address this, this set or the feasible set comprises of points with coordinates (σ_V^2, μ_V), where your $\mu_V = W_1\mu_1 + W_2\mu_2$ and $\sigma_V^2 = W_1^2\sigma_1^2 + W_2^2\sigma_2^2 + 2W_1W_2c_{12}$, where $W_1, W_2 \in \mathbb{R}$ or are just real numbers assuming the short sales are allowed and $W_1 + W_2 = 1$.

So what this really means is the following that I choose say one W_1W_2 and this will give me one set



(Refer Slide Time: 31:29)



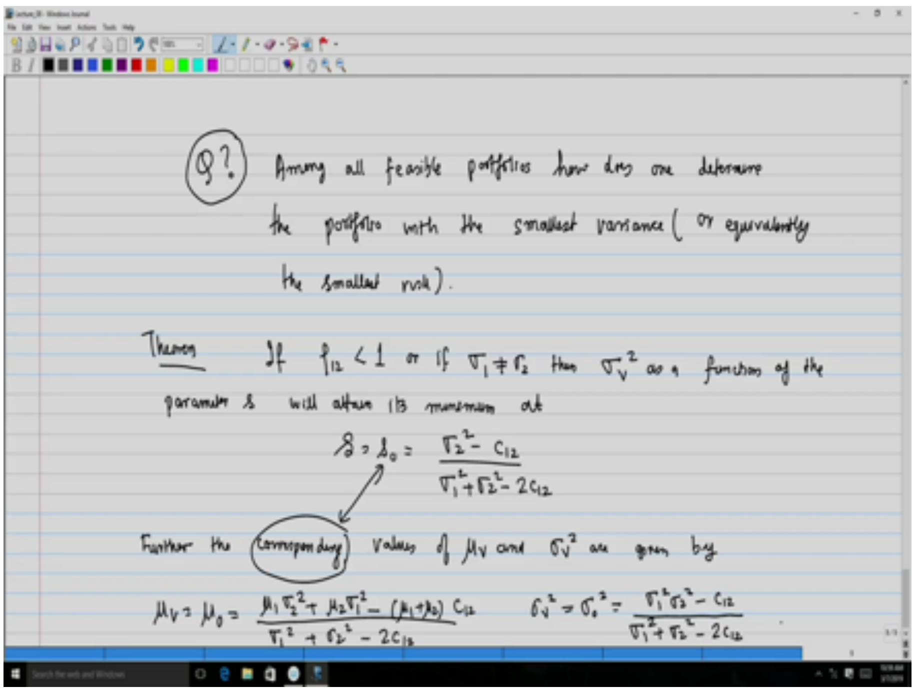
of μ_V and σ_V making use of this particular formula. Then I will have another set, (say) $\tilde{W}_1$, $\tilde{W}2$ and this will give me $\mu_{\tilde{V}}\sigma_{\tilde{V}}$. I might have another 3rd pair of weights $\bar{W}_1$ and $\bar{W}_2$ and this will give me the expected return $\mu_{\bar{V}}$ and $\sigma_{\bar{V}}$ as the risk. So, the feasible set essentially is the collection of all such portfolios which will result in these points being plotted on the mu sigma diagram with the sigma on the X axis and μ on the Y axis.

When you are talking about the question about what is the best choice, I mean effectively when I have said that W_1 and W_2 can be real value. So for all practical purposes, you are essentially getting an infinitely very different possible such portfolios and you want to figure out what is going to be the best choice of the portfolio particularly from the context of minimization of risk and motivated by this we know parameterize mu_V and σ_V by writing as follows.

That since $W_1 + W_2 = 1$, we choose $W_1 = S$ and $W_2 = 1 - S$. So remember in this expression for μ_V, your μ_1 and μ_2 are known. Likewise our estimated making use of the historical data for each of the assets and $\sigma_1^2,\ \sigma_2^2$ and c_{12} are the known terms in case of σ_V^2. So accordingly, the only two terms that actually can vary are W_1 and W_2 is equal to 1. We rewrite $W_1 = s$ which obviously gives $W_2 = 1 - s$. So what this does is that this will result in parameterization of the feasible set by one parameter.

And accordingly I can rewrite μ_V as $s\mu_1$ by writing this as s and I write this as $s\mu_1 + (1-s)\mu_2$. And I get my σ_V^2 to be equal to, so I replace this by $s,\ 1-s,\ s,\ 1-s$. So I replace this with $s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)c_{12}$ with $s \in \mathbb{R}$. So now what you have is that you have the advantage that earlier where mu_V was parameterized in terms of two variables, W_1 and W_2 and you have now rewritten the same thing in terms of one parameter, namely this s.

(Refer Slide Time: 35:43)



Now, what is the reason for me switching to a single parameter form of writing the expected return and risk of the portfolio. So let me address this by posing a question and the question is that, among all feasible portfolios, how does one determine the portfolio with the smallest variance, smallest variance obviously means the smallest standard deviation. So I can write or equivalently the smallest risk. So what we do here is that we are considering this two asset portfolio and what we can do is that we can sort of laboriously work on trying out various combinations of W_1 and W_2 and calculate the μ_V and σ_V square and check on a case-by-case basis which will be the lowest σ_V^2.

However, given the freedom that W_1 and W_2 are real value, this from a practical point of view, is impossible for us to make a decision on the part of $W_1,\ W_2$ that is going to give with the smallest σ_V^2. So that is the reason why we chose $W_1 = s$ and $W_2 = 1 - s$ so that the variance, σ_V^2 which was previously dependent on two variables, namely W_1 and W_2, now gets reduced to something that is dependent on one variable or in practice, σ_V^2 has become a function of a single variable. So if I am trying to minimize σ_V^2, then all I am trying to do is minimize a function of a single variable, namely s in this case.

So it is the amenability or the convenience of using basic differential calculus to a certain the value of its W_1 and W_2 that will give us the minimum risk portfolio. That is why we have chosen $W_1 = s$ and automatically I will get my $W_2 = 1 - s$ after I determine the s which minimizes the σ_V^2. So we can put this now as a theorem. If $\rho_{12} < 1$ and when it is equal to 1, we will consider the case separately or if $\sigma_1 \neq \sigma_2$, then σ_V^2 as a function of the parameter s will attain its minimum at $s = s_0 = \frac{\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}$. Further, the corresponding values and by corresponding, I mean the one that is corresponding to this particular s_0 which minimizes σ_V^2.

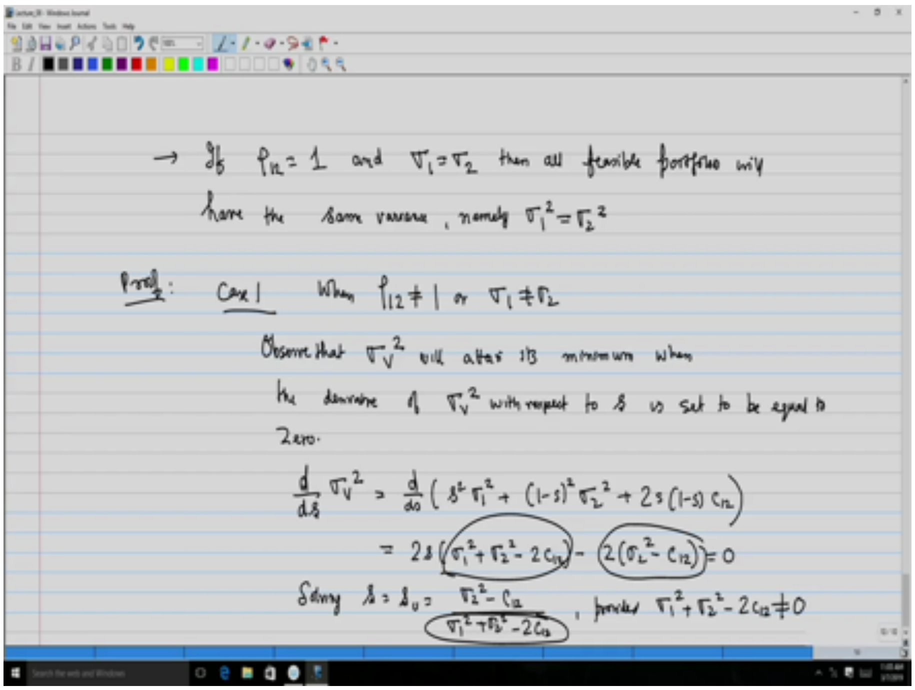
So the corresponding values of μ_V and σ_V^2 are given by μ_V and we call this particular

$$\mu_V = \mu_0 = \frac{mu_1\sigma_2^2 + \mu_2\sigma_1^2 - (\mu_1 + \mu_2)c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}.$$

And likewise,

$$\sigma_V^2 = \sigma_0^2 = \frac{\sigma_1^2\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}.$$

(Refer Slide Time: 40:51)



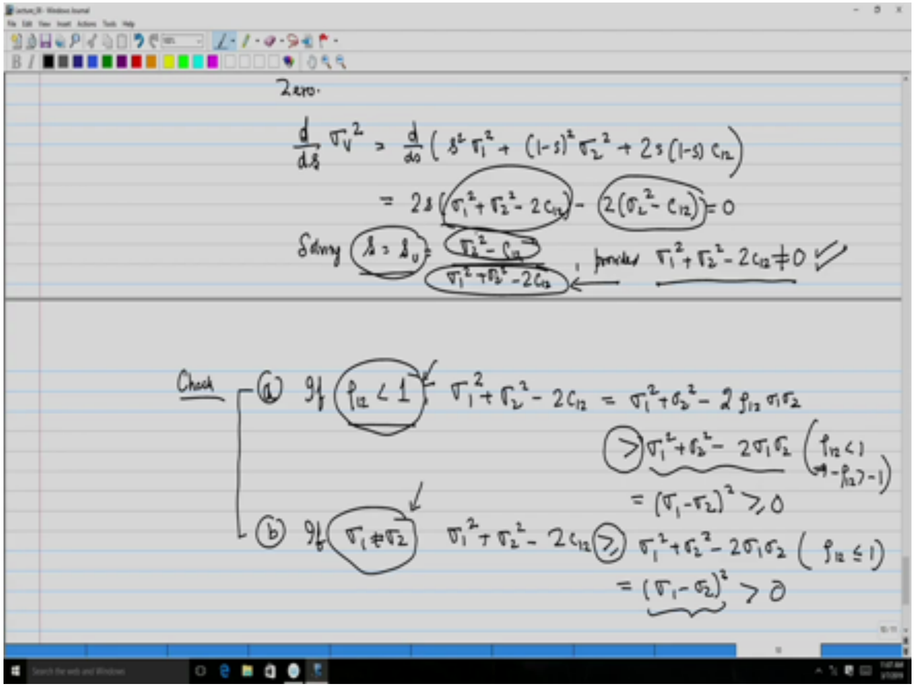
Now, we started off by saying that what you know if $\rho_{12} < 1$ and this condition holds, so if this condition does not hold which means either this is true or this is true. So that means the left out condition is that when ρ_{12} or the correlation coefficient is 1 and both the variances of the assets are identical, then all feasible portfolios will have the same variance, namely $\sigma_1^2 = \sigma_2^2$. So let us look at the proof of this. So let me split the proof into two cases.

So I will consider this to be the 1st case and this to be the 2nd case. So case 1, when $\rho_{12} \neq 1$ or $\sigma_1 \neq \sigma_2$. So observe that σ_V^2 which is synonymous with the risk because that is just the square of the risk which is given by the standard deviation. So 1st observe that σ_V^2 which is a function of a single variable s, will attain its minimum when the derivative of σ_V^2 with respect to $s = 0$.

So basically σ_V^2 which is now a function of a single variable. The minimization of that will be done by making use of the derivative test that you have seen in differential calculus where the derivative must be equal to 0 at the point where the minimum is attained. Accordingly, what we do is that we start off with taking the derivative of σ_V^2 which now is a function of s. So I take the derivative with respect to s, so this is $\frac{d}{ds}(s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)c_{12})$. So I have just used the definition of σ_V^2 in terms of the parameter s.

So the derivative of this turns out to be $2s(\sigma_1^2 + \sigma_2^2 - 2c_{12} - 2\sigma_2^2 - c_{12})$. And in order to find the value of s at which σ_V^2 is minimized, we set this derivative equal to 0. So then, solving we get $s = s_0$. It will be this term divided by this term which is $\frac{\sigma_2^2 - c_{12}}{\sigma_1^2+\sigma_2^2-2c_{12}}$ provided the denominator is not equal to 0. So provided $\sigma_1^2 + \sigma_2^2 - 2c_{12} \neq 0$.

(Refer Slide Time: 45:24)



So, I just do a quick check. One, if $\rho_{12} < 1$ which was one of the conditions for this particular case. Then $\sigma_1^2 + \sigma_2^2$. So which is that? So let us just examine what is going to be the denominator or what is going to happen to the denominator under the situation when $\rho_{12} < 1$ and under the situation when $\sigma_1 \neq \sigma_2$. So, first of all, accordingly I start off with $\rho_{12} < 1$. Then the denominator which is $\sigma_1^2 + \sigma_2^2 - 2c_{12}$, what is this going to be?

This is $\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2$. And this is going to be strictly greater than $\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2$. Why am I claiming this? Because my $\rho_{12} < 1$ which implies that $-\rho_{12} > -1$. So I multiply both sides by $\sigma_1\sigma_2$ to make this conclusion. And what is this expression? This expression is $(\sigma_1 - \sigma_2)^2 \geq 0$, because I have assumed only that $\rho_{12} < 1$.

So that means, the denominator under this assumption is going to be strictly greater than 0. So obviously this condition is going to be satisfied. And B, if the other condition that $\sigma_1 \neq \sigma_2$ holds, then $\sigma_1^2 + \sigma_2^2 - 2c_{12}$, this is going to be greater than or equal to $\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2$. So I have just made use of the fact that $\rho_{12} \leq 1$. Strictly less than 1 is not required. And that is the reason why I have the greater than or equal to sign here. And this is equal to $(\sigma_1 - \sigma_2)^2$.

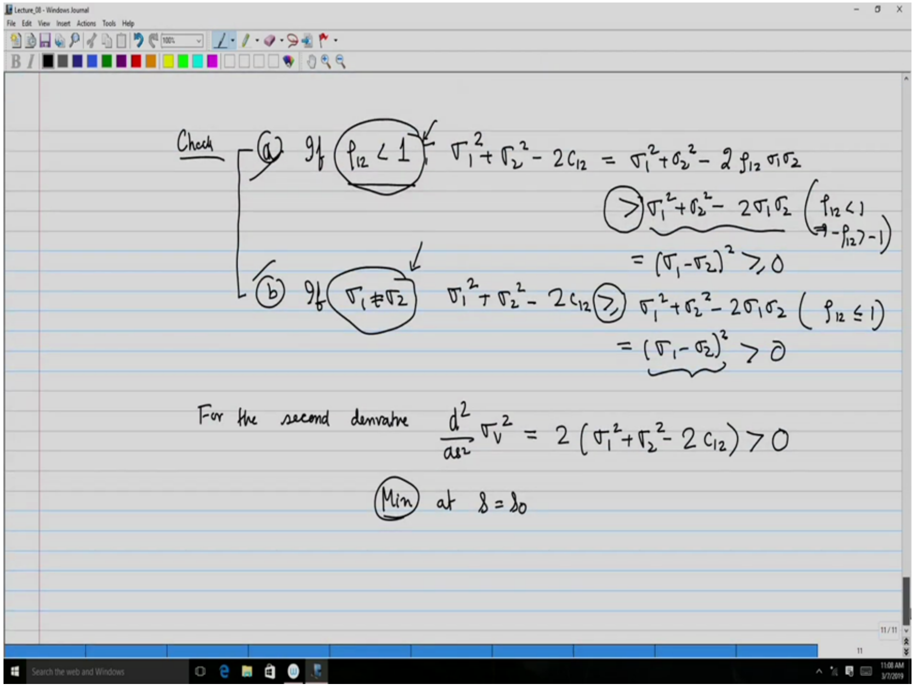
And because $\sigma_1 \neq \sigma_2$, this is going to be strictly greater than 0, which again satisfies this condition. So, under both the premises of the theorem, we are able to prove that yes indeed, this condition is satisfied. So that means, S naught will always exist. Remember that s_0 that we have here, have the terms σ_1^2, σ_2^2 and c_{12} and all of them are known quantities. Now when I am doing the minimization, it is not enough for me to just take the derivative and set it equal to 0. We need to do the 2nd derivative steps also.

Therefore, for the 2nd derivative that is $\frac{d^2}{ds^2}(\sigma_V^2)$, so we go back to this expression. So, the 2nd derivative is going to be just this term here. So accordingly, we get the 2nd derivative to be $2(\sigma_1^2 + \sigma_2^2 - 2c_{12}$, which is nothing but the denominator. And we have already shown for both the case (a) and (b), that this is always going to be strictly greater than 0. And hence, minimum is attained at $s = s_0$ that we have derived above.

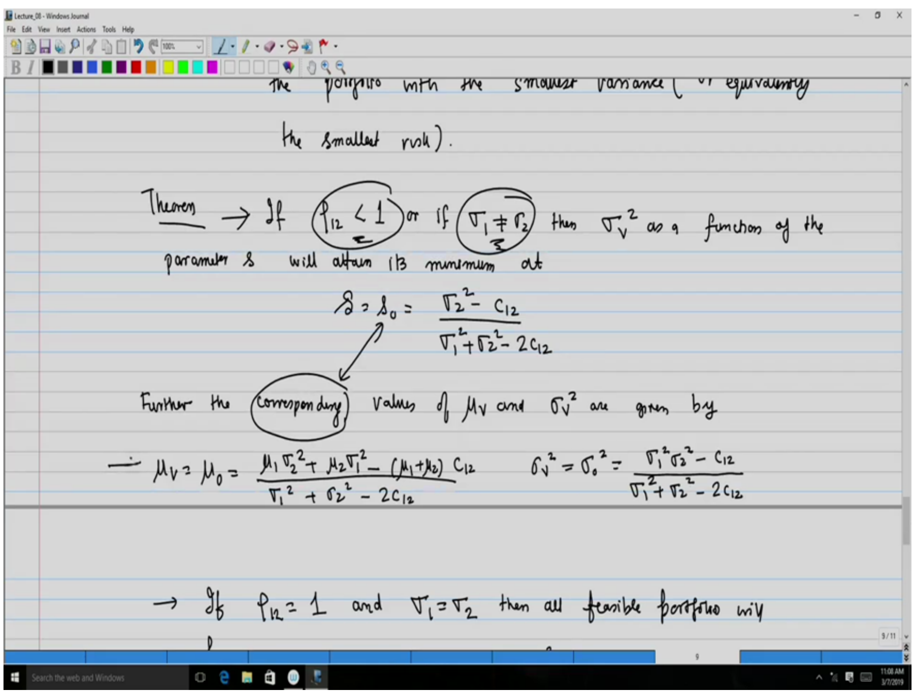
Now in this case, so if you go back to the statement of the theorem, I said that these are going to be the corresponding values for μ_V and σ_V. So, we can obtain these values. So these are nothing but specific values of μ_V and σ_V when we replace $s = s_0$.

So what do we need to do here is that in order to obtain, so to obtain μ_0 and σ_0 or σ_0^2, all you have to do is $\mu_0 = s_0\mu_1 + (1 - s_0)\mu_2$. So we just put the value of s_0 in both the places and we will get the expression of μ_0 that is given in the statement of the theorem.

(Refer Slide Time: 48:49)



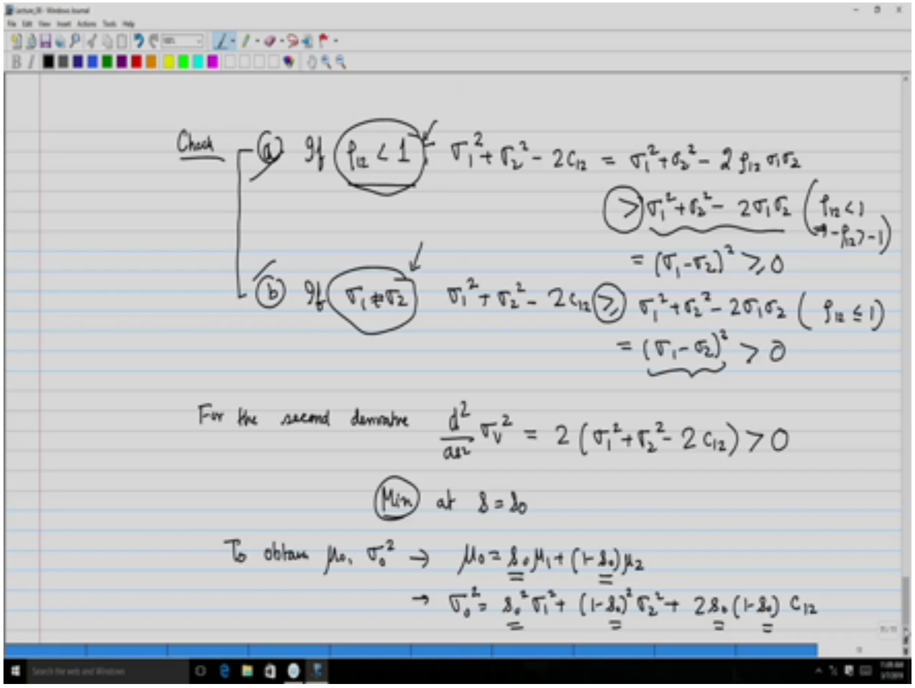
(Refer Slide Time: 49:55)



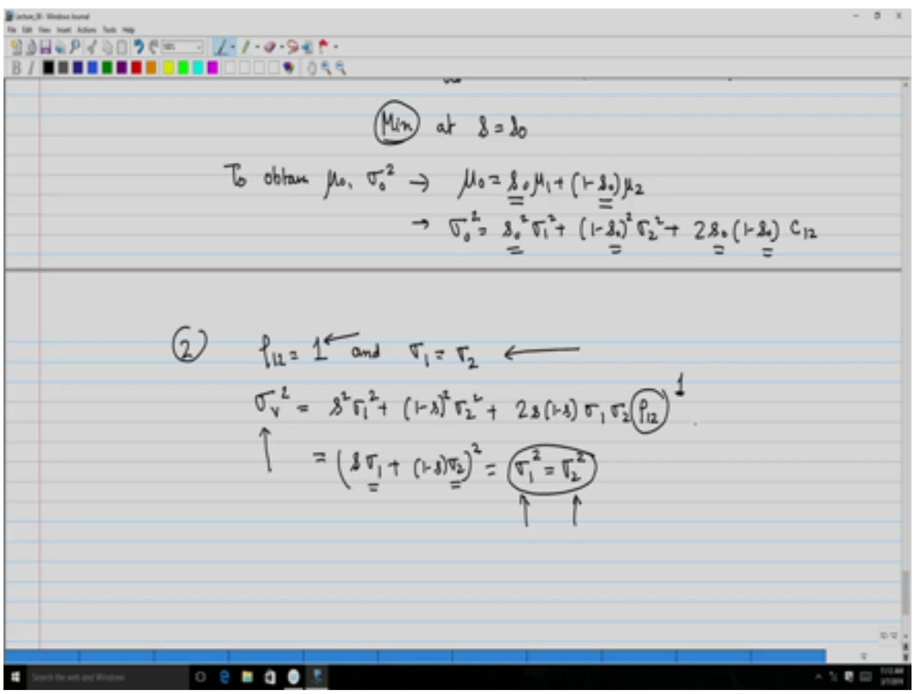
Likewise, you will get σ_0^2 will be $s_0^2\sigma_1^2 + (1-s_0)^2\sigma_2^2 + 2s_0(1-s_0)c_{12}$. So you replace these values of s_0 in these three places or four places here and do the simplification and then you will be able to get back the expression of σ_0^2 that is given in the statement of the theorem. Now, remember that this was the 1st case when I assumed that $\rho_{12} \neq 1$ and $\sigma_1 \neq \sigma_2$. So I look at the 2nd case now.

When ρ_{12} or the correlation coefficient is 1 and $\sigma_1 = \sigma_2$, both these conditions hold. So what happens in this case? In this case, σ_V^2 which is given by $s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)\sigma_1\sigma_2\rho_{12}$. And remember that in this case $\rho_{12} = 1$. So, this entire expression becomes $[s\sigma_1 + (1-s)\sigma_2]^2$. Now, observe carefully that here we have $\sigma_1 = \sigma_2$.

(Refer Slide Time: 50:19)



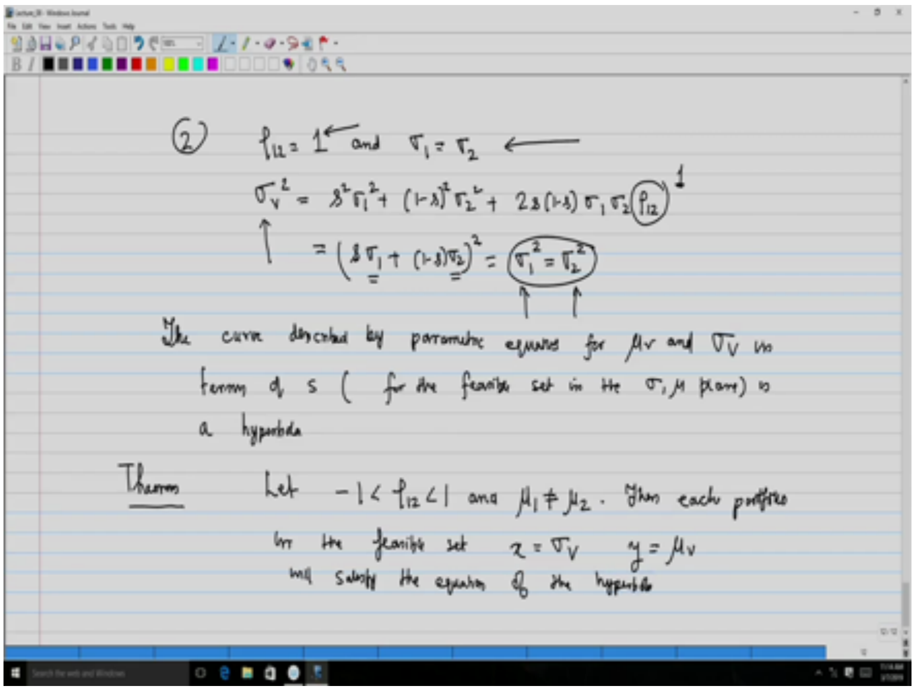
(Refer Slide Time: 51:34)

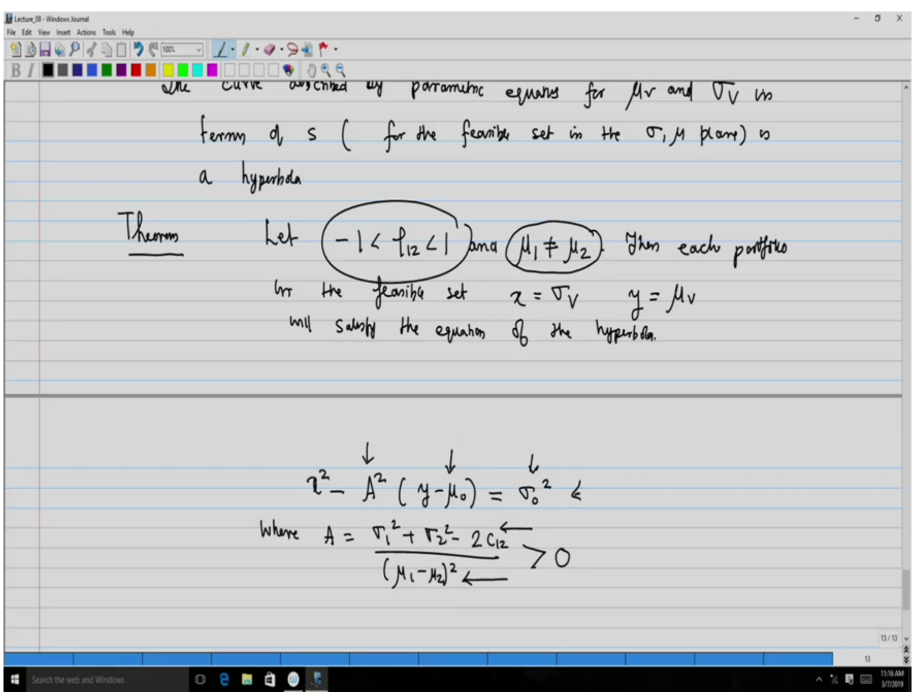


So that means, if I replace $\sigma_1 = \sigma_2$ or if I replace $\sigma_2 = \sigma_1$, then the s and $1 - s$ terms will cancel out. So you end up getting this to be equal to σ_1^2 or σ_2^2. So this is sort of a very interesting observation that basically irrespective of what the weights are, you end up getting the same portfolio risk or variance which is equal to the variances of either of the individual assets. So it does not matter in what proportion you invest in each of the assets, you may invest all your money in the 1st asset or the 2nd asset or you can take a combination of both.

In all cases, the variance is going to be exactly identical which again makes sense because you understand that $\sigma_1 = \sigma_2$ means in terms of this, there is very little or in fact, there is nothing to choose between two and $\rho_{12} = 1$ means that they are expected to behave in exactly the identical way in terms of the patterns of return. So that means that from the financial point of view, it is nothing to distinguish between both of them. So effectively, they are synonymous with each other. And that means that it does not matter in what proportion invest in either of them or if we invest completely in either of them, we end up facing the same amount of risk.

(Refer Slide Time: 56:13)





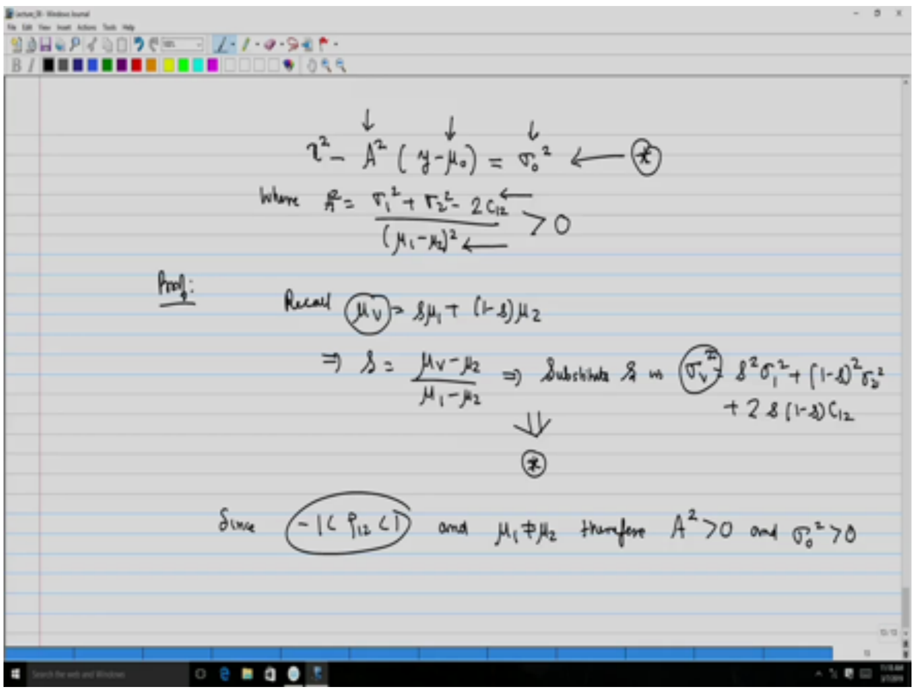
So now, coming back to the feasible set. So, the curve described by parametric equations for μ_V and σ_V in terms of s for the feasible set in the $(\sigma,\ \mu)$ plane is a hyperbola. And we will see how this is a hyperbola. We will state this as a theorem. So let the correlation coefficient lie strictly between -1 and 1. And let the returns for the two assets be non-identical, then each portfolio in the feasible set and by this I mean σ_V and

μ_V. So it means that corresponding to each value of $W_1,\ W_2$ or equivalently s and $1-s$, you will get a corresponding μ_V and σ_V.

I will call this $\sigma_V = x$ and $\mu_V = y$. So every portfolio $\sigma_V,\ \mu_V$ or equivalently xy, this will satisfy the equation of the hyperbola given by $x^2 - A^2(y-\mu_0)$. Remember that μ_0 was the return corresponding to s naught at which σ_0^2 becomes or rather σ_V^2 is minimized and is equal to σ_0^2. So, this is equal to σ_0^2. Now here, we already know what is μ_0 and σ_0. These are the μ_V and σ_V respectively when $s = s_0$ that we have already obtained.

And one additional term that is not introduced that I have included here, so I specify that now. Where $A = \frac{\sigma_1^2+\sigma_2^2-2c_{12}}{(\mu_1-\mu_2)^2} > 0$. So, I needed a condition that $\mu_1 \neq \mu_2$, because of the denominator here. And I needed this condition for this numerator here, so that I end up getting $A > 0$, so that I can look at this as a hyperbola.

(Refer Slide Time: 57:37)



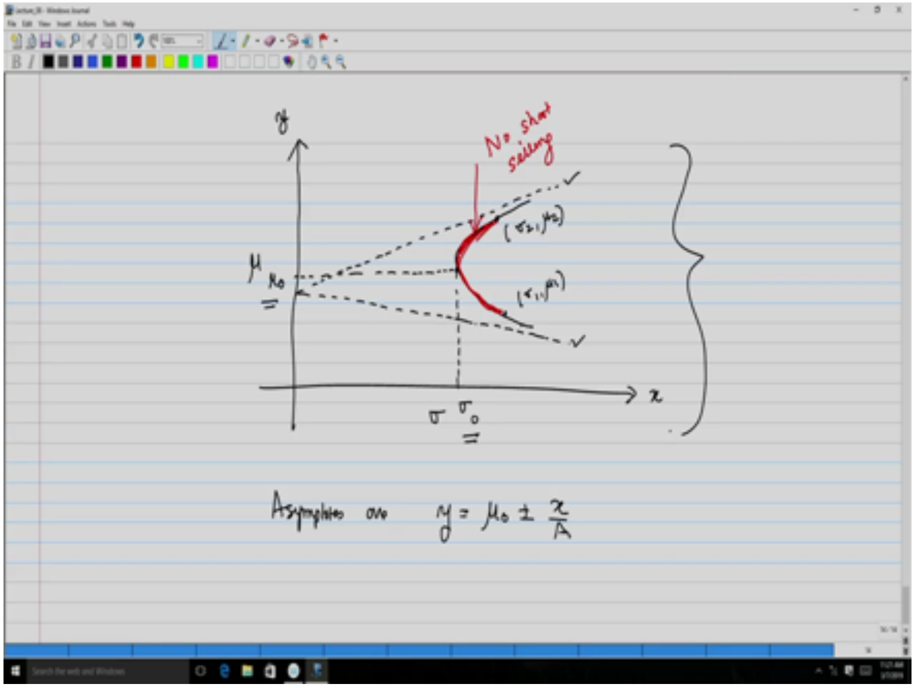
So the proof of this is fairly simple, but a little tedious. So recall, $\mu_V = s\mu_1 + (1-s)\mu_2$. So this implies that $s = \frac{\mu_V-\mu_2}{\mu_1-\mu_2}$ and then I substitute s in $\sigma_V^2 = s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)c_{12}$. And this will give you your expression. Let us recall this expression star. So this will give you an expression star.

So this means that the expression star can be a obtained by eliminating s from the parametric equations or expressions for μ_V and σ_V^2. Eliminate s and you will end up getting the relation connecting μ_V and σ_V. Now so as I have already pointed out before that since rho lies strictly between -1 and 1 and $\mu_1 \neq \mu_2$. Therefore $A^2 > 0$. Actually this is A^2. $A^2 > 0$ and $\sigma_0^2 > 0$.

So now graphically this looks like as follows. See, on the x axis and this is the y axis. On the x axis, we have σ? Remember, this is σ_V and on the y axis, we have μ. Then the shape of the hyperbola will look something like this. And these are the asymptotes of this hyperbola. So this hyperbola basically is graphically represented here and the hyperbola, I mean the one that I had indicated as star. And here this is the point where there is a minimum risk and so therefore the corresponding risk is going to be σ_0 and the corresponding return is going to be μ_0.

So this particular point here is the portfolio for which $s = s_0$ and therefore the corresponding return is σ_0 and the corresponding risk is σ_0. So therefore, now here this region now I identify two points on this hyperbola, one of them is $\sigma_2\mu_2$, this means that we have invested completely in the 2nd asset and likewise, there is the point (σ_1, μ_1). So then this particular region on the hyperbola, this is the region which has the

(Refer Slide Time: 59:42)



portfolios with no short selling.

And we have two asymptotes here, this one and this one. The asymptotes are $Y = \frac{\mu_0 \pm x}{A}$. So we have concluded the 1st part of our discussion on minimization of the risk for a two asset portfolio and constructing feasible set and giving a graphical representation. So we will continue this discussion and do a little more analysis on this two asset portfolio before moving onto the multi-asset portfolio. Thank you for watching.