Mathematical Finance

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Module 2: Time Value of Money and Risk Free Assets
Lecture 3: Price Yield Curve and Term Structure of Interest Rates

Hello viewers, welcome to the 6th Lecture of this course on Mathematical Finance, which is going to be the 3rd Lecture of Module 2. So, in this lecture, we will essentially look at yield curve and term structure of interest rates in details and illustrate this with a few examples.

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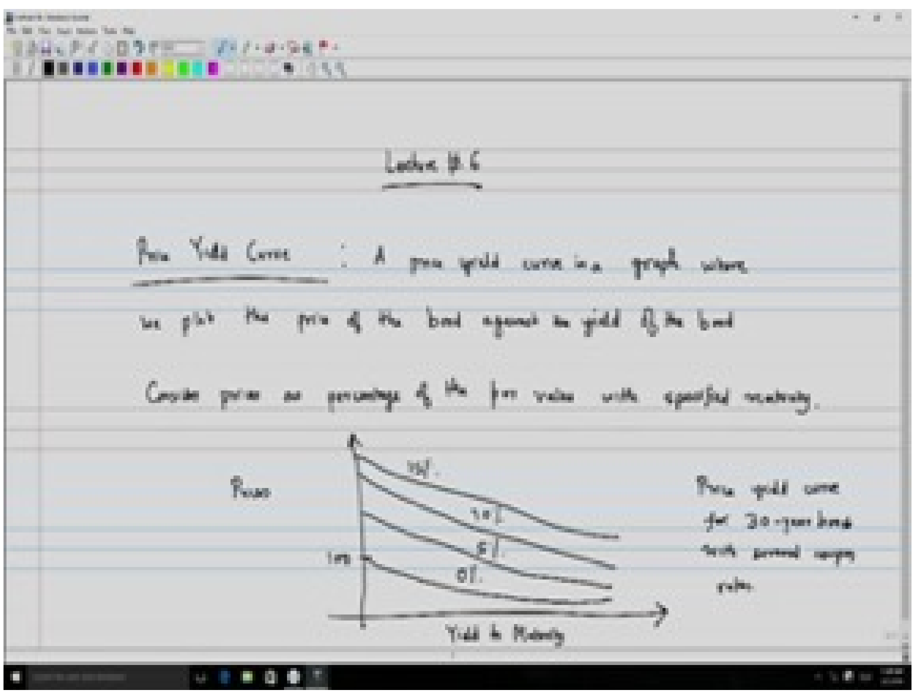

So, let us begin this lecture with what is known as the price yield curve. What is the motivation of price yield curve? Motivation of price yield curve is to ascertain essentially the return that you expect to get from various yields in case of both pure discount bond as well as coupon bond. So, formerly we will define a price yield curve as follows. A price yield curve, a two-dimensional graph where we plot the price of the bond on the Y axis against the yield of the bond. So, consider prices as percentage of the par value with specified majority. So a price yield curve will typically look something like this. On the Y axis we will consider the prices as percentage of par value and on the X axis we look at the yield maturity.

So, this is essentially what we are going to do is the price yield curve for 30 year bond with several coupon rates. So, we will look at several cases where we have different coupon rates and we will just see how the price changes depending on what is the yield to maturity. So, first let us look at a graph when the coupon is 0%. So, in this case, the initial time point the price is basically going to be 100%. Then suppose we have the coupon is at 5%, then this is going to be the graph of 5%. Likewise, we have the graph for 10% and (say) 15%.

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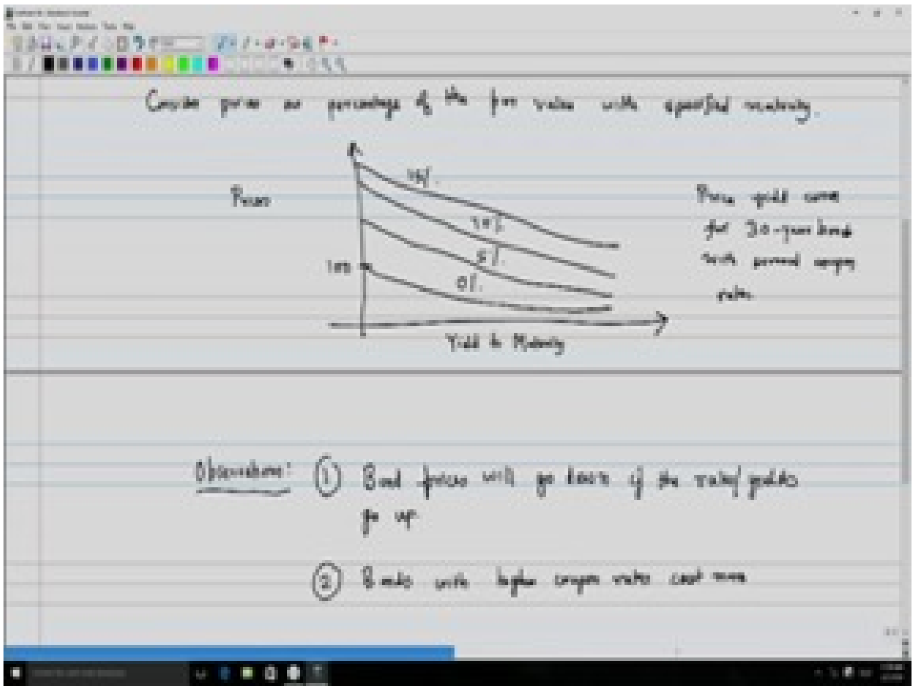

We will now make a couple of observations from the graph that we have just seen. So, 1st observation that we will make is that the bond prices will go down, that means as the yield increases as you move from left to right on the X axis, the curve keeps going down. So, the interpretation of this is that the bond price will go down if the rates or the yields go up. And 2nd observation that you make is that bonds with higher coupon rates cost more.

So, this means that if you go back to the graph, let us try to see how this interpretation is to be looked at in a meaningful way. So, if you look carefully in the graph, the 1st observation that I said is the bond prices will go down if the rates or the yields go up. That means that if you are moving from left to right of the X axis that is the yields are going up, then the bond prices are going down.

Then if you start looking at all the graphs in a comparative manner, you observe that as the coupon rates are increasing that means that in this case, say, I have 0%, when I go to 5%, then corresponding to the same yield to maturity I will have a higher bond price. Likewise, if you go up to 10%, then for every yield to maturity, the price of the bond is going to be higher than in case where the coupons are 0% and 5%, and this trend continues. So, what this means is that as the yields of the rates go down, the bond prices will go up and in case the yield of the rate goes up then obviously the bond prices are going to go down.

And the 2nd case, it is intuitively sort of obvious that as you are moving along this graph and as the coupon rates go up, the prices of the bonds are going to go higher. And the reason for this is that as compared to (say) for example a 0% bond, or a bond with 0% coupon, when you invest in a bond with 5% coupon, then the payments that you receive from the person or the entity which has issued the bond, the payments are going to be collectively a larger sum and hence it is obvious that at the initial time point, you would have made a larger investment in terms of the price of the bond as compared to when you are looking at a bond, where the coupon rates are smaller, in which case you are obviously going to get a lesser amount of money until maturity.

Now, this motivates us to talk about something known as the term structure of interest rates. So, the next topic we look at is the term structure of interest rates, and this brings us to the observation that it is essentially to look at the rates for all maturities as they progress over time.

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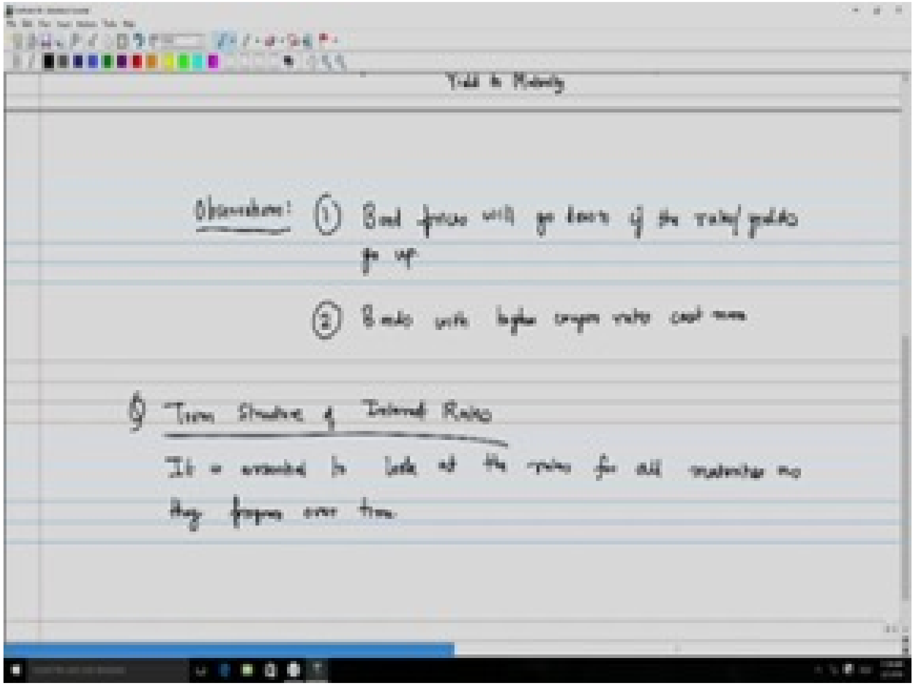

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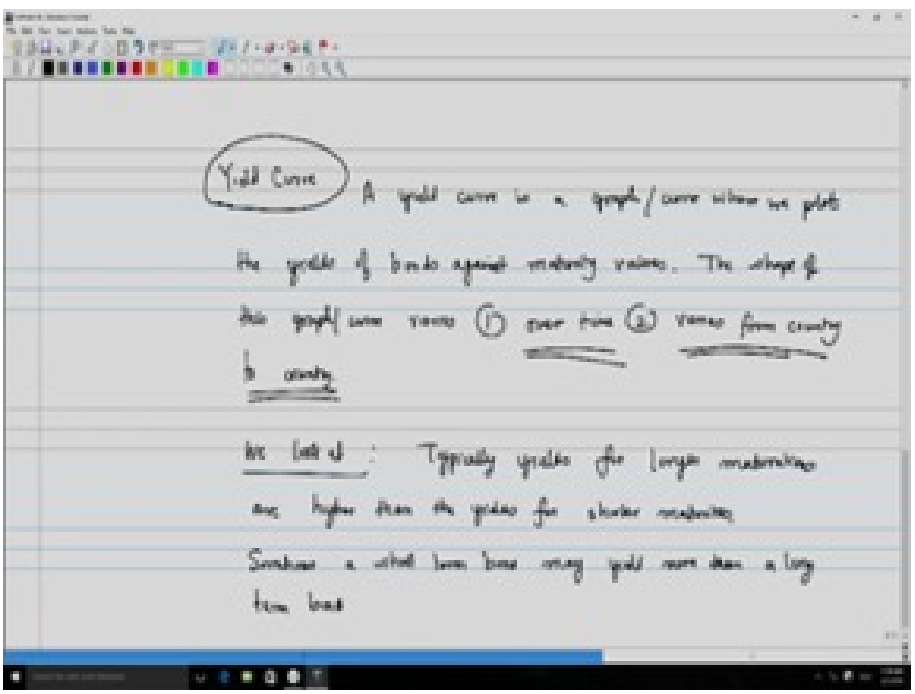

So, this brings us to what is known as the yield curve. So, yield curve essentially will give you relationship between the majority and the rates. Yield curve is a graph or a curve where we plot the yields of various bonds against maturity values and by maturity values you mean the actual maturity. And typically the shape of this graph or the curve varies on two things or varies according to two things, namely, varies over time and secondly, varies from country to country. So, for the 1st observation, what you can say that when I said it varies over time, that means for different maturities, you will have different rates that are applicable and my 2nd observation that it varies from country to country, this means that the interest rate that is prevailing for a certain period of time or certain maturity will keep changing as you look at bonds for various different countries.

Then we look at the following observation that typically yields for longer maturities that means long-term bonds are higher than the yields for short term maturities or rather for shorter maturities. This means that if you are for example purchasing a bond sale for a period of 6 months, then typically you will observe that interest rate that you are getting or the yield that you are getting on the bond of 6 months is going to be lower than what you typically get for a longer duration or maturity of (say) 5 years. Having said so what I have said just now is a typical situation however, it is possible that once in a while this particular trend of relationship between the yield and the maturity may not hold in a similar fashion. And for this I mean that sometimes a short-term bond may yield more than a long-term bond.

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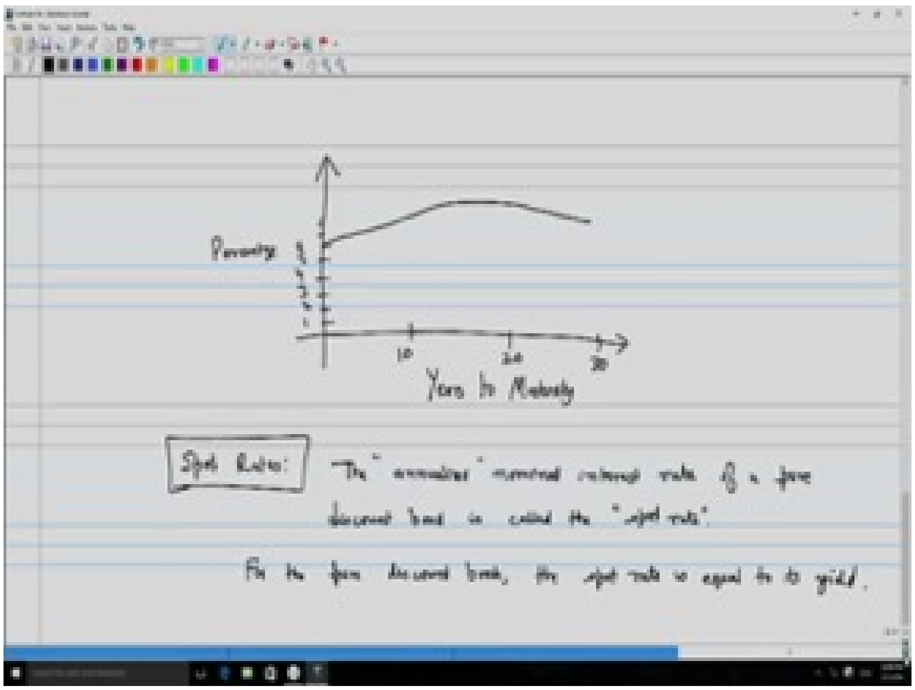

So, graphically if we observe this, this is going to look something like this. So if I have this as years to maturity, that means the remaining duration of the bond, may be 10 years, 20 years, 30 years, so on and the yield in terms of percentage, say, this is 1, 2, 3, 4, 5, 6, amd so on. So typically the graph will look like this, but occasionally the graph might dip that is to say that for a longer duration you might actually get lesser yield as compared to shorter duration, but typically you will have this graph always going up. So, next I will start looking at what is known as the spot rate and this is a very important concept in interest rate modeling.

So, the spot rate is defined as follows. The annualized nominal interest rate, remember I am talking about the annualized nominal interest rate of appeared discount bond is called the spot rate. Now for the pure discount bonds, the spot rate is equal to its yield, so it is obvious that the spot rate is going to be equal to the yield in case of a pure discount bond because you are making an initial payment which is the price of the bond and you receive a final payment. So whatever is the prevailing interest rate that is applicable for this particular bond is obviously going to match the yield of the bond.

Next, we will illustrate this through an example. In this example, we consider a pure discount bond that will mature in 6 months, so this is basically a six-month 0 coupon bond. Now, what are the characteristics in this case? It has the price of 98 and the nominal value of 100. So, this means that today at time t is equal to 0 the price of the bond is 98 and at the end of 6 months, you receive an amount of 100 which is your nominal value. Then the annualized compound rate which I have denoted as r_{6m} for this example is obtained as follows. So, 98 is going to be, the final value $\frac{100}{1+r_{6m}}$, which is the annualized interest rate raise to half, because this interest rate is applicable for the maturity of this bond namely six months. So if you solve this, this will have solution of $r_{6m} = 0.41233 = 4.1233\%$.

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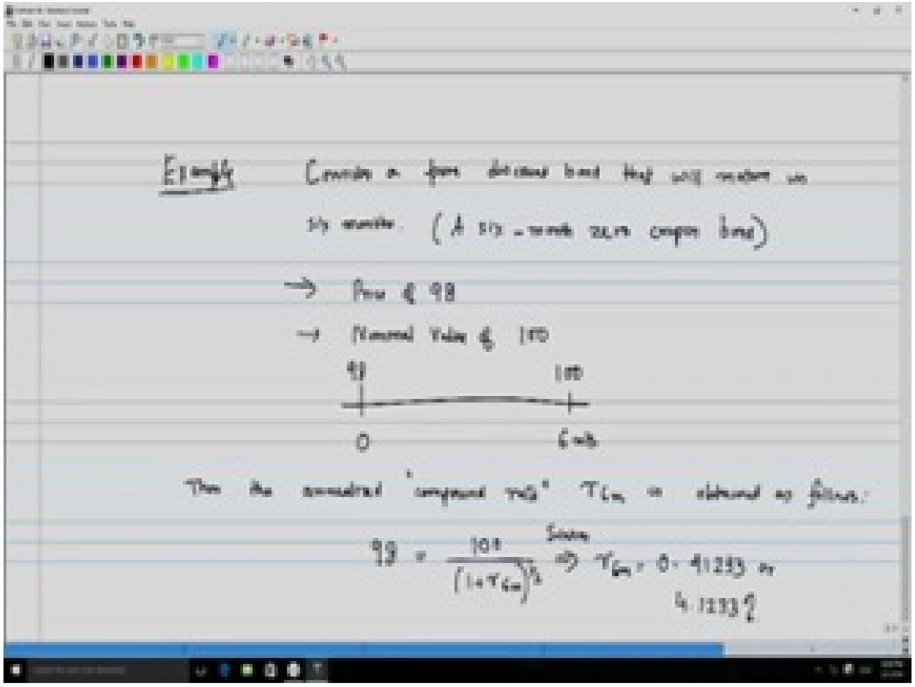

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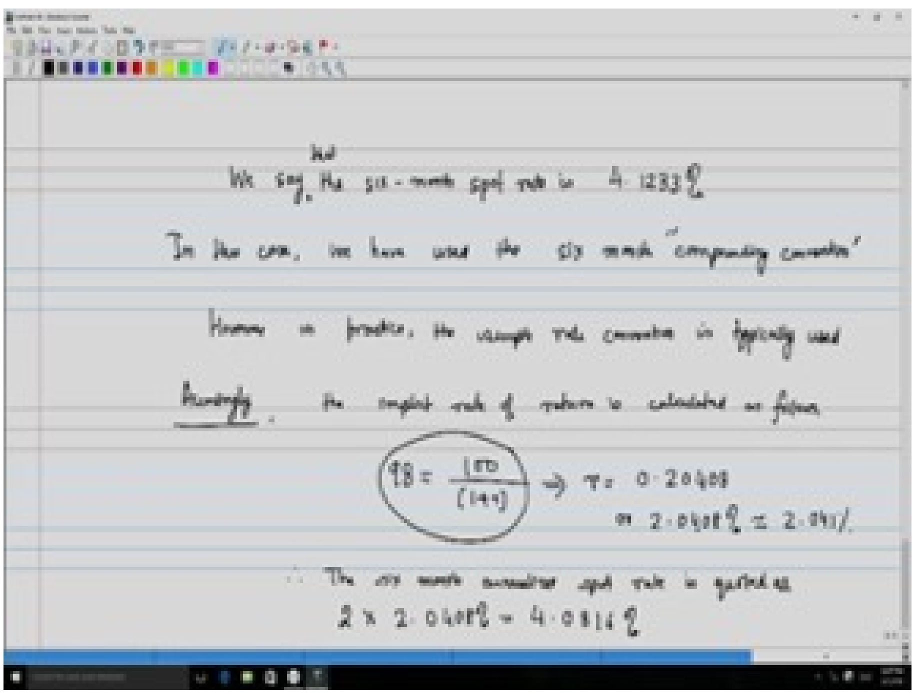

So, in this case, we say this formally that the six-month spot rate is 4.1233%. So in this case we have used the 6 month compounding convention. However, we have used basically the compounding interest formula in order to ascertain this percentage however, in practice what happens is that, so however in practice the simple rate convention is typically used. So accordingly in this example, the implicit rate of return is calculated as follows. So, what I do in this case, we will essentially in order to ascertain the rate using the simple interest convention which is typically what is done, we will take the initial price of the bond and we will look at the nominal value and we will just calculate the interest rate that is applicable for 6 months and then use the simple interest rate convention to convert it into annualized rate.

So, accordingly, we will calculate this as follows that $98 = \frac{100}{1+r}$, which gives you $r = 0.20408 = 2.0408\%$ or approximately 2.041%. Now from the formula it is obvious that this is the interest rate that is applicable for 6 months. Now if you are going to convert this into an annualized rate using the simple interest convention then you simply multiply the interest rate of 6 months by a factor of 2, so that it gives you an equivalent annual interest rate. So that means the 6 months annualized spot rate is quoted as $2 \times 2.0408\% = 4.0816\%$.

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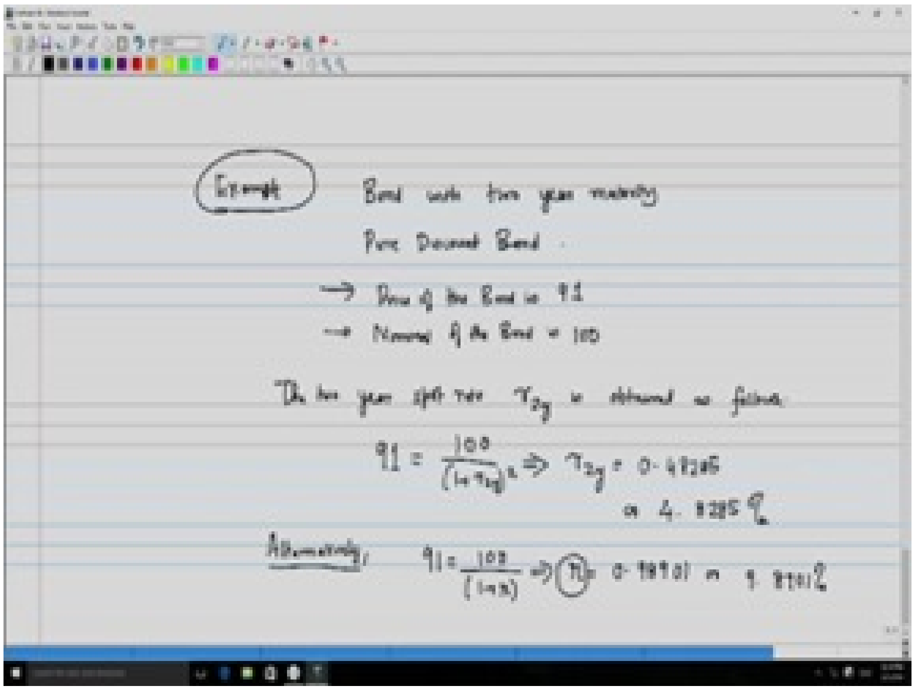

We will now look at another example where we will look at a case where the maturity is more than one year that is it is a long duration bond. Now, we consider a bond with two-year maturity, and as previously this bond being considered is going to be appear discount bond and the characteristics of this are that the price of the bond is 91 and the nominal of the bond is 100. Then the two-year spot rate which we will denote as r_{2y} is obtained as follows. That means I take the amount 91 which is the price of the bond, this is equal to $\frac{100}{(1+r_{2y})^2}$ and I am using continuous compounding convention and this gives you $r_{2y} = 0.48285 = 4.8285\%$.

So, basically this is going to be the two years interest rate making use of the continuous compounding convention. However, making use of the simple interest convention we have this alternative result. Alternatively, $91 = \frac{100}{1+r}$, which is the implicit rate, and if you solve this $r = 0.98901 = 9.8901\%$, the interest rate that you are considering this is the interest rate that is prevalent for 2 years or rather applicable for 2 years and then we have to turn into an annualized rate making use of the simple interest convention.

So that can be simply done that since this is a 2 years of rate. Therefore, the simple interest rate driven annual rate is going to be equal to 9.8901% which is applicable for 2 years, so for one year the interest rate will be this multiplied by 0.5 which is 4.9451%. Now the relationship between the yields of bonds for various maturities is of practical importance to investors and practitioners. Now, since this is of great interest that means that we need to have sort of 1 point or 1 stop representation of all these rates and this is of great interest? A simple example in our day-to-day life is that when you are going to a bank to purchase a fixed deposit, what is presented to us is basically the interest rates for various maturities and then decide on whatever interest rate is of relevance to us and typically a rational investor will choose the maturity of the bond that they are going to buy to be the one that offers the highest interest rate.

So even at retail investor level, it is critical that they know what is going to be essentially the term structure of the interest rate or what is going to be the yield against the maturity. So accordingly, so this

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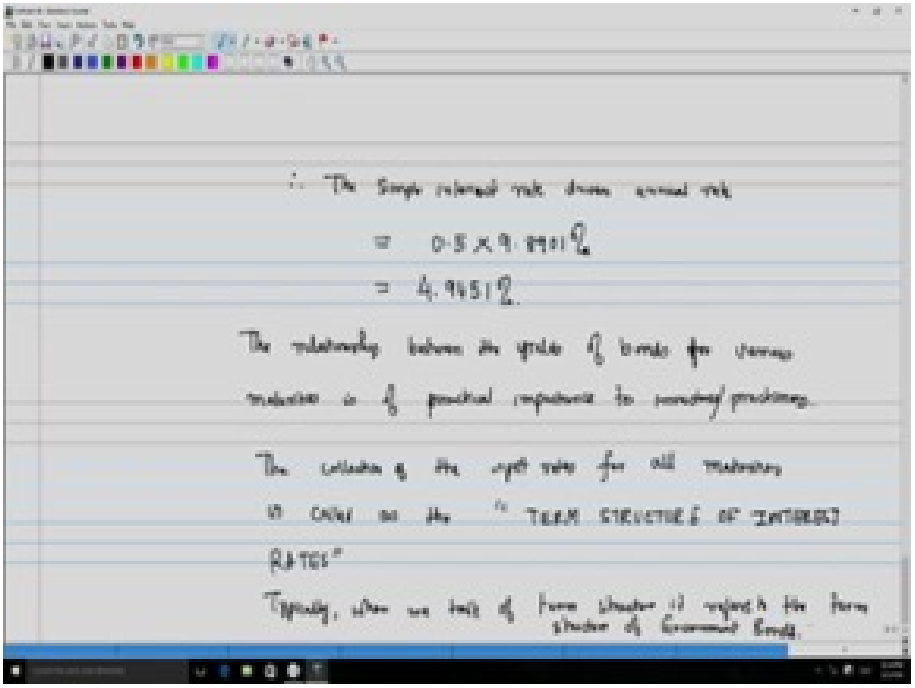

collection of the spot rates for all maturities that are available is called as the "term structure of interest rates". Typically when we talk of term structure, it refers to the term structure of government bonds. So now that we have spoken about term structure, let us now look at an example on how to derive a term structure.

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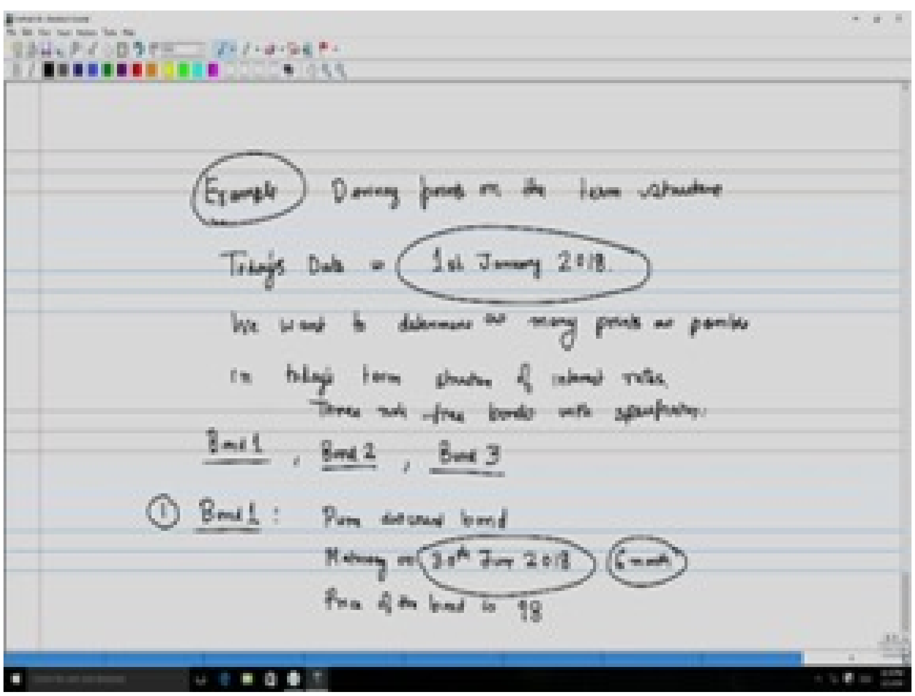

This example will be on deriving points on the term structure. So suppose todays date is 1st January 2018 and we want to determine as many points as possible in todays term structure of interest rates. So the question that you have to answer here is that how do we ascertain the term structure for various maturities and we want to essentially get as many maturities as possible so that we can actually graph as many as possible, corresponding interest rate that is applicable for those maturities. So suppose that now this is basically then going to be decided on the basis of the available bonds in the market. So in this example, suppose we consider that there are 3 bonds in the market, so accordingly, we will be able to ascertain 3 parts of the term structure.

Let me begin by certifying the characteristics and the nominal price, etc. for each of those 3 bonds and see how we can you make use of the concept of the interest rates that we have studied so far in order to determine a term structure with 3 points corresponding to those 3 maturities. Therefore, let us look at these 3 bonds, I have called them as bond 1, bond 2 and bond 3.

So there are 3 bonds, 3 risk free bonds with specifications as given; so I have bond 1, bond 2 and bond 3. So, first, let us take the characteristics of the 1st bond which I call as bond 1, so bond 1 is a pure discount bond and this is maturing on 30th June 2018. So remember I had taken todays date as 1st January 2018 and it is measuring the 1st bond, bond one is maturing on 30th June 2018, so that means that this is a 6 month pure discount bond. And thirdly, the bond is currently selling or the price of the bond today which is 1st January 2018 is 98.

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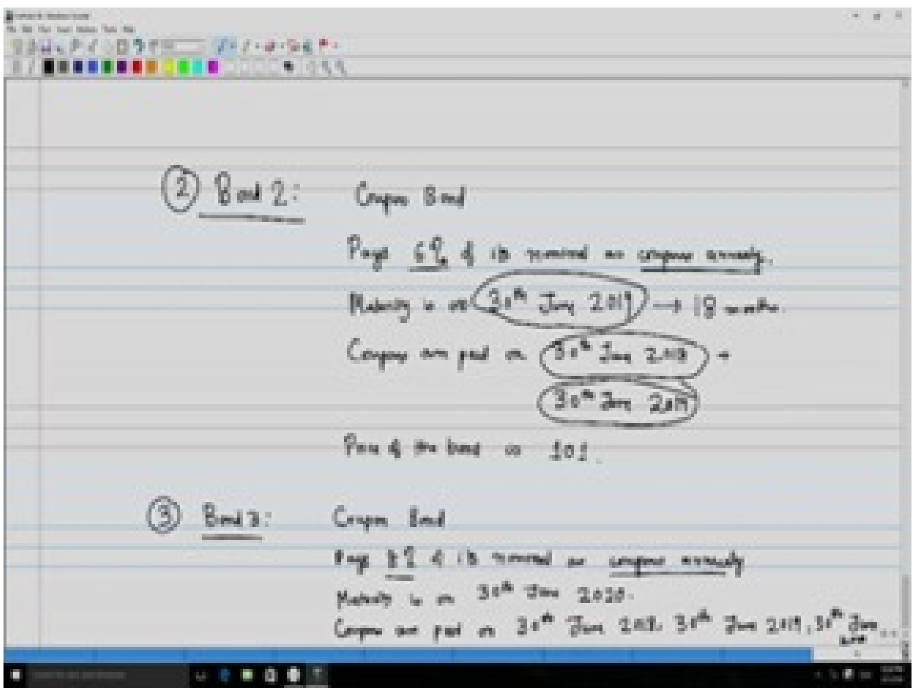

Now, let us look at the 2nd bond or bond 2, this is going to be a coupon bond. Now since we talk about coupon bond we need to specify what are going to be the coupons that are going to be paid and at what durations. So this coupon bond pays 6 percent of its nominal as coupons annually and this maturity is on 30th June 2019. So remember todays date was 1st January 2018 and this bond is measuring on 30th June 2019 which means that the maturity of the bond is going to be one and a half years or 18 months. So now here I have to specify one thing, I have said that it pays 6% of his nominal as coupons annually. So what I mean here is the following that the question that naturally arise that when I am talking about coupons being paid annually, this might seem to be somewhat contradictory that this bond is only has remaining duration of 18 months.

But one can view this as a bond that has already been issued at a prior time point and that these coupons annually are going to be coupons that are going to be paid on 30th June of every year. So this means that here the coupons are paid on 30th June 2018 and 30th June 2019 and the bond is selling today at a price, the price of the bond is 101. Finally, we come to the 3rd bond which is designated as bond 3, so this is also a coupon bond and it pays 8% of its nominal as coupons annually and the maturity is on 30th June 2020.

So if it is annual coupon bond, this means that the coupons are paid on 30th June 2018, 30th June 2019 and 30th June 2020.

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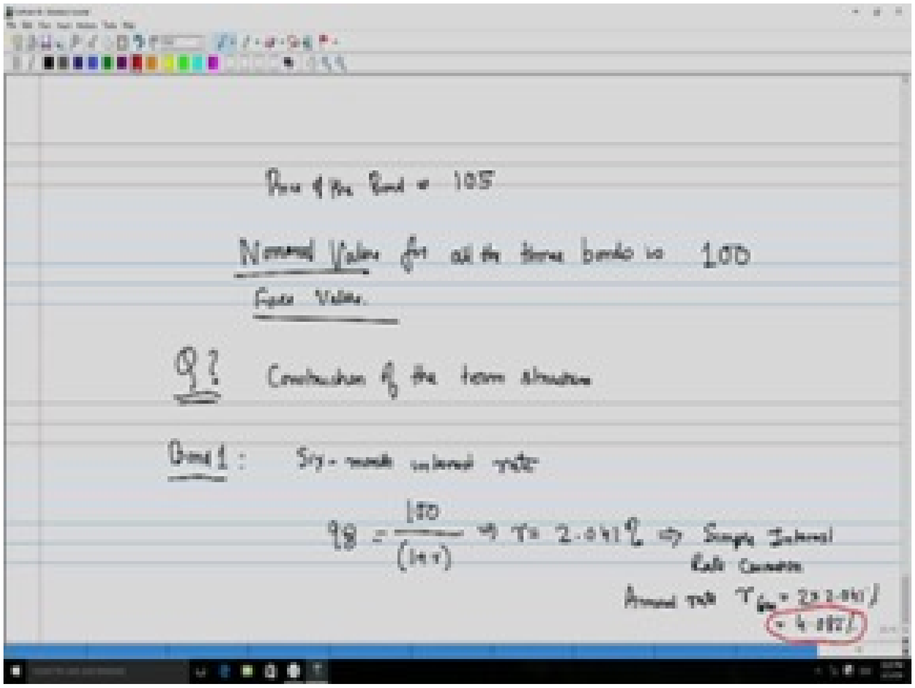

And a price of the bond is 105, so there is one last missing piece of information that you have not included as since you are talking about coupon is an pure discount bonds, we need to have the nominal value. The nominal value for all 3 bonds is 100 or the face value is 100. Okay, so with this piece of information what is the question that you want to answer? The question that you want to answer is that what is going to be the construction of the term structure. Let us take the 1st bond and go one by one. So, we 1st compute the six-month interest rate and how do you do it? Remember the nominal value of the face value is 100 and the price of the six-month bond is 98 so that if r is the interest rate, then $r = 2.041\%$.

So this is exactly the same example that we have already done, the 1st example that we have seen in todays class. So what this will mean is that if we use the simple interest rate convention, then the annual rate $r_{6m} = 2 \times 2.041\% = 4.082\%$, so this is one number we will like to actually note down for the 1st bond, this is going to be the interest rate annual interest rate that is going to prevail.

Next, we will look at the 2nd bond, so what are the characteristics of 2nd bond? The 2nd bond is of maturity 18 months and what are the payments that you are going to receive? The payments that you are going to receive are that at the end of six months, you receive a coupon of 6% calculated on the face value of 100 that means the coupon payment is going to be equal to 6 and at the end of 18 months, you are going to receive again a coupon of 6 plus the nominal value or the face value 100, that is 106. Then the interest rate r is calculated as, remember what was the price? The price was 101, this is equal to the present value of the first coupon and the present value of the 2nd coupon.

Now, for the 1st coupon we have already calculated, the 1st coupon was paid after 6 months and we have already calculated the interest rate for the 1st 6 months in case of bond 1 and we are going to utilize that number and so this is going to be $\frac{6}{1+0.2041}$, so we have basically taken this value from here, because this is already decided as 6 months interest. And we have to also ascertain the 18 month interest rate, because 106 is the amount that you will receive 18 months after you have purchased the bond, so this will become $1 + r$.

Solving this you get $r = 11.438\%$. So you make use of the simple interest rate convention and so we basically then get the annual rate is going to be equal to, remember 11.438% that we have figured out here, that is the interest rate that is going to be applicable for 18 months.

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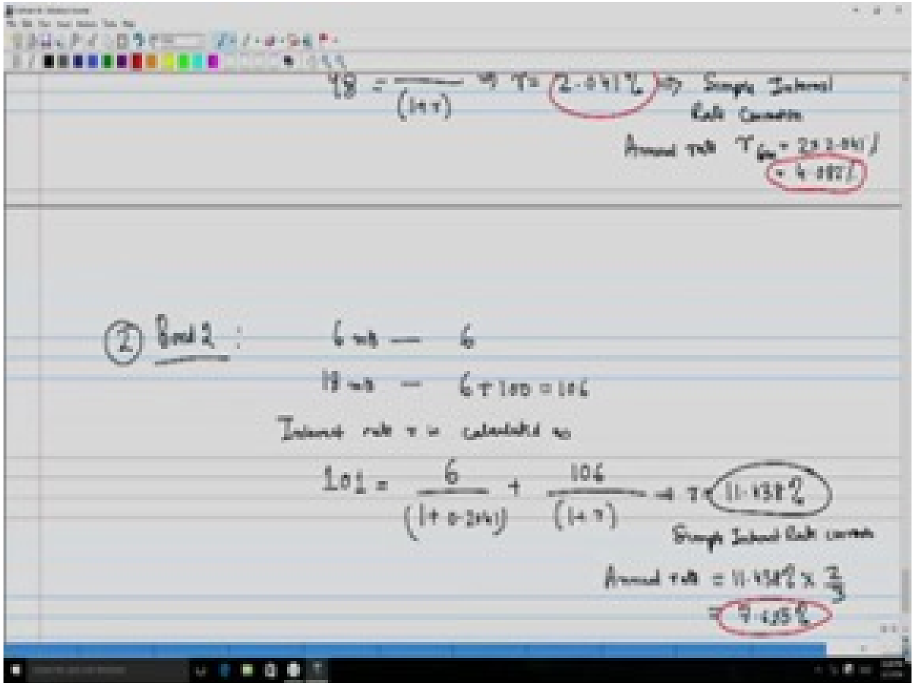

Accordingly, the annual rate can be obtained by scaling it down to 12 months or 1 year. So accordingly, 11.438%, this has to be multiplied by a factor of $\frac{2}{3}$, which I got by taking. Basically, this is going to be factor of $\frac{12}{18} = \frac{2}{3}$ and therefore this annual rate turns out to be equal to 7.625%. So this is another key number, so I basically found out the 2nd interest rate in the term structure which is applicable in case of 18 months.

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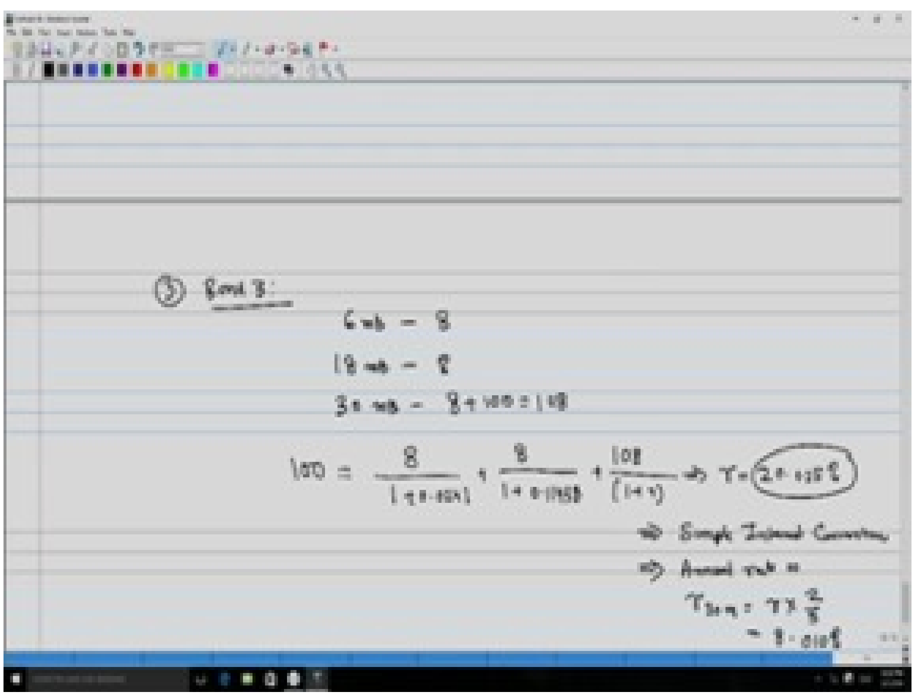

Now let me come back to the 3rd bond. In case of 3rd bond, Bond 3 what are the characteristics? Remember in case of bond 3 the nominal face value is again 100 and the coupon that are being paid those coupons are of an amount of 8 at the end of 6 and 18 months and at the final time point that is at the end of 2.5 years you end up getting $8 + 100$. So that means at six-month you basically get an amount of 8, at 18

months to get an amount of 8 and at the end of 30 months you get an amount of $8 + 100 = 108$. So now, remember what is the price of this bond? The price of this bond was 100 and this is going to be the present value of the amount of 8 that you received at the end of 6 months plus the present value of the amount of 8 that you receive at the end of 18 months and the present value of the amount of 108, that you receive at the end of 30 months or 2.5 years.

Now for 8, since the coupon of 8 is being paid at the end of 6 months, so for in order to calculate the present value we will make use of the interest rate that we have determined for the bond with 6 months, because this can essentially be treated as a six-month pure discount bond and this is going to be $1 + 0.241$. Now next in order to calculate the present value of the coupon payment of 8, that is actually being made at the end of 18 months, we will make use of the interest rate for 18 months that you have determined here, that is 11.438%. Accordingly, we will divide this or discount this by a factor of $1+0.11438$. And finally 108 will be discounted and the interest rate prevailing for 30 months which is what you are going to determine in case of the 3rd bond, so this will be divided by $1 + r$.

So if you make use of this you end up being getting r is equal to 20.025%. So, this again you make use of simple interest convention and this is going to give you. Now, this is 20.025%, what does this interest rate signify? This is basically the interest rate that is applicable for a period of 2.5 years or 30 months, so that means $\frac{5}{2}$ years, so that means when you have to convert this to an interest rate that is applicable for a period of one year, then you multiply this by a factor of $\frac{2}{5}$. Therefore, under the simple interest convention, the annual rate is $r_{30m} = r \times \frac{2}{5} = 8.010\%$.

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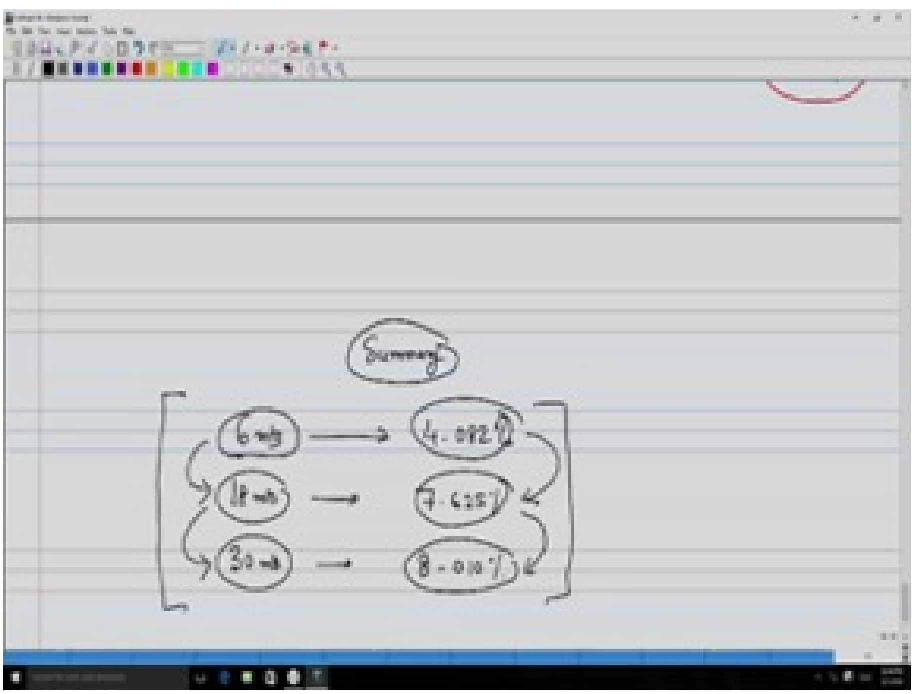

So, to sum up, we just present all the interest rate, because we have started with the problem of the setting of the term structure by making use of the information or the characteristics of those 3 bonds. So, to summarize this, we have got that in case of 6 months annual interest rate that is prevalent is 4.082% alright. So, you go back, remember we had this number. Then in case of 18 months, the interest rate is 7.625% percentage, remember this is the annualized rate that we have got in case of the bond of 18 months. And finally we have the rate of 8.010%, that was determined for the coupon bond lasting 30 months. So, for 30 months, we have 8.010%.

And if you observe carefully, as my maturity of the bond increases from 6 months to 18 months, the corresponding annual rate goes up from 4.082% to 7.625%, and further as I go from 18 months to 30 months, the corresponding interest rate goes up from 7.625% to 8.010%. Remember that these interest rates determined for the bond that are applicable at an annual level so these are annualized interest rates and based on this one little example we will see that this example shows us that this one example shows us that this is consistent of the observations of this example are consistent with an earlier statement made that typically depending on the maturity the interest rate, you can expect them to typically go up that is the bonds with lesser maturity or shorter maturity will have lower interest rates as compared to bonds with longer maturity.

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So, this completes our 2nd Module which is essentially focused on interest rates, so in the 3rd Model we will start discussing more in detail about the basic principles of financial derivatives in general. Thank you for watching.