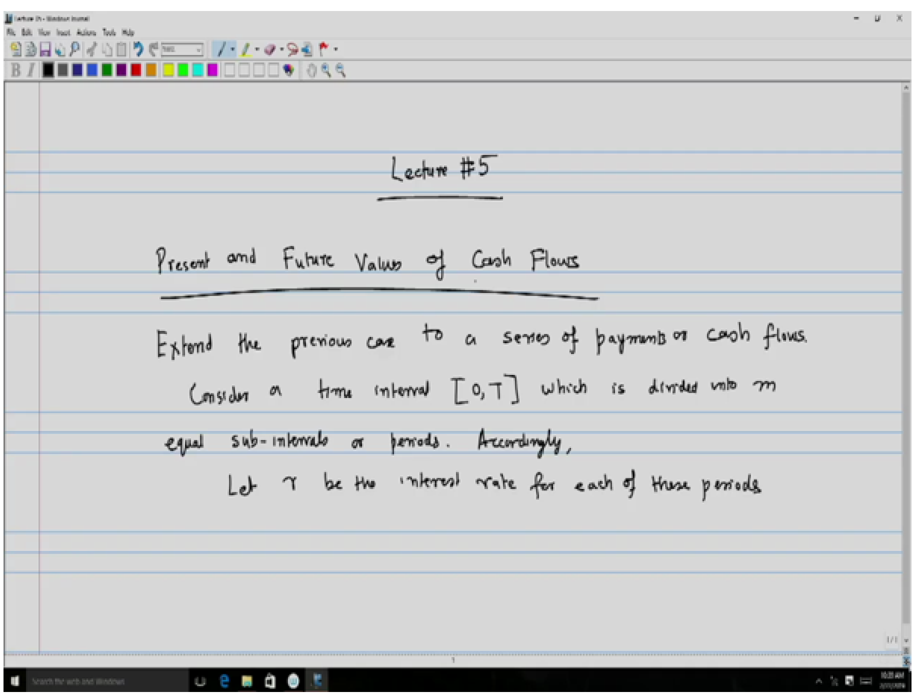
Mathematical Finance

Professor N. Selvaraju
Professor Siddhartha Pratim Chakrabarty
Department of Mathematics
Indian Institute of Technology Guwahati

Module 2: Time Value of Money and Risk Free Assets
Lecture 2: Present & Future Values, Annuities, Amortization and Bond Yield

Hello viewers, welcome to Lecture 5 of this course on Mathematical Finance. This is going to be the 2nd Lecture of the 2nd Module. Recall that in the previous lecture, we had discussed about interest rates. In particular, we had looked at simple interest, compound interest and we also looked at continuous compounding and we had concluded by speaking about discussing on present value of future payment. So, we will continue with this discussion and go a little forward and discuss about the future value, present value and also regarding amortization and finally, we will discuss in this lecture about the internal rate of return.

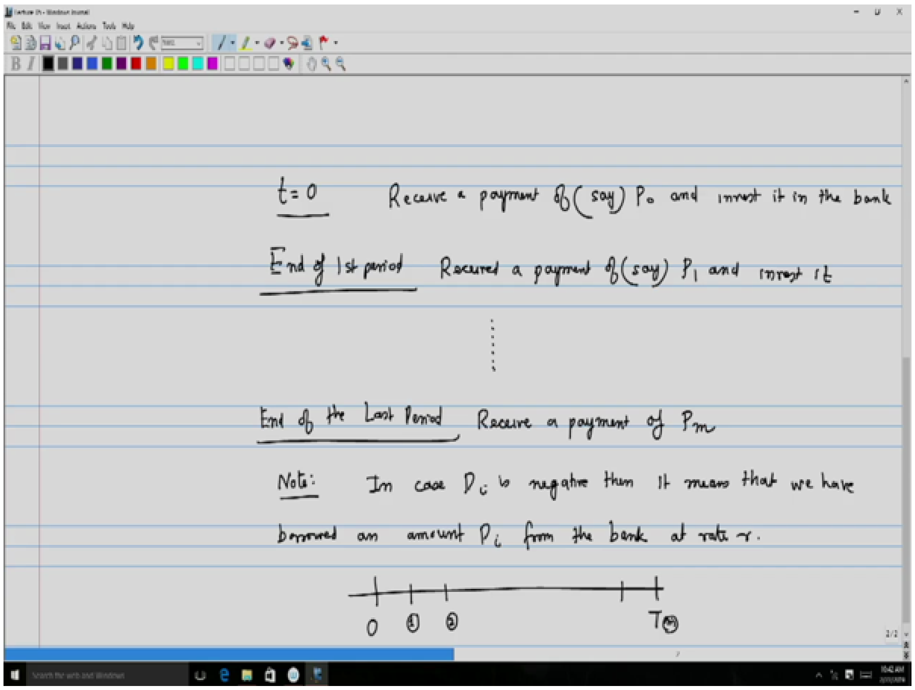
(Refer Slide Time: 01:17)



So, let us begin this lecture, so we will talk about present and future values of cash flows. So, what we will do now is we will extend the previous case to a series of payments or cash flows. So, we consider a time interval $[0, T]$, which is divided into m equal sub-intervals or periods. So, you would remember that earlier we were just looking at essentially a single period that there is initial investment at time $t = 0$ and then there is a payment being made at time $t = T$. However, in todays class, we will look at payments being made with periodically from time to time and accordingly, we consider our time window of $[0, T]$ and this time window is going to be divided into m periods or sub intervals each of which is of equal length.

So, accordingly, let r be the interest rate for each of these periods. Remember that when I talked about r here, I do not mean yearly, but I mean that I have divided the entire interval $[0, T]$ into m sub intervals, r is going to be the interest rate, that is applicable to each of those sub intervals.

(Refer Slide Time: 04:05)



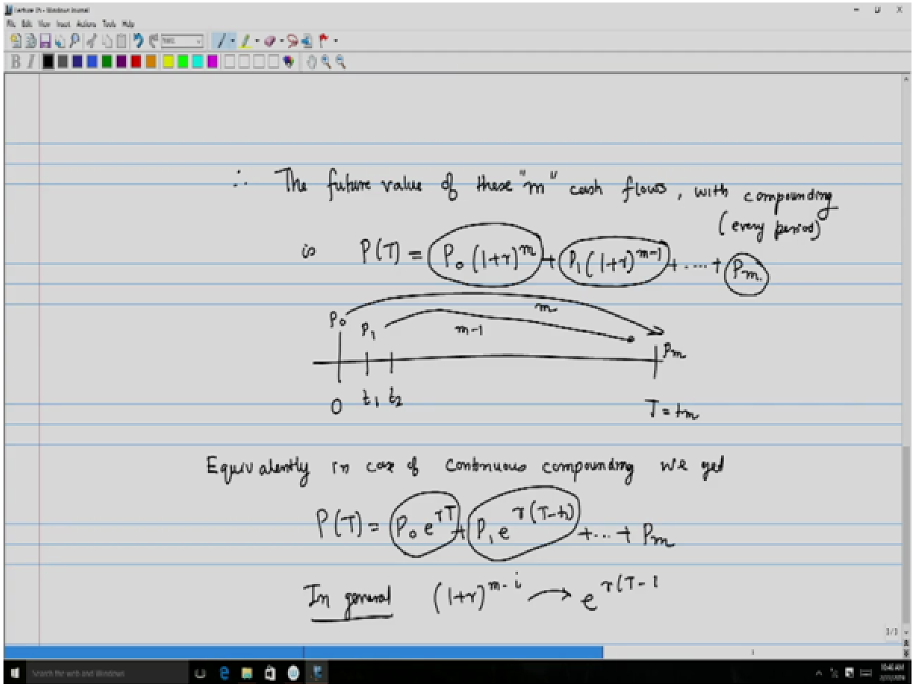
Now, let us start with time $t = 0$. At time $t = 0$, we receive a payment of (say) P_0 and invest it in the bank. At the end of 1st period, we receive a payment of (say) P_1 and invest it. So, we go on like this and at end of the last period which is time $t = T$, we just receive a payment of P_m. So, just to note here is that in case P_i is negative, then it means that we have borrowed an amount P_i from the bank at rate r. So, this is what it looks like, so basically, we will have some time initial time $t = 0$, final time $t = T$ and we will essentially divide this into m sub intervals, so this is the end of period 1, this is end of period 2, all the way to T which is the end of period m.

Therefore, the future value of these m cash flows which are happening at end of the 1st period, 2nd period, 3rd period, all the way to m period. What is going to be the period future value of this m cash flows, what are the cash flows? P_0, P_1, all the way to P_m. So the future values of this m cash flows with compounding, that means I use the compound interest convention of calculation of interest is $P(T)$, and the compounding happens every period and remember that each period is of identical length. Then $P(T)$ is going to be $P_0(1+r)^m + P_1(1+r)^{m-1} + \cdots + P_m$. So the logic for this is that if I go back to my time interval $[0, T]$ and there are M sub intervals, let t_1 with the time at which the 1st period ends, t_2 with the time at the 2nd period ends, all the way to $t = t_m$ when the last period ends.

So, that means that if I receive an amount of P_0 at time $t = 0$, then the number of remaining periods is m. So, that is the reason I have used the term $P_0(1+r)^m$ to calculate the future value for the payment of P_0. Likewise, at time t_1 the payment received is P_1. However, by the time the payment P_1 is received, one period has elapsed and so the number of remaining periods is $m - 1$, which is why we have amount of P_1 being compounded by a factor of $(1+r)^{m-1}$. And the final payment P_m received is not invested anymore, but is just received, so there is no compounding for this particular term. So, this means that now equivalently, in case of continuous compounding, we get $P(T) = P_0e^{rT} + P_1e^{r(T-t_1)} + P_m$.

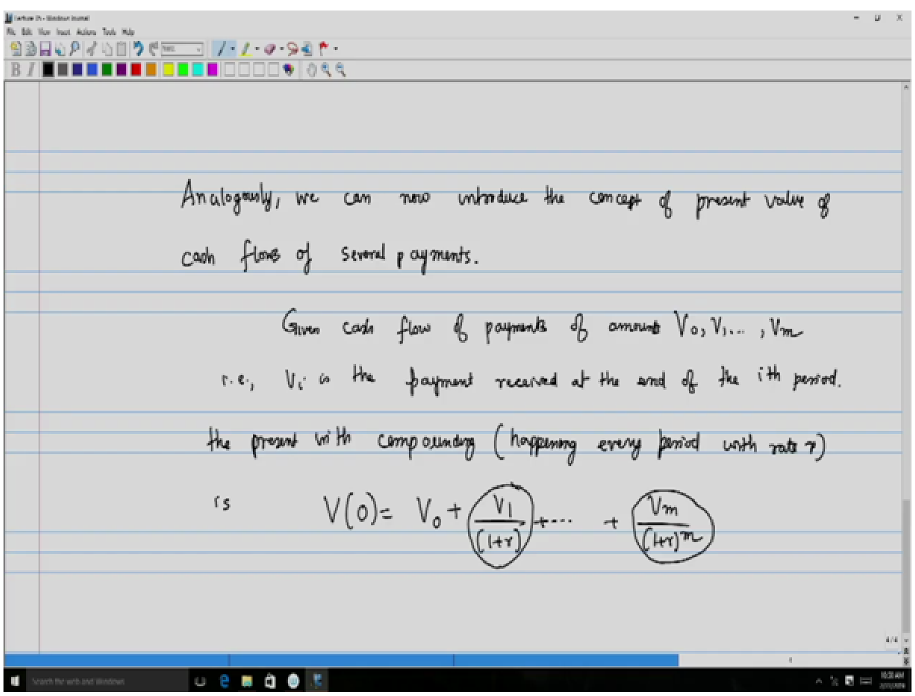
So, observe that in this case, P_0e^{rT} using continuous compounding arises from the fact that your initial investment of P_0 is being made for the entire duration of length T. For the 2nd term, by the time we receive

(Refer Slide Time: 06:23)



the payment of P_1 at time t_1 and invest it, already at time t_1 has elapsed and the remaining time left is only $T - t_1$. So, accordingly, here it is multiplied by a factor of $e^{r(T-t_1)}$ and so on. So, in general, in case of compounding, we have $(1+r)^{m-i}$ for the payment P_i, and in case of continuous compounding this becomes $e^{r(T-t_i)}$.

(Refer Slide Time: 10:39)

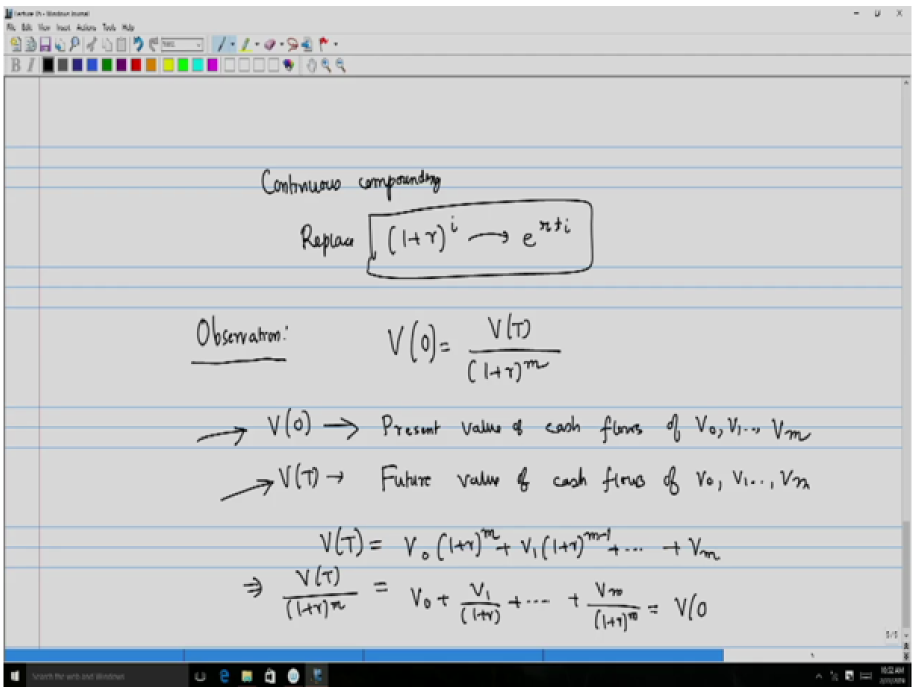


So, in an analogous manner, we can now introduce the concept of present value of cash flows of several payments. So, this means that when you are looking at the future value, what you do is that we took into account the payments that are actually being received and investing that in the bank. What we will do now analogously is that we will introduce the concept of present value in case of several payments that are happening on the lines of the present value that you have looked at in case of single period setup.

So therefore given cash flow of payments of amounts $V_0, V_1, \ldots, V_m$, that is, V_i is the payment received at the end of the i-th period, the present value with compounding (happening every period with rate r) is $V(0) = V_0 + \frac{V_1}{1+r} + \cdots + \frac{V_m}{(1+r)^m}$. So, you can view this as an extension of the present value in case of a single period model where each of these payments can be treated separately as the present value of this individual payment.

That means that for the 1st period, we will see the payment V_0 immediately at time $t = 0$, so the value itself is going to be the present value. For the 2nd term, I receive the payment of V_1 and remember this V_1 is the payment that is received at the end of first-time period. So, it has to be discounted by the factor of $1 + r$. Hence, I have $\frac{V_1}{1+r}$. Likewise, for the final payment of V_m, since m periods have elapsed, I will have to discount in by a discounting factor of $(1 + r)^m$ for the number of periods.

(Refer Slide Time: 14:04)



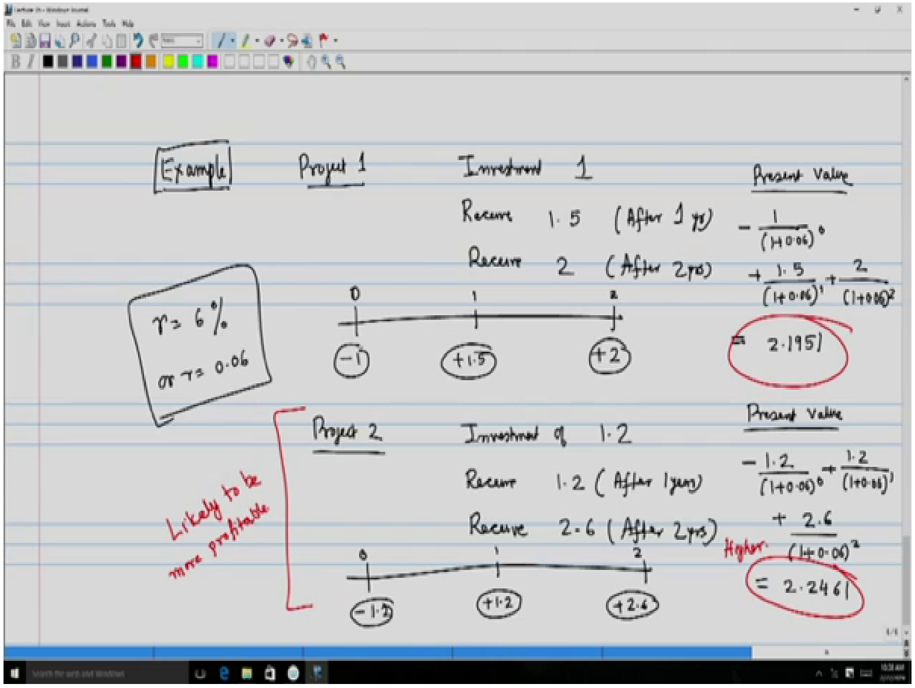
Now, if you want to move this in case of continuous compounding, then you replace $(1 + r)^i$ in the denominator to e^{rt_i}. So here we have introduced the concept of future value at the present value, so let us see how we can actually connect these two things and make the following observation. And I am making a statement that $V(0)$ is going to be equal to $\frac{V(T)}{(1+)^m}$.

So, for V_0, as already seen, this is the present value of cash flows of V_0, V_1, all the way to V_m, and $V(T)$ is the future value of cash flows of $V_0, V_1, \ldots, V_m$. So this means that in the 1st case we will receive cash flows of $V_0, V_1, \ldots, V_m$ at times $t_1, t_2, \ldots, t_m$, and $V(T)$ is going to be the future value of the cash flows that is again of $V_0, V_1, \ldots, V_m$.

So, how do I connect both of these? So we will make use of the two formulas using compounding. Therefore, $V(T)$ which is the future value is going to be $V(0)$ invested for a period m, so $V_0(1 + r)^m + V_1(1 + r)^{m-1} + \cdots + V_m$. Now, I can rewrite this as $\frac{V(T)}{(1+r)^m}$, this is going to be $V_0 + \frac{V_1}{1+r} + \cdots + \frac{V_m}{(1+r)^m}$, which is equal to simply $V(0)$, so this proves this particular result.

Now, to illustrate the utility of present and future values, let us consider the example. Remember that we started off with the motivation that we are looking at present value and future value and we will look at present value in particular as a way of assessing and comparing various investment opportunities that are presented to us. So, accordingly, let us consider an example, where we consider two projects and we are trying to ascertain which projects will be more attractive from the investment point of view. So, firstly, let us

(Refer Slide Time: 16:55)



consider a project 1. So, suppose you make an investment say of 1 million and then you receive 1.5 million after one year and then you receive 2 million after 2 years.

So, this means that cash flows are -1 at $t = 0$, because you have made investment of 1. So, at the end of one year you will receive an amount of $1 + 0.5$, so that is basically a cash flow of 1.5 and at the end of 2 years, you will receive an amount of 2, so that is the cash flow of 2. Now, we assume that the interest rate r is going to be 6% or that means $r = 0.06$ per year. Then let us consider another project, project 2, where you have made an investment of (say) 1.2 million, you receive an amount of 1.2 million after one year and you receive an amount of 2.6 million after 2 years.

Then what are you going to be your cash flows at initial time 0, 1 and 2? So, the initial cash flow at time 0 is going to be minus 1.2, because that is going to be your investment and at time 1, you have received the amount of 1.2 million. So, the cash flow is going to be 1.2 and after 2 years you, will receive an amount of 2.6, so that is the cash flow of 2.6.

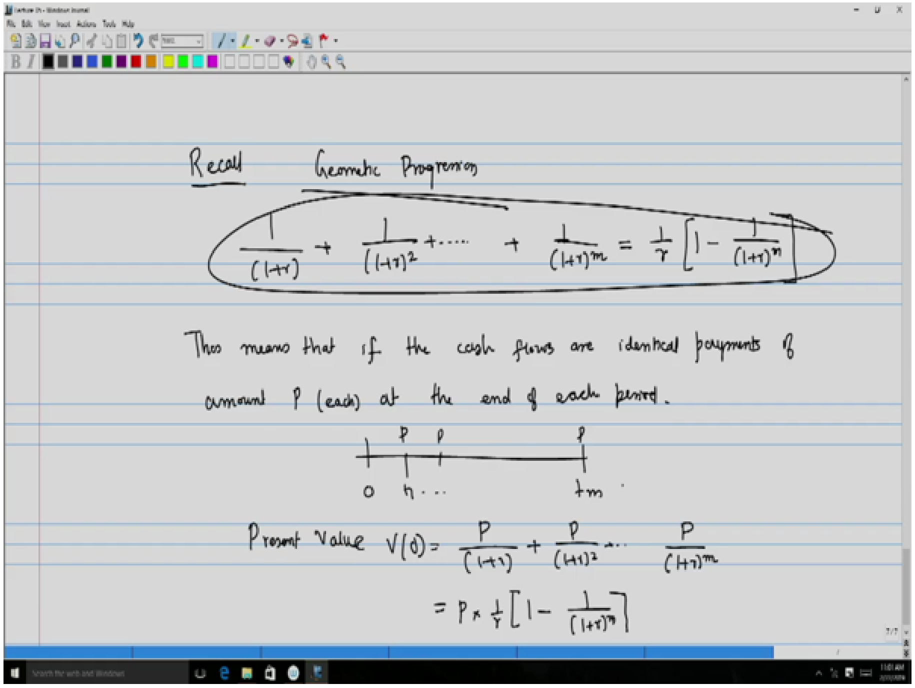
Now, we are faced to the question of comparing these two investments. So, accordingly, let us look at what is going to be the present value of both these investments. So, 1st case, let us look at what is going to be the present value. So, in this case, the present value is going to be $-\frac{1}{(1+0.06)^0} + \frac{1.5}{(1+0.06)^1} + \frac{2}{(1+0.06)^2} = 2.19951$.

Now, for the 2nd case, the present value is going to be calculated as follows. So, the initial investment of $-\frac{1.2}{(1+0.06)^0} + \frac{1.2}{(1+0.06)^1} + \frac{2.6}{(1+0.06)^2} = 2.2461$. Now, you observe that this value, you compare this value and this value and you notice that this value has a higher present value and therefore, based on the calculation of the present values we recommend that it is probably wiser to basically get into project 2, because this is likely to be more profitable.

Alright, there was discussion regarding the present and future values and how they have practical utility, when it comes to investment decision. Now, we briefly recall a geometric progression formula which is $\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^m} = \frac{1}{r}\left[1 - \frac{1}{(1+r)^m}\right]$. Now, what does this mean?

This means that if so as you see that you know the terms actually appear in the left-hand side of this geometric progression, these are nothing but the discounting factors that was applied when you were calculating the present values. Now, this particular formula is obviously going to be useful in case all the cash payments are identical so that you can factor it out and you can obtain a simplified formula which is what you are going to do now.

(Refer Slide Time: 22:01)



So, this means that if the cash flows are accordingly motivated by this, we assume that if the cash flows are identical payments of amount P each at the end of each period that means that these are the payments, 0, $t_1, \ldots, t_m$ and the payments are all identical payments P being made, then what is going to be the present value?

In this case, the present value $V(0)$ is going to be $\frac{P}{1+r} + \frac{P}{(1+r)^2} + \cdots + \frac{P}{(1+r)^m}$. That means $V_0, V_1, \ldots, V_m$ are all taken to be identically equal to P. So, finally, we get $\frac{P}{(1+r)^m}$. And so now we can make use of this above formula. So, we factor this P out and then we get $\frac{P}{r}\left[1 - \frac{1}{(1+r)^m}\right]$, where r is basically the interest rate per period.

Now, what is the reason that we actually choose $V_0, V_1, \ldots, V_m$ is equal to all P, because typically such payments are referred to as annuities. So, this means that basically it is a stream of cash flows in the future, all of them are identical. An example of this could be (say) pension that you are promised fixed amount of pension P, pension amount P that is being promised to you every month effectively is nothing but annuities being paid out to you for an amount of P on a monthly basis.

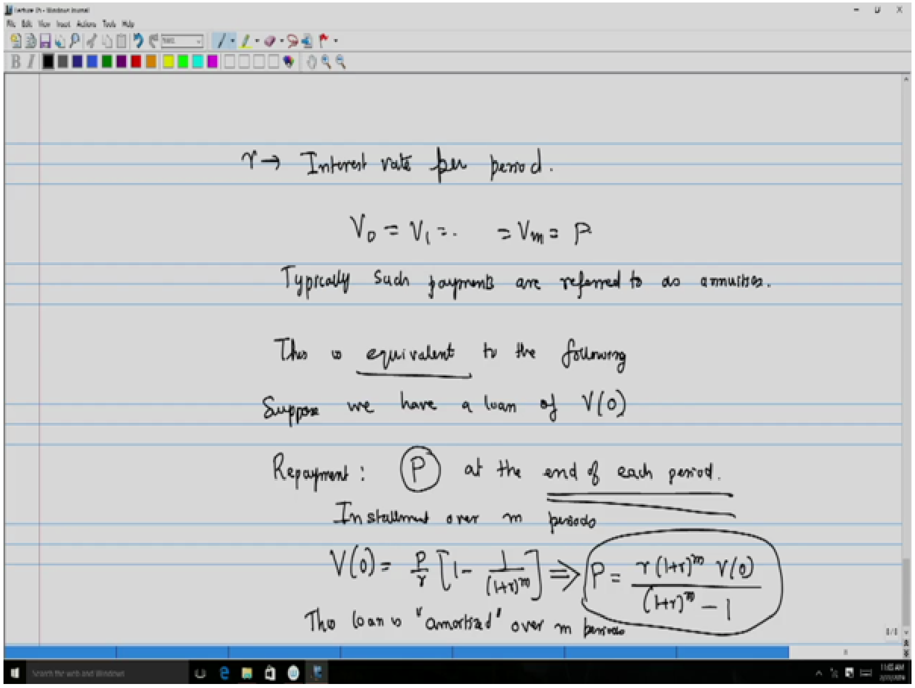
Now, let us make another observation, so this is equivalent so what is the equivalent interpretation of this formula? So, this is equivalent to the following, suppose we have a loan of $V(0)$, then when you look at the repayment. Suppose the repayment is P at the end of each period. So, you can view this as that, suppose you invest an amount of $V(0)$ or rather you borrow an amount of $V(0)$ and then you make payment of P at the end of each period. So, this basically means that an amount of P that you paid at the end of each period typically on a monthly basis.

So, how do you reconcile these two? So, then what you do is that that means that you are making an EMI payment or installments over (say) m periods, then $V(0) = \frac{P}{r}\left[1 - \frac{1}{(1+r)^m}\right]$, this can be solved to obtain $P = \frac{r(1+r)^m V(0)}{(1+r)^m - 1}$.

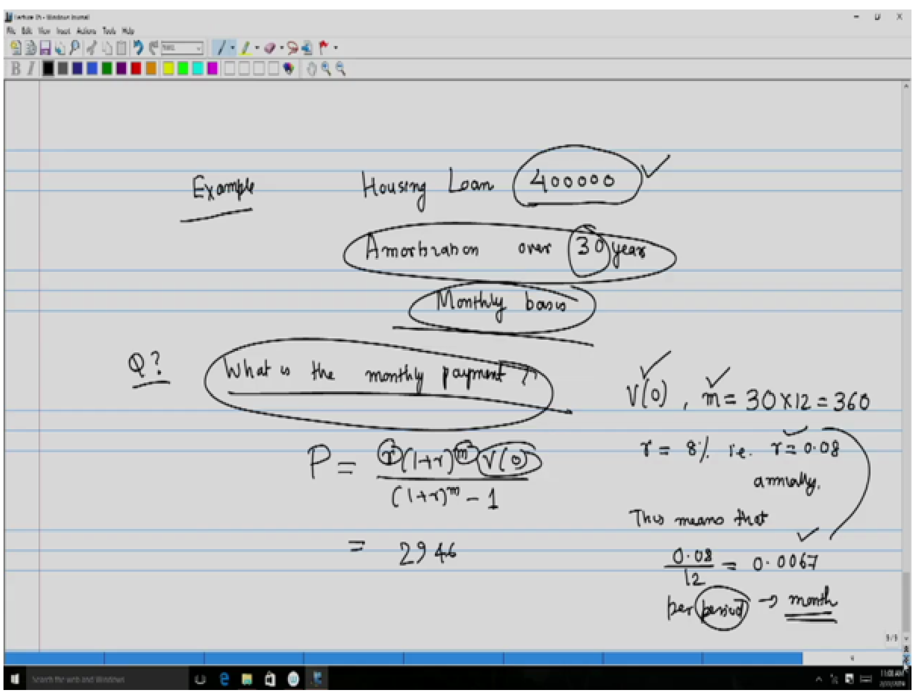
So, this effectively means that if you borrow an amount of $V(0)$ and you have to make a monthly payment of P for m number of periods with the interest rate per period being r, then the monthly payment can be determined by this formula and then you say that this loan is amortized over m periods.

So, next we look at an example. Let us consider a housing loan (say) of $400,000$, so that is the money that you actually borrowed and amortization will be done over a period of 30 years and the payments will

(Refer Slide Time: 25:23)



(Refer Slide Time: 28:56)



be basically done on a monthly basis. So, the question that I want to address here is that what is the monthly payment that you have to make or the EMI that I have to make.

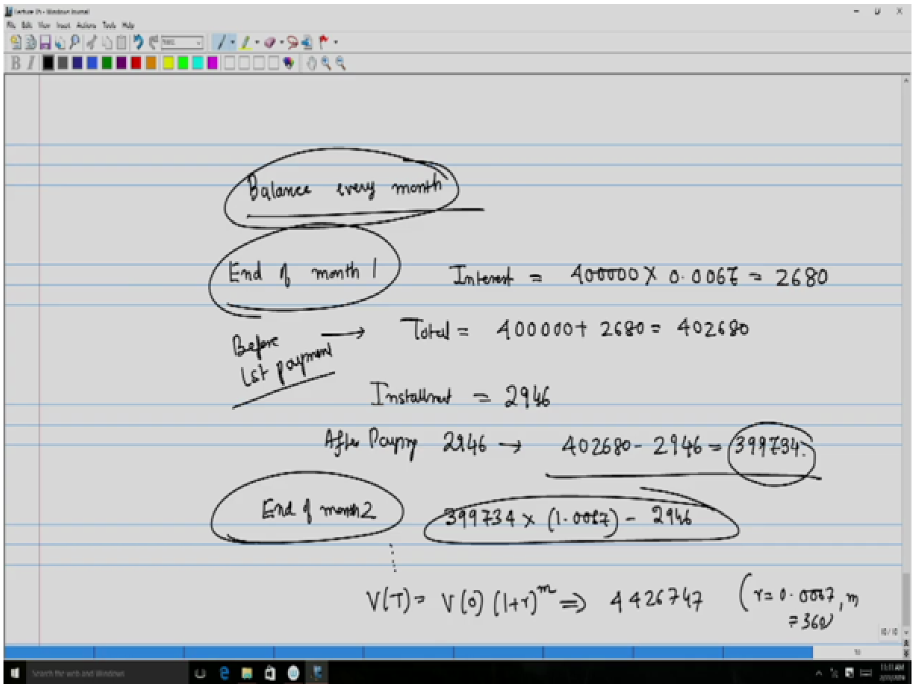
So, what do you identify here? So, what is the piece of information you need? We need to know what is going to be $V(0) = 400,000$. We need to know number of periods over which the payment is being made. Now, since the payment is being made on monthly basis and the payment has to be made over a period of 30 years. So, the number of periodic payments that you will make is going to be 30 years $\times$ $12 = 360$ periods.

Now, the 1st thing that you need to know is what is the r? So suppose my $r = 8\%$, that is, $r = 0.08$ annually. So, this means that I have to pay $\frac{0.08}{12} = 0.0067$ per period which is basically every month. Alright, now I have my $V(0)$, I have my m, I have my r annually and consequently I have my r for each period. So, the question as I said what is going to be my monthly payment?

What is the monthly payment? Monthly payment was this amount of P. So all I need to do is just simply substitute these values and I get $P = \frac{r(1+r)^m V(0)}{(1+r)^m - 1}$. So, when you substitute $r = 0.067$, $m = 360$, $V(0) = 400,000$, this number turns out to be 2946. So, that means, for a 30 year loan of $400,000$ monthly payment and the loan is being extended at 8% per annum, the monthly payment has to be 2946.

So this is very useful and provides the way you actually calculating your EMI yourself and deciding you know what is going to be your monthly payment. Now, looked at from a different perspective and let us look as slight more detail as what is going to be the balance every month. That means that how much money do you still owe at the end of each month. So, in order to calculate the balance, we will look at what happens at end of month 1.

(Refer Slide Time: 32:48)



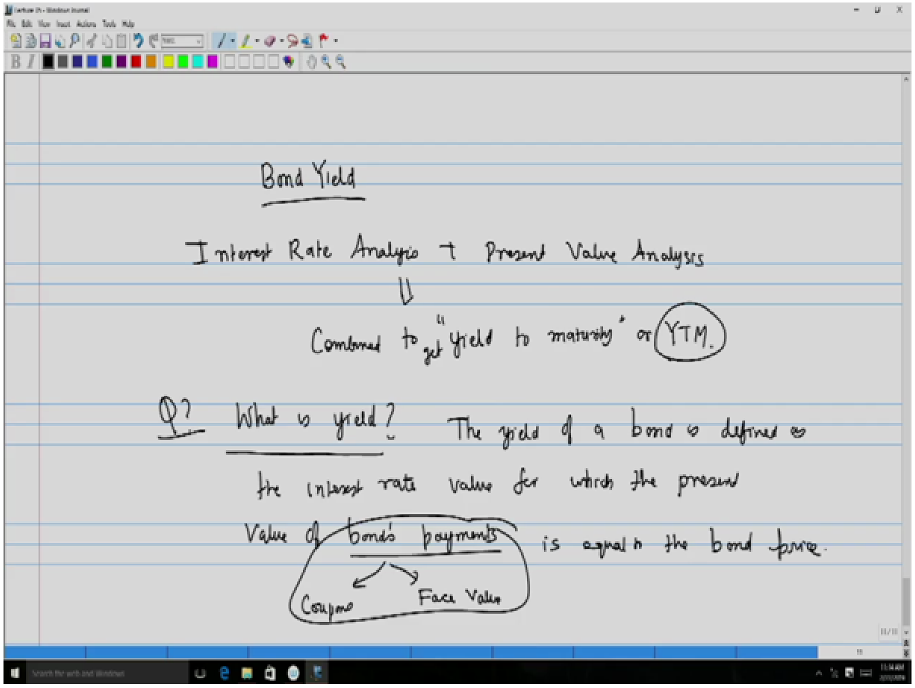
At the end of 1st month, what is the interest for the month? The interest is going to be the original amount of $400,000 \times 0.0067 = 2680$. Now, that means your total liability at the end of 1st month before you made the 1st EMI payment is going to be the original amount of $400,000 + 2680 = 40,2680$. So, this is basically before 1st payment or rather 1st repayment. Now, after paying the installment, what is the installment? It is 2946.

So, after paying 2946, you will basically have an amount of $402680 - 2946 = 399734$. So, that means at the end of 1st month, the amount that you actually owe to the bank after making the 1st payment is going to be 399734. Now, let us see what happens at the end of month 2. At the end of month 2, you will have to make your liability becomes 399734×1.0067 and then from that you basically make a payment of 2946 and you can obtain your new balance.

So, we can carry on like this for the entire period. So, that means the future value $V(T) = V(0)(1+r)^m$, this gives you, in this case, it turns out to be 4426747, because $r = 0.0067$ and your $m = 360$. So, this amount is approximately 4.5 million. So, now you see that you started off with an initial loan of $400,000$ and this 31 loan means that you end up paying more than 10 times that amount of money as a consequence of this monthly installment of 2946.

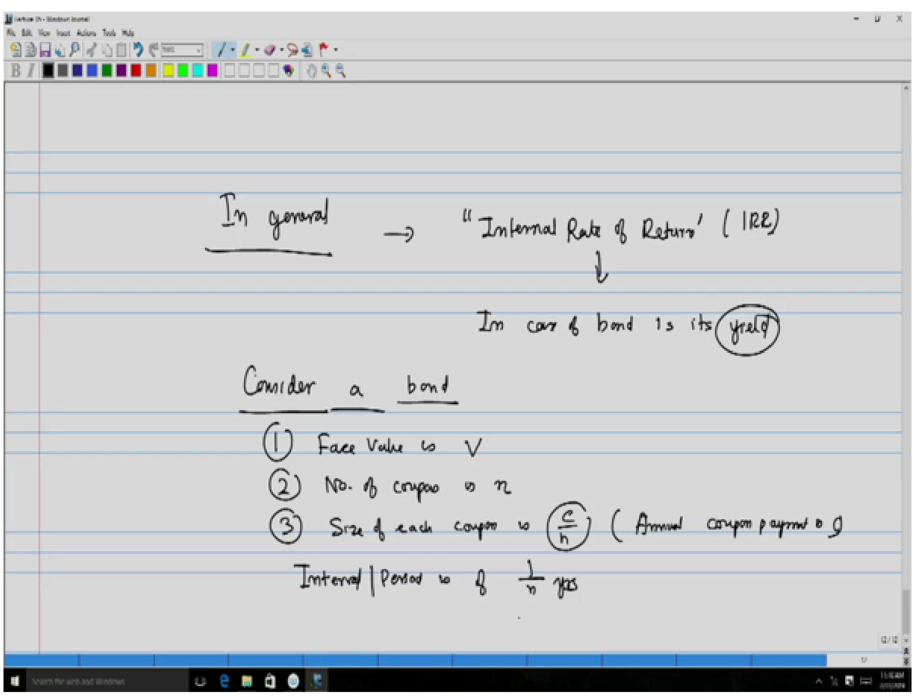
Let us come to the last topic for the day that is Bond yield. So, what we will do here in this bond yield is that we will do interest rate analysis plus present value analysis. This will be combined to get yield to

(Refer Slide Time: 35:58)



maturity or sometimes we acronym this as YTM. So question is I have introduced this new term yield, so what is yield? So, let us define what is yield. The yield of a bond is defined as the interest rate value for which the present value of bonds payments (that means the bond future payments). What are the bonds payments? Bond payments are your coupons and the face value for which this is equal to the bond price.

(Refer Slide Time: 37:39)

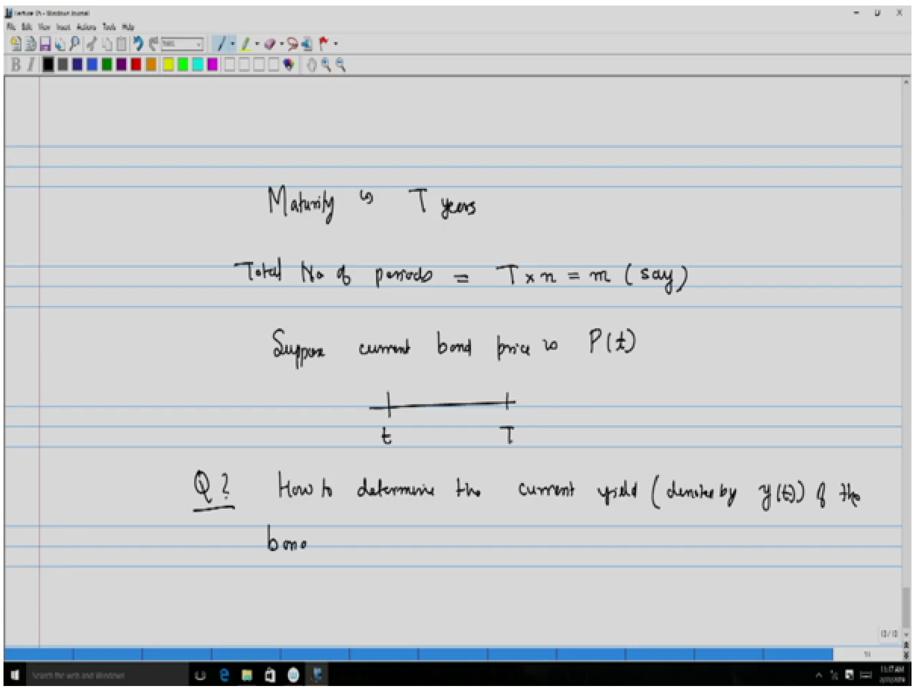


So, let us look at this little more closely. So, what happens is that in general, this is called the internal rate of return or IRR which in case of bond is yield. So, yield is a specific case applicable to bonds as far as the internal rate of return is concerned. So, internal rate of return is called a bond yield, where you are actually trying to see that what is the kind of yield that you are actually getting out of the bond.

Now, let us look at the concept of yield. Now, we consider a bond with the following characteristics. So, I will consider a bond with a certain face value and some coupon payments. So, with the characteristics of this bond are that its face value is V, number of coupons is n and size of each coupon is $\frac{C}{n}$.

So, essentially what I mean is that basically there are n number of coupons being paid in a particularly year, so that means that annual coupon payment is C. So, what I mean to say here is that when each period, each you know equally sized periods at the end of which the coupons are being paid, so this each interval or period is of $\frac{1}{n}$ years. So, that is why I said that the coupon is $\frac{C}{n}$, so that means over a period of one year, we receive a total amount of C, that is the total coupon payments for a particular year.

(Refer Slide Time: 39:53)



Then I need to look at what is the maturity, so maturity is T years. What is going to be total number of periods, right? This is going to be T number of years multiplied by n number of payments being made each year and we will call this as some m. Now, suppose current bond price is $P(t)$, that means I am setting here some point t and T. What do I want to find is that I want to find the following is that how to determine the current yield which is denoted by $y(t)$ of the bond. So, how do I determine this?

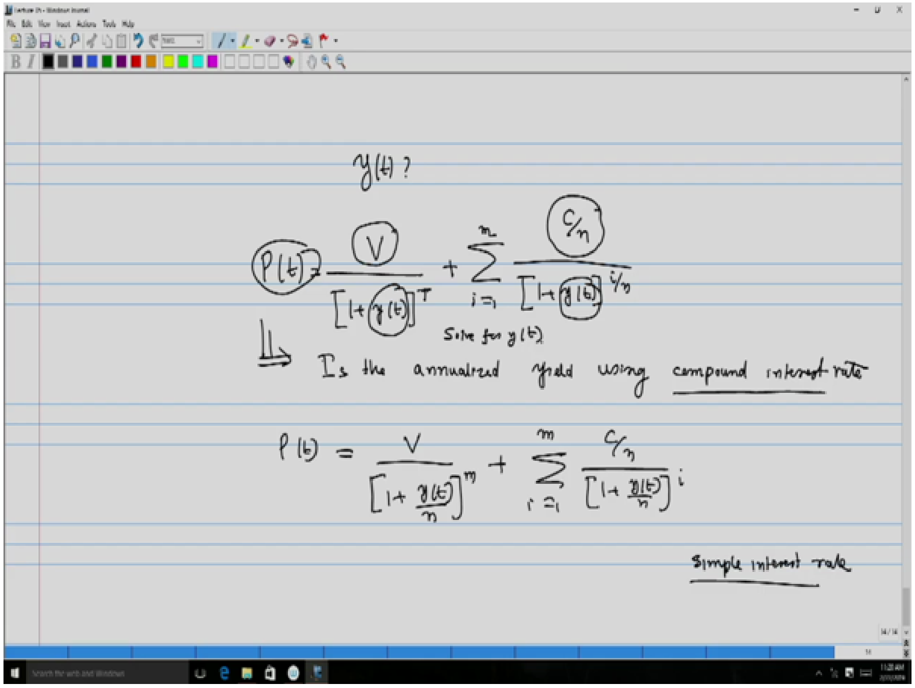
So, it can be determined very easily by making use of the concept of present value for multiple periods that we have already seen. So $y(t)$, this means that what are the payments that you are going to receive? You are going to receive a payment of V at the end of m number of periods, so its present value is going to be $\frac{V}{[1+y(t)]^T} + \sum_{i=1}^{m} \frac{\frac{C}{n}}{[1+y(t)]^{\frac{1}{n}}}$. Now, this expression that you have obtained, this is the annualized yield using compound interest rate.

Now, in case of continuous compounding, basically this is equal to $P(t)$, so your $P(t)$ is known, because this is the current bond price, your V is known, because this is the face value, your $\frac{C}{n}$ is known, because this is the coupon. So, the only thing which is unknown is $y(t)$, so you have to solve this for $y(t)$. So, likewise we have used the compound interest rate, so likewise in case of continuous compounding this formula gets modified to $P(t) = V\left(1+\frac{y(t)}{n}\right)^m$.

I am sorry let me just correct myself, not compound interest but rather simple interest, this becomes $\frac{V}{\left[1+\frac{y(t)}{n}\right]^m} + \sum_{i=1}^{n} \frac{\frac{C}{n}}{\left[1+\frac{y(t)}{n}\right]^i}$.

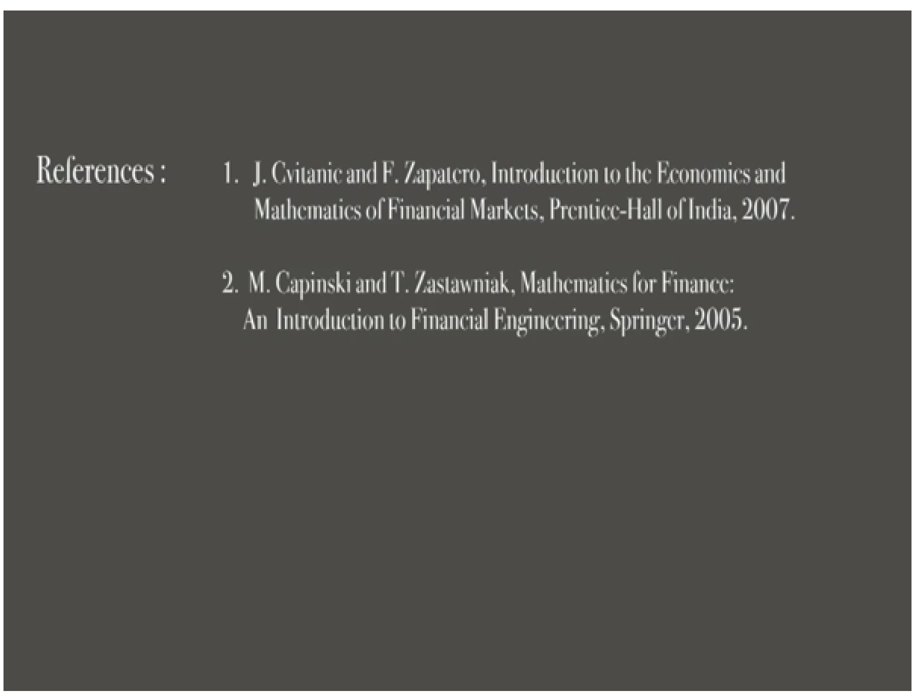
So, just to sum up what we did today, we started looking at what is going to be the future value for the series of payments that you are receiving from the bank and investing them. Now, in analogous manner, we

(Refer Slide Time: 41:12)



looked at what is the present value that means the current value for a series of future payments that we will leave and then we connect both of these two and extend it to the case of EMIs or monthly payments that you are going to make.

(Refer Slide Time: 44:39)



So, we looked at the concept of the present value in order to compare two different investment strategies and then we looked at this concept in the context of calculating the monthly payments on loan or what is commonly known as EMI and then finally, we concluded by introducing the concept of internal rate of return in the particular case, when you are considering a coupon bond. Thank you for watching.