Mathematical Finance

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Module 2: Time Value of Money and Risk Free Assets
Lecture 1: Interest Rates and Present Value

Hello viewers, welcome to Lecture 4 of this course on Mathematical Finance. This is the beginning of the second module where we will look at time value of money and we will look at various convention of calculating interest with an emphasis on simple and compound interest. We will also look at the internal rate of return and the present values and finally, we will talk about annuities looking at some specific examples.

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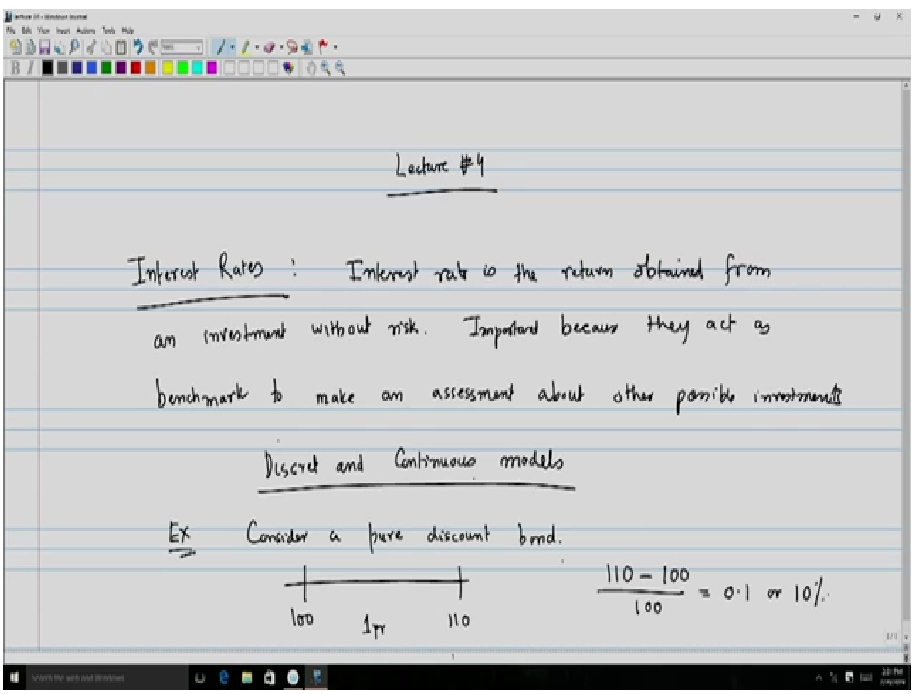

We begin with interest rates. What is an interest rate? Interest rate is the term that you are very familiar with, because all of us are involved in banking. An example of interest rate would be that suppose we deposit 100 rupees into our saving account and after 1 year the saving accounts our balance grows to 104. So, that means during this 1 year period, we have earned an amount of 4 rupees for our initial deposit of 100 and we say that we have received an interest of 4% on the money that was deposited in the bank. More specifically, interest rate is the return obtained from an investment without risk and these are important, because they act as benchmark to make an assessment about other possible investments. What do you mean by benchmark? What I really mean here is that, we know that when you are making a risk free investment, the interest that we earn or the rate at which you earn the interest that is risk free and so any other investment that you look at particularly when you are looking at a risky investment, then it is always essential to make a comparison of the return that you expect to get on a risky investment as compared to the interest rate that you would get by putting your money in the bank.

And obviously you want that if you are getting into risk, you obviously expect that the expected return that you get on your risky investment is significantly higher than the interest rate that you can actually get from the bank and which is why this interest rate plays a vital role in acting as some sort of a basic benchmark or a year stake when you are assessing several other ways in which you can earn particularly the risky ways for example, stocks or swaps or forwards or futures and so on.

In this context of discussion, we will mostly talk about both discrete and continuous models. So, to begin with, we first look at an example of a pure discount bond. You recall that a pure discount bond is where you pay the money to purchase the bond and then you essentially get one single final payment at the expiration time or the maturity. Suppose I invest an amount of 100 over for a period of 1 year and at the end of one year I receive an amount of 110, then $\frac{110-100}{100} = 0.1$ or 10% is the interest rate.

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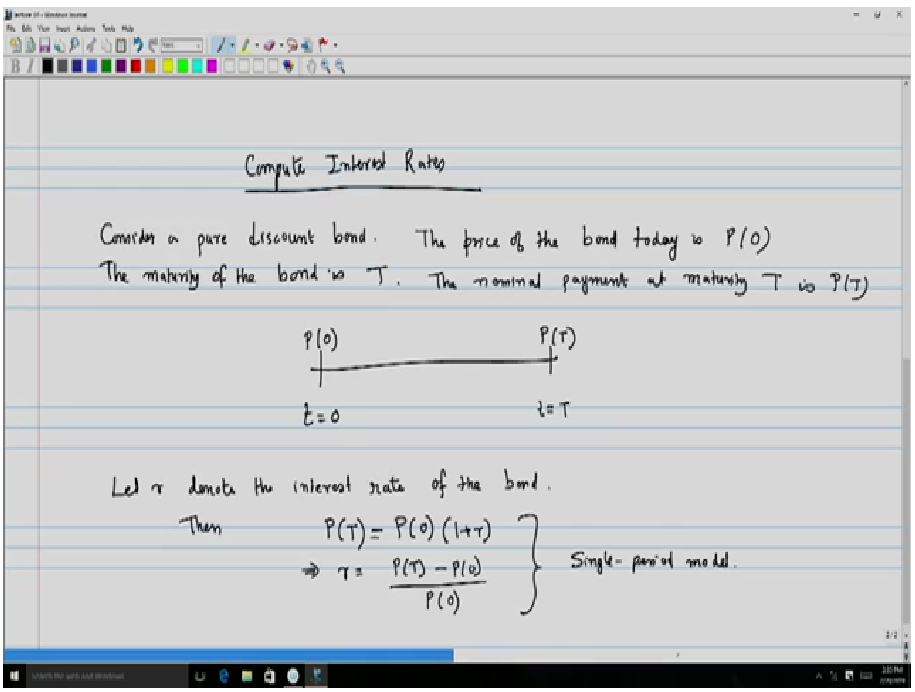

Let us now look at more on how you can actually look at and compute interest rates. So, in order to look at this, let us first consider a pure discount bond. The price of the bond today is $P(0)$. So, that means I have today's time is at $t = 0$, the price of the bond is $P(0)$ the maturity of the bond is T, so that means $t = T$ is the maturity or the time at which we will receive the proceeds the original amount and the proceeds from our investment in the bond.

Then the nominal payment at maturity T is $P(T)$. So, that means at time $t = T$ the amount that you receive is $P(T)$. So, how do you bring the interest rate into the picture here? So, let r denote the interest rate of the bond. Please remember when you talk about r being interest rate of the bond, it typically will be a number that will lie maybe between 0 and 1. But if you want to view this in terms of percentage when you say r is the interest rate, if you want to make the statement in terms of percentage, you have to multiply this by 100. For example, if r is 0.07, then we multiply it by 100 which is 0.07×100 and we say that the interest rate is 7%. So, then I start off with an initial amount of $P(0)$ and I multiply this by $1 + r$ and this is basically the amount that I will get at time $t = T$, namely, this is the same as $P(T)$. So, this will imply that $r = \frac{P(T)-P(0)}{P(0)}$.

Now, please remember that what we have done here is basically we have taken a time $t = 0$, when the initial investment has been made and a time $t = T$, when a nominal payment is received. So, this means there is only one initial time period and one final time period and so this is what is we will call as the single period model, because it involves only one period.

So, an example of a bond as we have already pointed out. An example of a bond is a savings account because in savings account you make an initial investment and after a certain period of time or periodically you keep getting interest on it.

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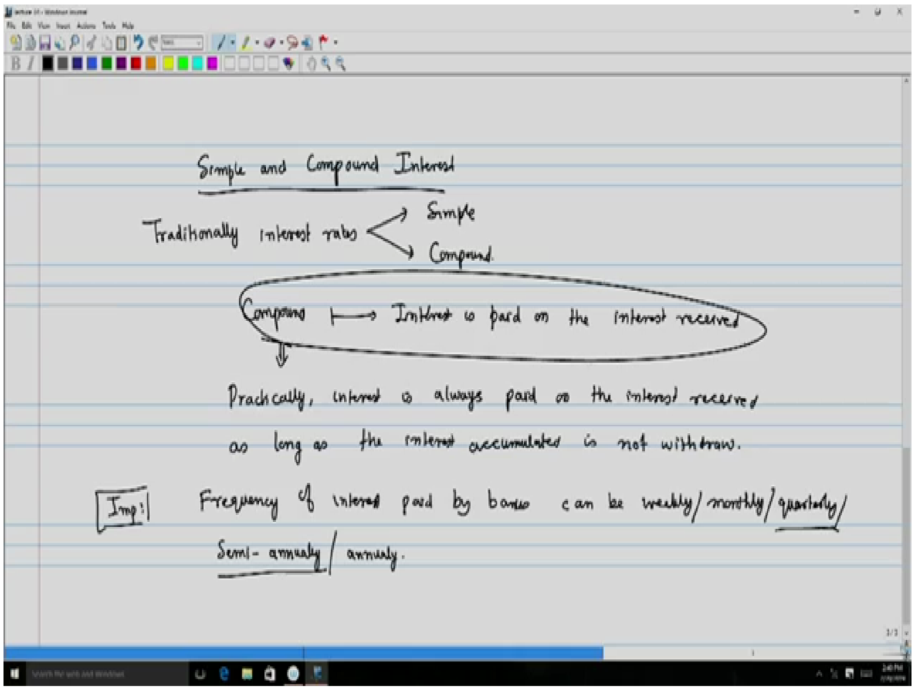

Now, let us look at some specific kinds of interest, in particular, we will look at the simple interest and compound interest and we will look at what is the convention and how you can calculate them. So, we start off with simple and compound interest. So, traditionally interest rates have been classified as simple or compound. So simple interest rate is basically you know as the name suggests it is just simply calculated in terms of percentages, but compound interest rate is when interest is paid on the interest received and compound interest is what is actually practically used which is why we will mostly focus on compound interest.

It basically says that suppose you have put money into a savings account and every 3 months or every 6 months you will get the interest as long as you do not withdraw the interest, you will then get subsequently paid interest on the interest itself. So, suppose that you have invested in amount of 100 and the interest rate for the whole year is 5%. So, every 6 months the interest rate is 2.5%.

So, for the first 6 months, the interest accumulated would be 2.5 on this 100, but after 6 months your principle amount changes to 102.5 and then the interest will now be calculated on 102.5 instead of the original 100, assuming that these 2.5 interest that you have received is not withdrawn by you. So, this is something that is practically used, so practically interest is always paid on the interest received as long as the interest accumulated is not withdrawn. So, this is what I meant by saying that you interest accumulated interest of 2.5 for the first 6 months and assuming that you have not withdrawn that 2.5 your interest for the end of the year for the remaining 6 months will be calculated on the amount of 102.5.

Now, here I just want to make an important observation that frequency of interest paid by banks can be weekly, say, monthly, quarterly, semi-annually or annually. So, our domestic banks typically will be paying interest either semi-annually which is every 6 months or they will be paying interest quarterly that means every 3 months.

So, let us now discuss simple and compound interest in little more detail. So, we come back to simple interest. Now, if r_Q denotes the quarterly simple interest rate that means if r_Q is the interest rate that is being

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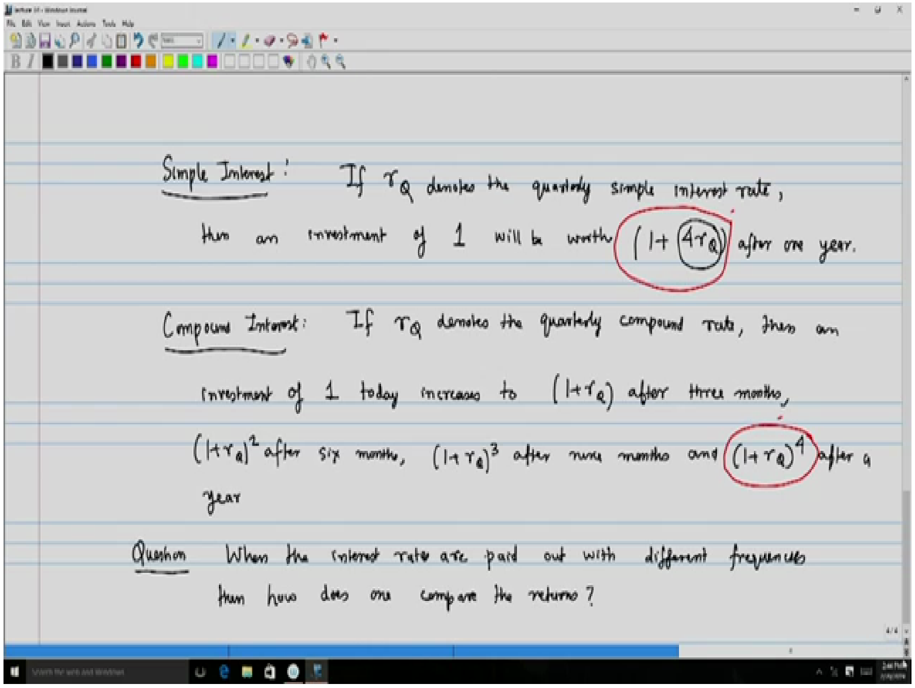

paid quarterly, then an investment of 1 rupee, for example, will be worth $1+4r_Q$ after 1 year. So, this means that we are not calculating interest on the interest, but rather we are saying that, ok every 3 months that is every quarterly r_Q is the interest rate that will be paid for those 3 months. Since the whole year involves 4 quarters, so that means that the interest that I will get is simply calculated will be $4r_Q$. So, investment of 1 rupee will grow to $1 + r_Q$ at the end of 1 year.

Now, let us see how does this change in case of compound interest. So, if r_Q denotes the quarterly compound rate, then an investment of 1 today increases to $1 + r_Q$ after 3 months. That means after one quarter (since this is continuously compounded), we have to calculate the interest on the interest. Therefore, after 6 months we will get $(1+r_Q)^2$, so the $(1+r_Q)^2$ after 6 months, $(1+r_Q)^3$ after 9 months and $(1+r_Q)^4$ after a year.

So you see, in this case, you start off with 1 and the amount that you get after 1 year is $1 + r_Q$, but in case of compound interest you start off with an amount of 1, but at the end of 1 year, you will get an amount of $(1 + r_Q)^4$ and this is a term that is obviously going to be larger than previous term.

Now, let us ask ourselves a question, when the interest rates are paid out with different frequencies, then how does one compare the returns? So, what are the meaning of this question? The meaning of the questions following that I have said that the banks could calculate interest rate weekly, monthly, quarterly, semi-annually. So now, if I have several investments, in one case, I might be getting the interest every month, another case, I might be getting every 6 months, in other case, I might be getting every 3 months.

So, having this different interest rate for different periods can lead to a certain amount of ambiguity and in order to attain and obtain us greater clarity or standardization what can one do? So, this is basically the question that we are posing, how does one take care of this situation when you have these different periods over which this interest rate are actually paid?

So, the answer to this question is follows that what you do is that we have all these different rates and we take them and we transform the rates into rates for a common or typically I might call it a standard time interval, interval typically a year. So, what we will do is that we will pretty much take all this interest rate and find you know some equivalent rate which is applicable for 1 year, so that would help us actually compare all the different rates for different frequencies that we actually encounter for a variety of investments.

So, this transformation is what is known as the annual or more commonly it is known as annualized interest rate, all right. So, one assumption before we move on to a little more about this annualized interest

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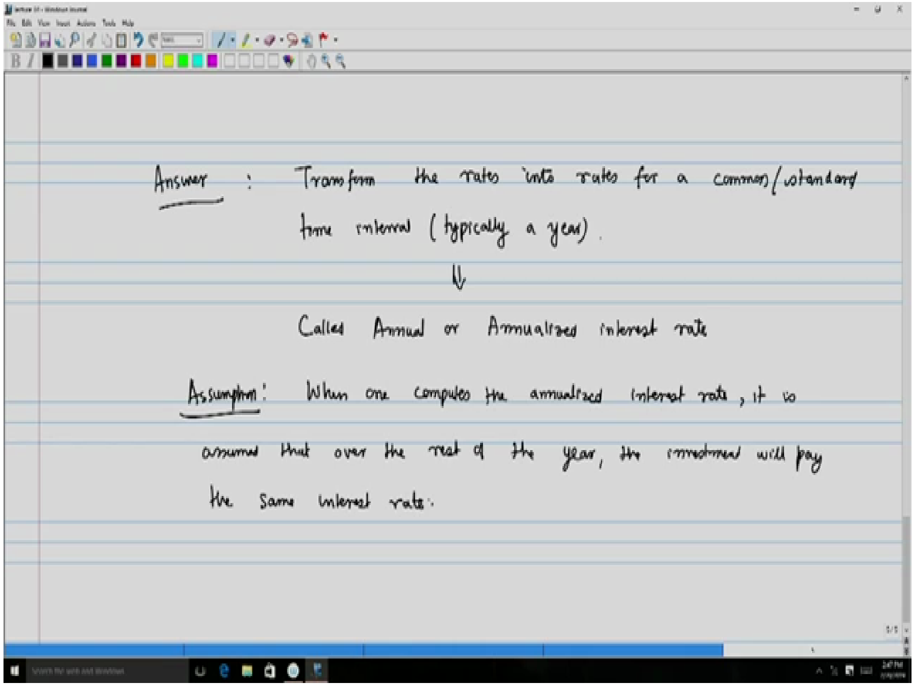

rate is that when one computes the annualized interest rate it is assumed that over the rest of the year the investment will pay the same interest rate. So, this means that suppose you are looking at an investment for 1 year period and the interest rate is actually being calculated quarterly or every 3 months, so that means we have this r_Q, what this assumption means that this r_Q will remain the same for each of the 4 quarters.

So, in other words, for the first 3 months suppose I get some 2% interest rate, then at the end of 3 months, when this is reinvested the assumption is that the interest rate has not changed, but the interest rate has remained at 2% from the 4th, 5th and 6th month and likewise the same for the remaining two quarters of 6 to 9 months and 9 to 12 months, ok.

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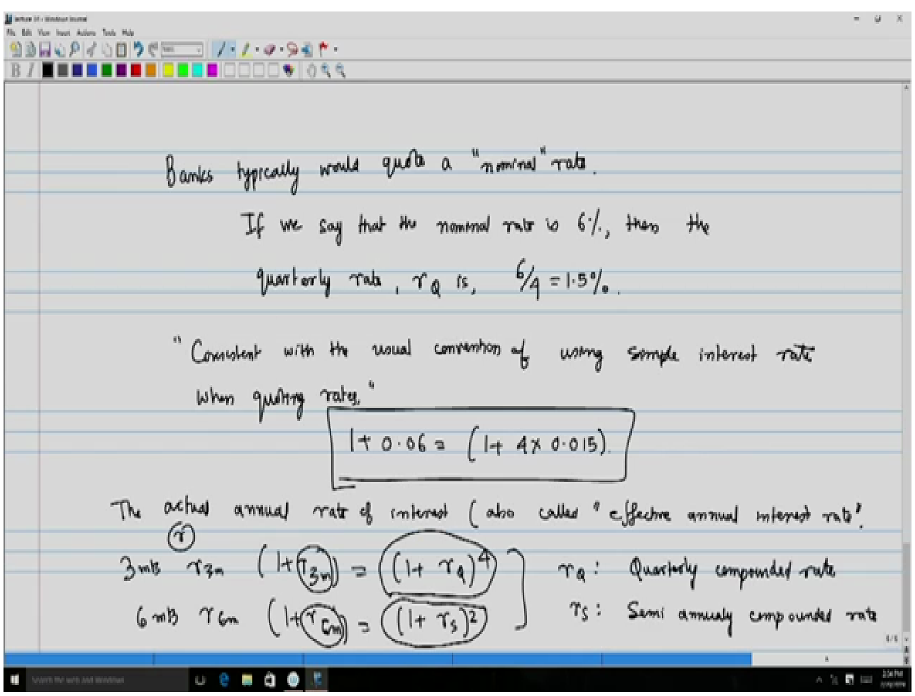

So now for this, what they do? Banks do, banks typically would quote a nominal rate. So, let me explain what this nominal rate is, what they will do is the nominal rate is some sort of a yearly rate, but it is not the actual rate that is applicable but it works in the following way. So, if we say that the nominal rate is 6%, then the quarterly rate which is applicable to all the quarters, so the quarterly rate r_Q is $\frac{6}{4}\% = 1.5\%$.

So this is basically means that if I say there is a nominal rate for a year of 8% and then the interest is been calculated every 6 months, so this means that I basically split that 8% interest into two equal parts, namely 4% will be paid for the first 6 months and 4% will be applicable for the remaining 6 months. So, this is consistent with the usual convention of using simple interest rate when quoting, so this is just a matter of convention.

So, basically, when I say $\frac{6}{4}\% = 1.5\%$, basically it is making use of a simple interest rate, which means that in this example $1 + 0.06$ is nothing but $1 + 4 \times 0.015$. So, effectively the nominal rate is quoted in a way that it basically reflects a simple interest.

Now, please remember this is what is known as a nominal rate alright or some sort of a notional rate, but this is different from the actual rate that you are interested in. So, the actual annual rate of interest also called effective annual interest rate. So, what will it look like? This will be and this rate is going to be this r, so if r is this actual annual interest rate then it will satisfy $1 + r$ is equal to now if I for 3 months and denote this r as r_{3m}, so this will be $1 + r_{3m} = (1 + r_Q)^4$ and for 6 months this will be r_{6m} and it will satisfy the relation $1 + r_{6m} = (1 + r_s)^2$.

So, what is this r_Q and r_s? r_Q is basically the quarterly compounded rate and r_s is the semi-annually compounded rate. So, basically what it means is that if I invest an amount of 1, then if it is quarterly compounded, then $(1 + r_Q)^4$ is going to be the amount of money that I will get at the end of 1 year. If it is semi-annually compounded at the rate of r_s, then $(1 + r_s)^2$ is the amount that I will get at the end of 1 year.

And I find some sort of an equivalent rate r_{3m} and r_{6m} for which if those rates are applied, then in a first case r_m is the rate if it is the annual rate which will give the same return as making and investment on a quarterly basis and r 6m is the interest rate that you would get if you were to invest and receive the interest annually and it will be the same as the interest rate that you would get when the semi-annually compounded rate is applied.

So, that means I can either I mean if I choose an amount of 1 and if the bank gives me an interest rate quarterly of r_Q then r_{3m} is the interest rate which will be applicable for you to make the same investment and end up getting the same amount of return likewise for r_{6m}, all right.

Now, to sum it up the same principle is applied in case of interest rates for periods longer than 1 year. So, in the previous two cases what you looked at was that we look at the quarterly and semi-annual interest rate and how to convert them to some sort of an annual as interest rate. Now, we are going to apply the same principle in case of the interest rate for periods which are now no longer less than 1 year, but for periods which are more than 1 year.

Next, suppose I have a situation where I have for 2 year period with interest rate r_2, the effective annual rate is $1 + r_{2y} = (1 + r_2)^{\frac{1}{2}}$. Remember r_2 is the interest rate for 2 years, so for 1 year it is going to be $(1 + r_2)^{\frac{1}{2}}$ and then the effective annual rate will be given by this relation the effective annual rate r_{2y}, will be given by this particular relation.

Now, we have done with the effective annual rate. Let us now talk about something which is called the continuous interest rate. Remember earlier we had mentioned that we will look at both discrete and continuous. We have looked at the discrete case. So, now, let us look at the continuous interest rate. So, suppose, the bank has a nominal rate r_2 paid every 6 months remember, this is the nominal rate, so nominal rate r subscript 2 paid every 6 months basically means that every 6 months the interest rate that will be applied to calculate the interest will be $\frac{r_2}{2}$.

Then, the effective annual rate is $1 + r = 1 + \frac{r^{(2)}}{2}$. Remember $\frac{r^{(2)}}{2}$ is going to be the interest rate, since $r^{(2)}$ is the nominal annual rate, that means if you are calculating the interest every 6 months, so the interest rate that will be applied for the 6 months will be $\frac{r^{(2)}}{2}$ and I take the square of this which will take into account the compounding.

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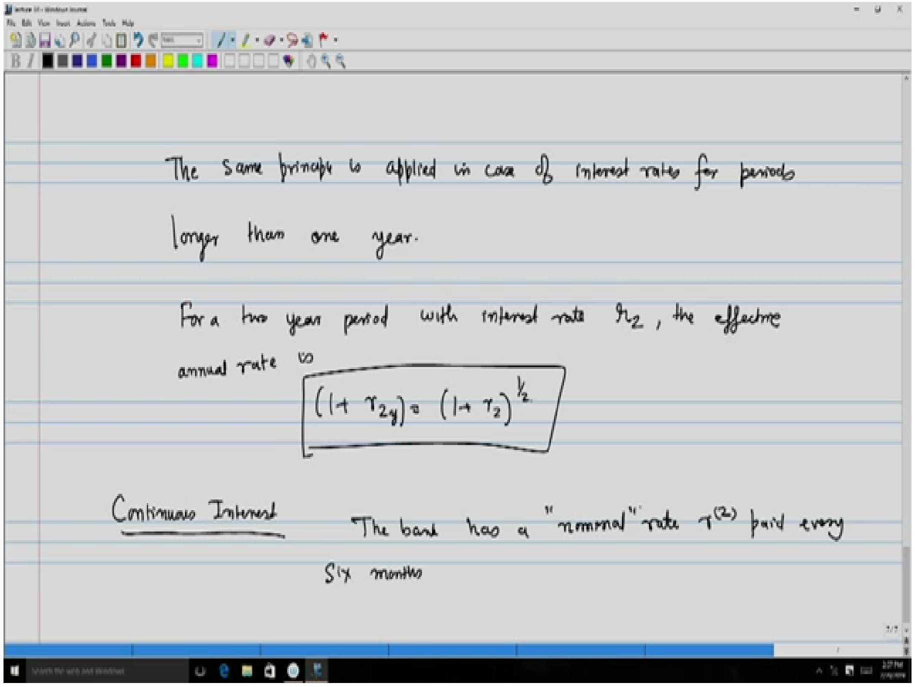

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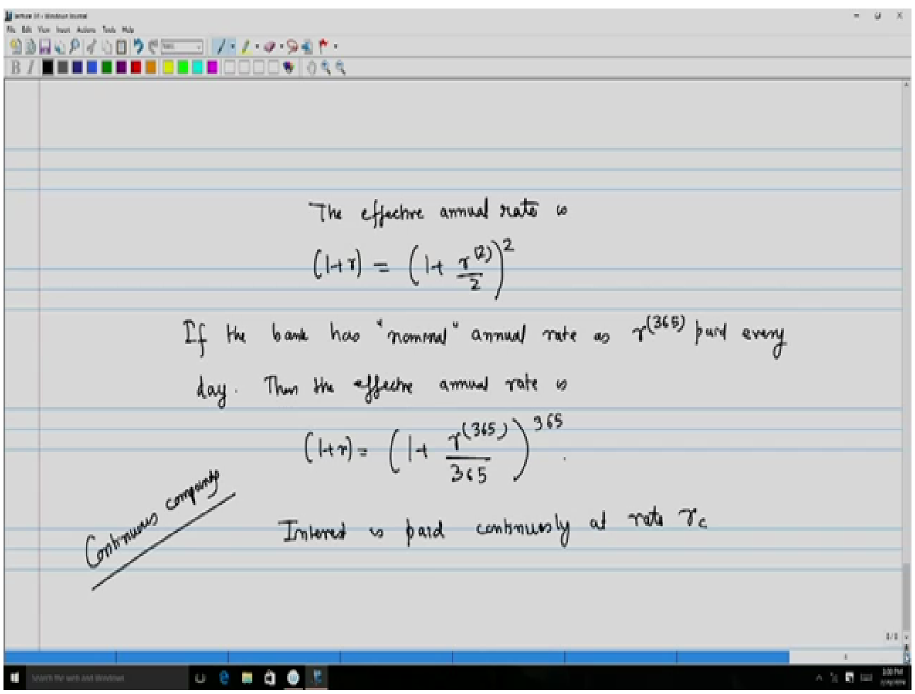

Now, if instead of a semi-annually interest rate that means interest rate paid every 6 months. If the bank has nominal annual rate as $r^{(365)}$ paid every day assuming that this is the year 365 days and the nominal days $r^{(365)}$, that means this is the interest rate that will prevail when you are calculating the interest rate every on a daily basis. Then the effective annual rate is given by $1 + r = \left(1 + \frac{r^{(365)}}{365}\right)^{365}$.

Now, let us move to continuous compounding. So, suppose, in case of continuous compounding the interest rate, the interest is paid continuously at rate some r_c.

So, in order to look at this in a more quantitative way what you do is the following that you assume that the interest, first you start off with that interest is paid n times, so that means $\left(1 + \frac{r_c}{n}\right)^n = 1 + r$, where r_c

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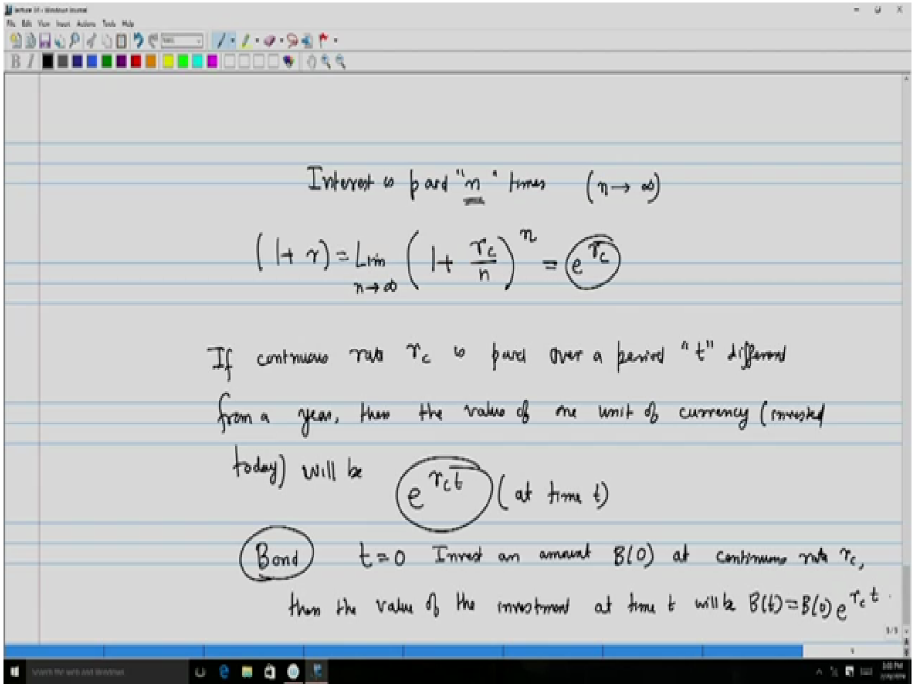

is the nominal. Now, in order to visualize continuous compounding what this means that? Instead of the interest rate is paid n times, so in the first case $n = 2$ and in the second case $n = 365$.

If we assume that the interest rate is continuously compounded, so that means I will have to look into the scenario where $n \to \infty$. Then I take the limit of this as $n \to \infty$ and this becomes equal to e^{r_c}, right the limit of this is equal to e^{r_c}. So, that means that if continuous rate r_c is paid, this is the case where I am assuming you just paid for a period of exactly 1 year however, you might encounter a situation where continuous compounding is actually applied for a period that is not equal to 1 year.

So, in that case, if the continuous rate is r_c is paid over a period t different from a year then the value of one unit of currency (say, 1 rupee invested today) will be the value of 1 unit of currency invested today will be $e^{r_c t}$ at time t. Now, we can use this in order to calculate what is going to be the value of the bond. So, if I have a bond where at time $t = 0$, I invest an amount $B(0)$ at continuous rate r_c, then the value of the investment (assuming of course that r_c remains constant for all subsequent time) at time t will be $B(t) = B(0)e^{r_c t}$. So, this covers the discussion about simple interest and compound interest including continuous compounding.

We next look at present value. So present value will essentially be just a concept that moves in the opposite direction of what we have looked at so far in this lecture. So, an important concept or requirement is to make a choice between two future payments or cash flow of several payments. So, this means that we might be looking at different investment, you are be looking at two different investments and we have we know what the payments are going to be or what the returns are going to be for those two at a future point, either they could be deterministic or if they are unknown or a random variable, then we could have an expected value of those.

So, sometimes it becomes important that when you are presented with two choices. So, sometimes such multiple choices it effectively becomes that you are making a choice between different kinds of investments with a certain expected cash flows and it is very important at that point to distinguish and make a choice between all these different possible cash flows and this is where the concept of present value comes in. This is something that we later see it is a very, very critical importance when you are looking at the option pricing problem.

What is the idea of present value? The idea of present value is that we apply the technique of interest calculation in reverse direction. So, that means what you have done so far is that we have transformed

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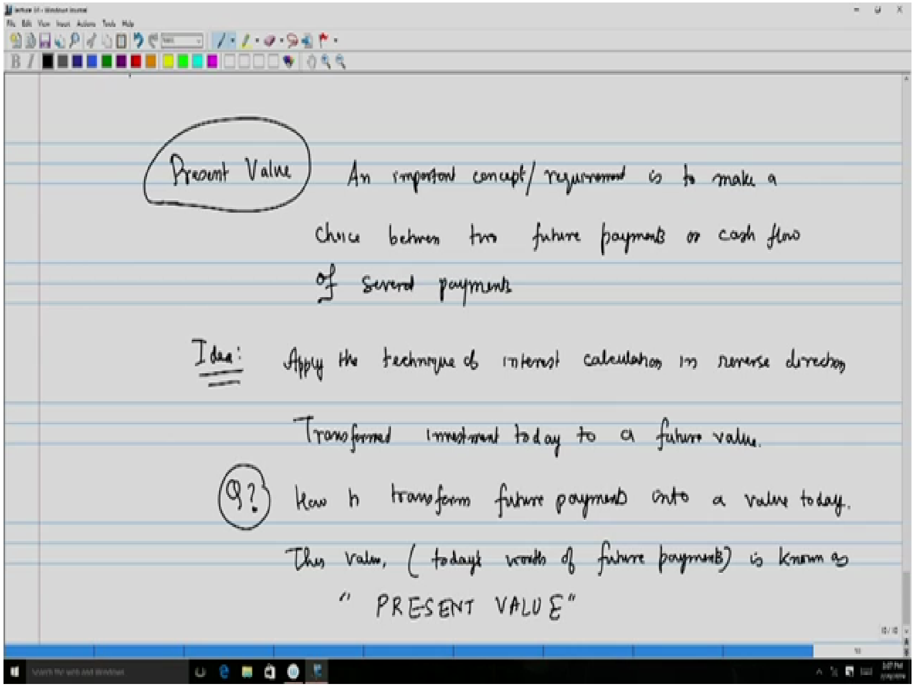

investment today to a future value. So, what we done so far is that we have made an initial investment and we look at what is going to be the future value, but what we were doing in case of present value is exactly the opposite that given a future expected value we need to find out what is going to be its present value. Therefore, the question is how to transform future payments into a value today? This value which is todays worth of future payments, is known as present value.

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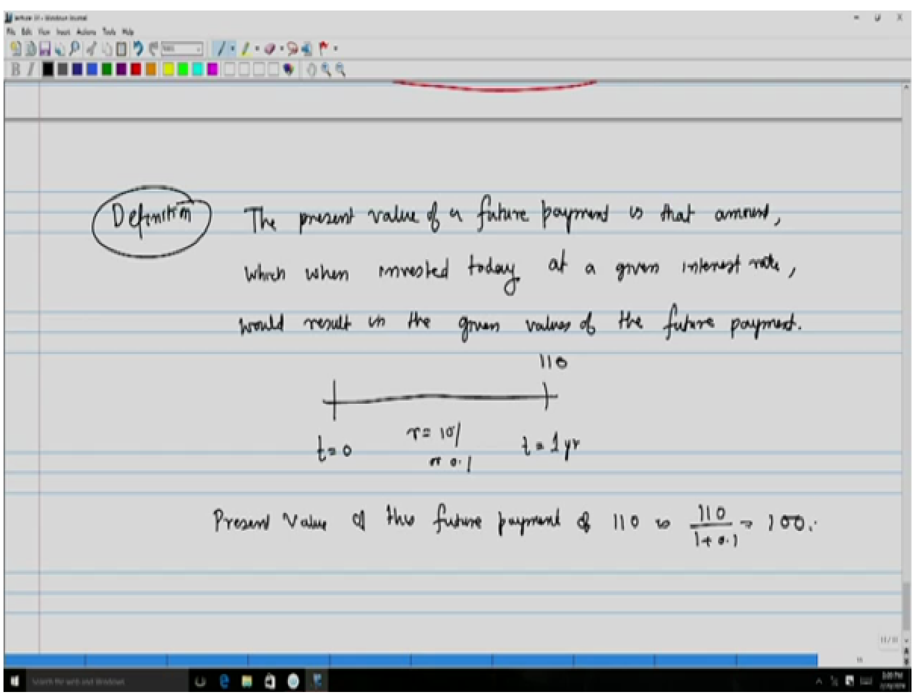

So, what is the definition of present value? So, the present value of a future payment is that amount which when invested today at obviously a given interest rate would result in the given values of the future payment. So, let us go back to our example of where today time $t = 0$ and I am looking at a future time point $t = 1$ year and the amount that I expect to receive is 110 and the interest rate is (say) 10% or 0.1, then the present value of this future payment of 110 is $\frac{110}{1+0.1} = 100$.

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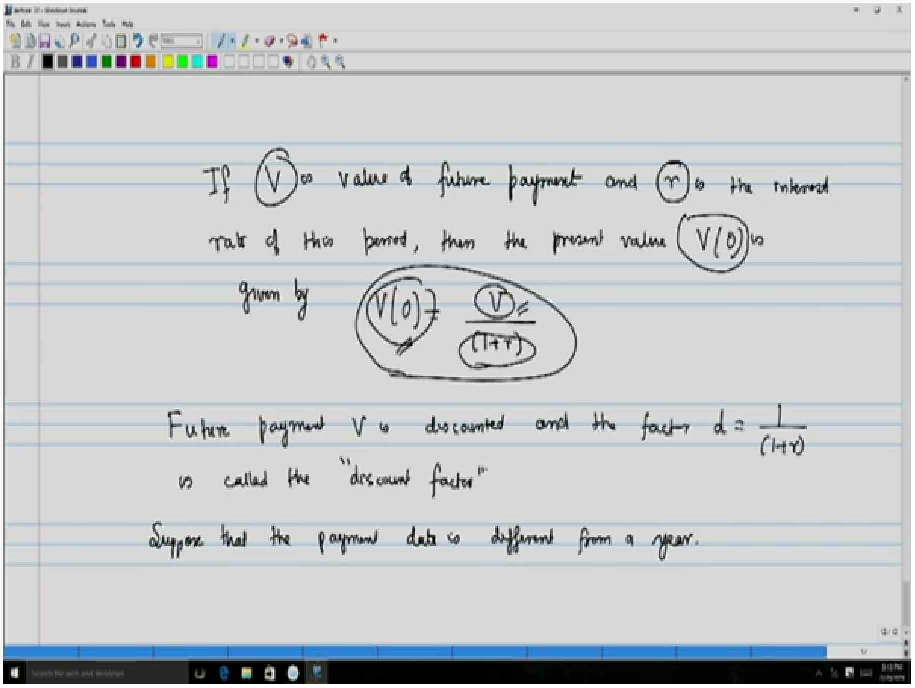

Let us now generalize this to a more you know broader setting. So, if V is the value of future payment intuitively, we can think of that this payment is this something that is being received at the end of 1 year and r is the interest rate of this period, then the present value $V(0)$ is given by $V(0) = \frac{V}{1+r}$. So this is in fact applicable to any period provided that the interest rate is r for the same period, but intuitively you can think that this is something that you can look at it from the point of view of a 1 year investment.

So that means that if you have an amount of V that is being paid at the end of your investment period and if the interest rate is r for this investment period, then the present value which you will denote as $V(0)$ is given by this particular relation. So, in our example, $r = 10\%$, $V = 110$ and consequently, we get $V(0) = 100$. So now, another interpretation of this is that the future payment V is discounted and we said that this is discounted because the future payment is V, but $V(0) = \frac{V}{1+r}$, which is a number, this denominator is greater than 1.

Obviously, the value $V(0)$ that we have here this is going to be less than the value of V that we have here. So, that means that this $V(0)$ is like a discounted or a lowered value. So the future payment V is discounted and the factor $d = \frac{1}{1+r}$ is called the discount factor. So, this is called the discount factor because $V(0)$ is obtained by discounting V and this discounting is happening, because you are dividing V by $1 + r$ or equivalently multiplying V by $\frac{1}{1+r}$. So, that is the reason why d is called as the discount factor. Now, suppose that the payment date is different from a year, that means it is not same as 1 year.

Then, the corresponding interest rate will be expressed in its equivalent annual rate. So, this means that, if payment takes place in 1 by n years for some n and r is the corresponding equivalent annual rate using simple interest, then the discount factor is $d = \frac{1}{1+\frac{r}{n}}$. So, this were the case when you are using a simple interest, so that means the discount factor will be $\frac{1}{1+\frac{r}{n}}$, because r is the annual rate, so $\frac{r}{n}$ is the rate that is applicable for the $\frac{1}{n}$ duration of the investment.

Now, when the compound rule is used to compute the equivalent annual rate, then the discount factor is $d = \frac{1}{(1+r)^{\frac{1}{n}}}$. So, this is the discount factor when the simple interest convention is used and this is the discount factor when the compound interest convention is used.

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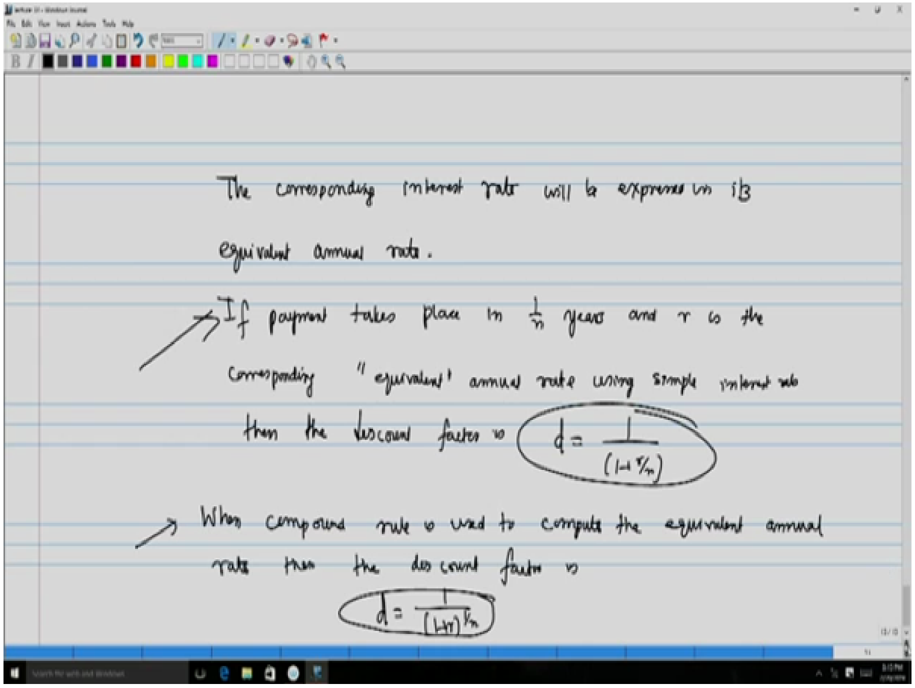

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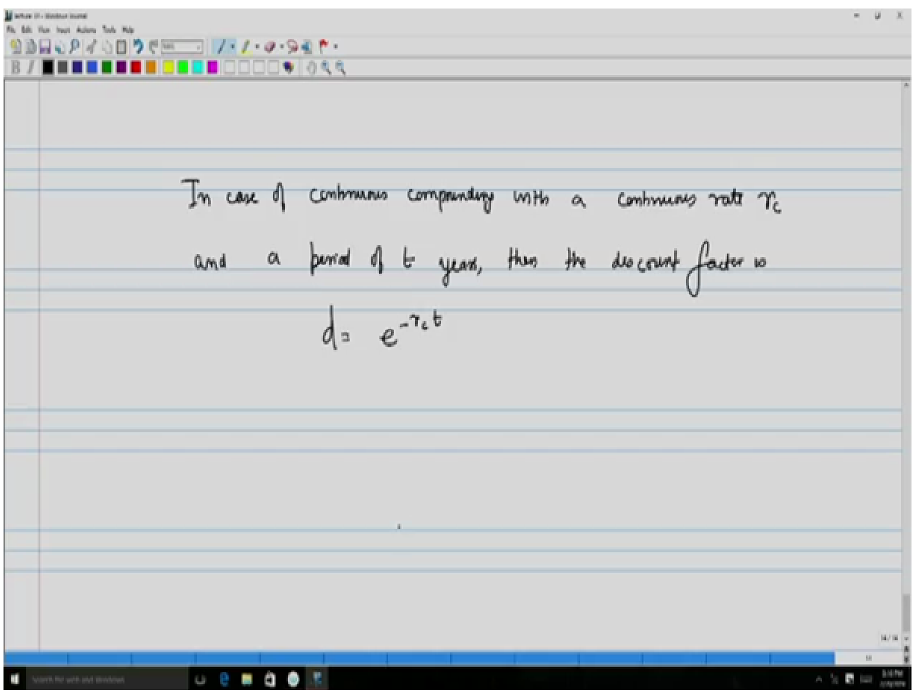

Finally, in case of continuous compounding with a continuous rate r subscript c which you have already seen and a period of t years. Then the discount factor is $d = e^{-r_c t}$. So that means if we have an investment of V, then the present value of that investment would become $Ve^{-r_c t}$. So, this concludes this lecture while we have spoken about simple and compound interest as well as we talked about present value and introduce a very important notion of discount factor. Thank you for viewing.