Mathematical Finance
Lecture 36 BSM Model with Dividend Paying Stocks

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Hello everyone, we will, next what we will deal with is, the classical BSM but with a change in the dividend paying stocks.

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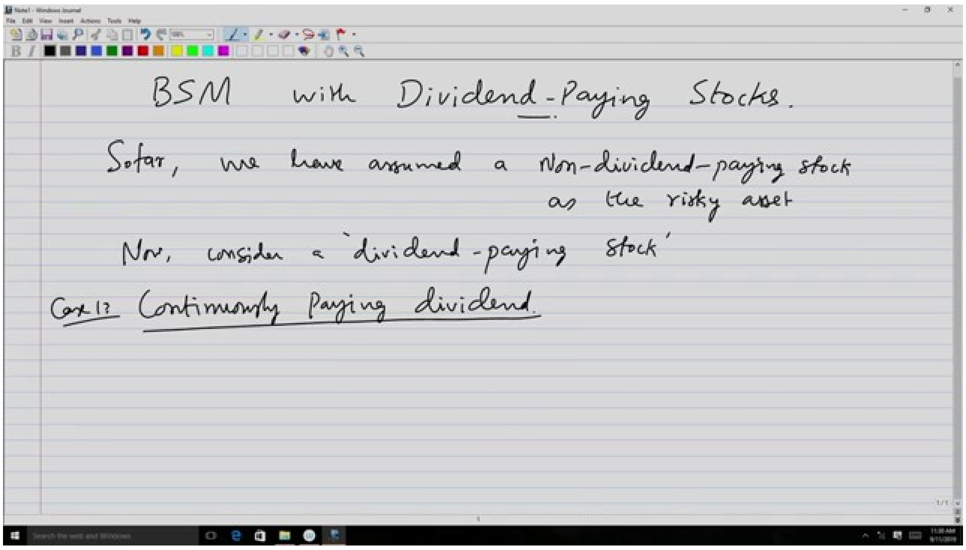

So, recall that so far whatever we have done, we have assumed a non-dividend paying stock as the risky asset. So, with this only like, whether it is binomial model or the BSM model, with one stock or BSM model with the M stocks, right. Whatever we have assumed so far is that those stocks do not pay a dividend, but it is generally the real case, reality case is that the stocks do pay dividends, and hence, it will be worthwhile to consider the risky asset as paying dividend and how that affects our risk neutral pricing formula, ok.

We will be in a simple and setup of exactly classical BSM with only change as dividend paying case is what we will consider. But whatever we are doing it can also be considered in more general with M stocks. And the Brownian motion case, which is what we call that the multi dimensional BSM market model, ok. So, now consider dividend paying stock, ok. So, this is what we will now consider, ok. So, that is the change and let us, we will consider here the 2 cases, but before that just to recall that the what we had or how we arrived at to this complete pricing formula.

What we did was we first showed that the discounted asset prices, which a discounted stock prices are Martingales under risk neutral measure, which we have constructed using Girsanov's theorem. and that implied that the self-financing portfolio value processes that you are considering, which the portfolio consisting of asset, this risky asset and the risky assets of the underlying model is also, the discount version of that is also Martingale under the same risk neutral measure.

And that led us to the risk neutral pricing formula. So, what we needed was this self financing portfolio process, the discounted version of it, we $\tilde{P}$ Martingale is all that needed in order to pin down the risk neutral pricing formula, right. But that was aided by the fact that they discovered asset prices are $\tilde{P}$ Martingale, right. So, that was the case when we consider a non dividend paying stocks. Now in the dividend paying stocks, right.

If we have to have the same scenario, of course, obviously, since the dividends and as you understand, the dividends reduces the stock prices right whenever it is being paid. And that stock prices if you look at only the stock price process alone, that may not be $\tilde{P}$ Martingale, right. So, (what we take is that) suppose if that gets reinvested because this is going to be a self financing portfolio, processor which means no money you are taking out in between, having said that, this is the initial price. So, after that once the process started that you are not taking your money not you are putting in any money. So, with a self financing portfolio process, if you look at it, then with dividend reinvested in the stock, right, the underlying stock prices process will be a $\tilde{P}$ Martingale. And that makes the discounted portfolio value processor Martingale. So, that is the point that people, right. Here is risk neutral measure, which we defined as if it makes the discounted value process portfolio value process a Martingale, that is the measure that we will take and it is exactly similar to what we have done in the case, ok.

Now, so, what is the major change that you might observe, is that stock price process need not be a Martingale under $\tilde{P}$. What are we defined, but the stock price process with the dividend reinvested in the same stock will be $\tilde{P}$ Martingale, we pick that $\tilde{P}$ and that would be if you make an analogy, it is as if you know the dividend has not been paid out, right. So, that is what will turn out to be, ok.

Now, we are talking about dividends. There are 2 cases really here, right. 1, the dividend is paying at continuously at some rate, the other discrete dividends, ok. Now, in general you would find most of these formulas assume that the dividend rate continuously paying dividend rate, is what we assumed, ok. The reason is that, for a well diversified portfolio, for example, if it has so many assets in it's, as its component and each asset might pay dividend at various random times. So, this is itself as if the whole portfolio is paying dividends at a continuous rate. So, this will be more appropriate, this will be more approximate to such a situation, but as far as single asset is concerned, it may be prudent to view as if the dividends are being paid at some discrete time points during your time of consideration, which is typically 0 to t, right.

So, you may expect, for example, if you take a typical star you may expect that 3 to 4 times in a year depending upon the financial situation that a stock may pay dividends. So, it may be assumed that discrete dividend. So, let us deal with both cases, first what we will do is that the case 1, which is basically continuously paying dividend, is what we will consider as the first case. So, our as a model everything is the classical BSM set up except that the stock price process, ok.

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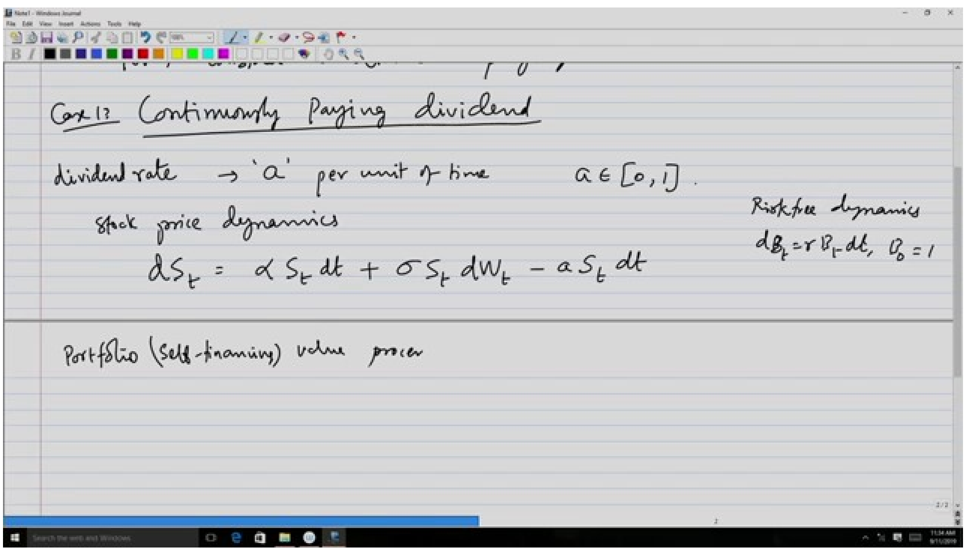

So, what we have now is that, so, stock is being modeled as a geometric Brownian motion, right. And the rate, which is what we call it as the dividend rate, is we pick it to be some quantity 'a'. This is 'a' per unit of time. So, what we have? So, 'a' is typically in the interval 0 to 1, if 'a' is equal to 0, if you put which means this corresponds to the case where there is no dividend is being paid out. and if 'a' is equal to 1, the whole stock price is being paid out as dividend and that is no more the stock exists in the market in that case. Typically the rate is what belongs to the open interval 0 1, which means that is some positive rate at, which dividend is being paid, right.

So, that is what the general assumption that you will have? Now, whenever the dividend is being paid at any point of time it reduces the stock price by that amount, right. So, the dynamics, so, then the stock price dynamics would be $dS_t = \alpha S_t dt + \sigma S_t dW_t$. As usual, (if as if the dividend,) this is the rate of return of this stock, minus, the dividend is being paid at a rate continuously at the rate 'a' per unit of time, then 'a' is the rate per stock, right.

So, this a St dt would be the amount that is getting reduced by this. So, you look at here, in a small interval, the stock that t, right. The change in the underlying stock price process would be the stock would grow at the rate of alpha, but it will also reduce by their dividend rate 'a' and volatility as usual is the same as the sigma parameter that you have it here. So, basically and you also have this risk free rate, ok, as usual.

So, this is the description, stock price dynamics and this is risk free assets dynamics or the bond or money market whatever you want to call these dynamics is as usual. So, this is what the 2 dynamics, every assumptions remains the same as in the classical BSM framework, ok. Now, as if you know, because this is reduced by the rate 'a', if you reinvest this dividends in the same stock then basically you will be adding an additional this 'a' rate, I mean you will keep buying more of that St by that rate 'a'. So, that means net return, the, the main rate of return of the stock where the dividends are getting reused in the same stock would then be equal to alpha as in the previous case, ok.

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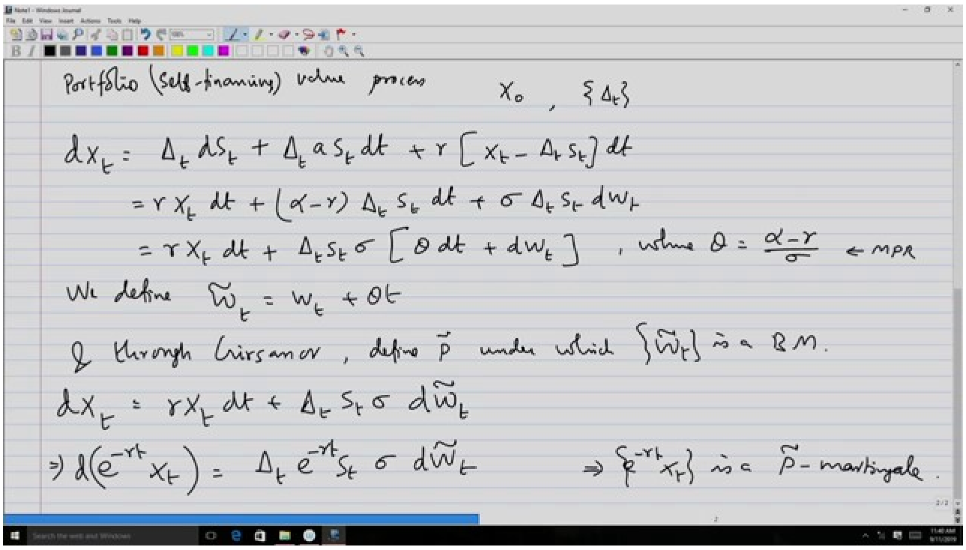

So, now, let us look at the portfolio value, which is basically a (self-financing) value process. So, what we have here, is that you know there is some X_0 and there is portfolio position given by Δ_t, which you all at each time t and its dynamics is what. Then, so, you are starting with the X_0 and this portfolio position plus the Δ_t. So, your dynamics of this X_t would be $\Delta_t dS_t$ plus now, you since you are to reinvesting.

So, this is the amount that you got, a $S_t dt$ is the amount that you got as dividend and since you hold delta t number of such stock. So, this is the total amount that you have and that you are reinvesting plus R, $X_t - \Delta_t S_t dt$. So, if I look at my self-financing portfolio values dynamics, right, this will be given by this, right. I cannot take out the dividend out of this portfolio value process, then the whole theory will fail.

Because it has to be in no arbitrage situation that starting with some initial naught and some Δ_t process given then you are looking at a self-financing value process, so that that equals the value of derivative at maturity that is what you look for. And since that is the requirement, right, so, I cannot take out the dividend. I have to reinvest the dividend and then look at the portfolio which does that and its value for replication, right, for making it equal interest, ok.

So, now this you can rewrite, now , $rX_t dt + (\alpha - r)\Delta_t S_t dt + \sigma \Delta_t S_t dW_t$, which can also be written as, is the usual. Now, so, this can be also be written as in the third line here, where θ is given, which is the usual market price of risk, ok. Now, we can define redefine $\tilde{W} = W_t + \theta t$, right, and through Girsanov, right. Define $\tilde{P}$ under which this is a Brownian motion, ok.

So, that implies that this could be written as $dX_t = rX_t dt + \Delta_t S_t \sigma d\tilde{W}$. So, what we do? It is exactly the same the marketplace of risk that we take as theta here as well, right. But here alpha is the rate of return of the stock with dividend reinvested. If you are really using the data to calibrate the model then that is what you do. With the dividend reinvested. What is the rate of return it is giving? That is what your alpha is, ok.

So, the rate of dividend is 'a', right. So, that is what we need to get and once we get that, then $\alpha - r = \sigma\theta$. that theta I will plug it in the Z_t that we define in Girsanov theorem to get to $\tilde{P}$ and under $\tilde{P}$, this particular quantity, which is return within the square bracket here will be a Brownian motion and hence you write this and now, with this what you can see, ok.

So, this implies now $d(e^{-rt}X_t) = \Delta_t e^{-rt} S_t \sigma d\tilde{W}$. So, this implies $\{e^{-rt}X_t\}$ is $\tilde{P}$ Martingale is what you see immediately, right.

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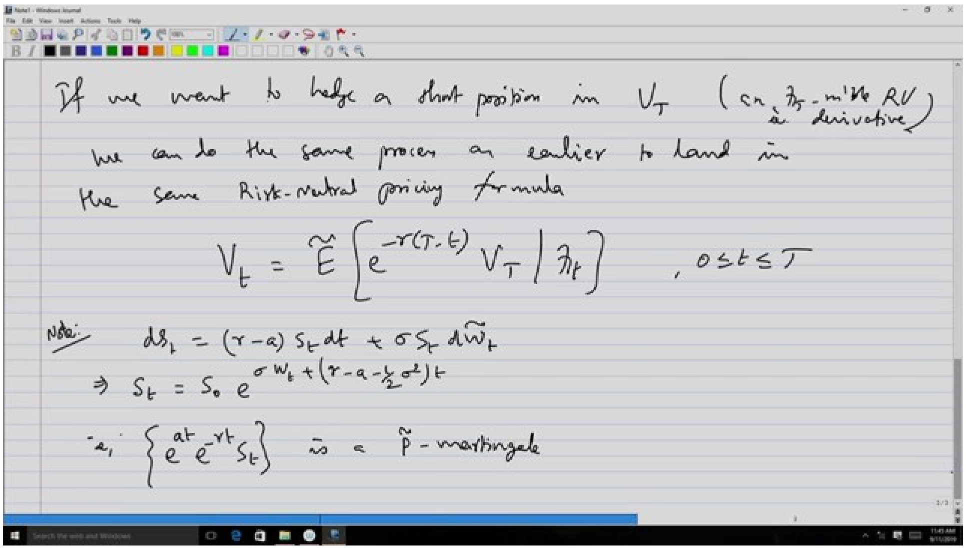

Then once we have this, so that now, if we wish. Now, if we want to hedge a short position in V_T and $\mathcal{F}_t$ measurable random variable that is a derivative, right. Now we can do the same process as earlier, right. What we did that we will need to choose an initial wealth X_0 and the portfolio process the Δ_t which gives us a portfolio value process X_t.

The fact that the discounted value of such a portfolio value process is a Martingale, gives us as earlier to land in the same, right, risk neutral pricing formula, right. So, such that it is X_t equal to V_t and that means that in your usual step of Martingale property will give us the risk neutral pricing formula Which is

$$V_t = \tilde{E}[e^{-r(T-t)} V_T | \mathcal{F}_t]$$

So, the process is similar once you use said that the discounted wealth process is a Martingale to show under some $\tilde{P}$ because, as you saw that is what gives us this neutral pricing formula. Once we have this step, the red underlined step, then from then onwards everything is the same as earlier. For any derivative you what the price then you will use the same idea to get to the risk neutral pricing formula which is given by this, right.

Now, as a note, so, you can see the difference between the dividend and no dividend cases is in the evolution of the stock price process, really did not make any change in the risk of neutral pricing but the stock price process S_t will have a different formula. now, that is what it is essentially. So, what we saw dS_t is essentially, your α, ok, if I want to write under this neutral measure. So this is we get. So, that implies that my S_t finally. is what you would get.

So, this is what it is and you know that this is not a Martingale in this particular case. because of this extra minus your term. So, if you discount even by that or you multiply I take out that term, ok. Then you could see, that is you could see that, this process e^{at} and the usual discounted price process. So, this is a $\tilde{P}$ Martingale, ok. So, the discounted value with growth with stocks reinvested would be a $\tilde{P}$ Martingale this what do you see, ok.

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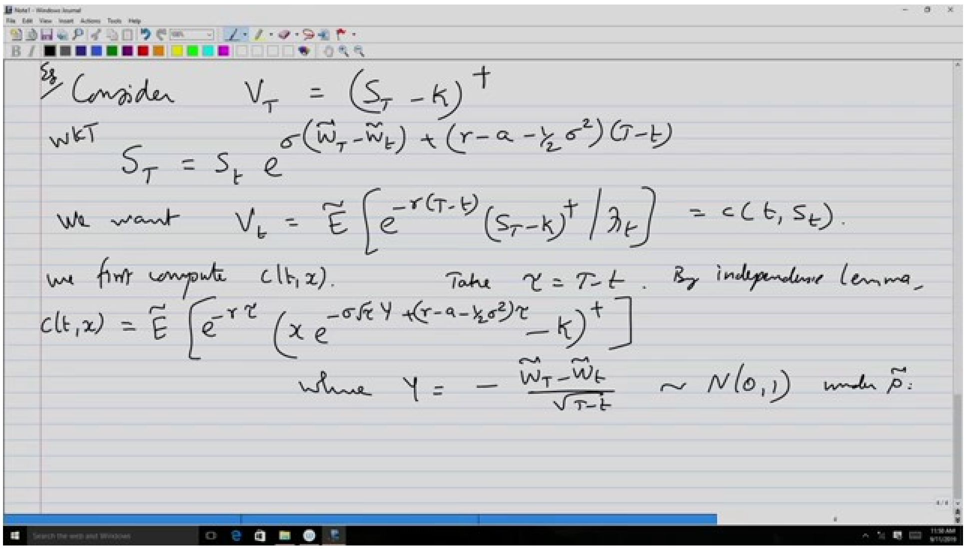

So, let us with this thing, then we will look at our typical the discounted pricing formula for the special case of European call option what happens, ok, right. So, what we have S_t is given by as given here, (oh sorry, this is the $t\sigma\tilde{W}$), ok. So, this actually, so, this is what we have here, this is the note part that is can observe that, this is what $\tilde{P}$ Martingale. Now, (we have something now,) we can consider this example my $V_T = (S_T - K)^+$.

Now, let us see what changes it makes to the original pricing formula that we derived in this particular case. So, let us look at this we are given we know that, my S_T is given above. So, look at, this is the solution or this is what the stock price process is given at time T, given that it is at S_t at time t. So, this is what the expression where what you see is an additional term of a, r was replaced by r minus a in some sense, ok.

Now, what we want, we want V_t, which is

$$V_t = \tilde{E}[e^{-r(T-t)}(S_t - K)_+ | \mathfrak{F}_t].$$

So, by our usual proper Markov property of this S_t, this will be some function of t and S_t, right. So, what we compute is that, we first compute the quantity $c(t,x)$ right, which we know how to do that. Now, S_t is as given the previous line and we need to compute this conditional expectation, then you can use the same process what we have followed earlier, right.

So, let us you know rewrite some of these crucial steps that we would have done in the earlier case, ok. Now for our simplicity take tau to t as usual T minus t, right. Now, if I apply by independence Lemma, ok. By independence Lemma I apply. I am not looking at the right side, but I am looking at this conditional expectation where the St is given by this where S_t is $\mathfrak{F}_t$ measurable and $\tilde{W}_T - W_t$ is independent of $\mathfrak{F}_t$ and hence, we can apply the independence Lemma to this conditional expectation expression.

Now, if I apply, then the functions $c(t,x)$ would be given by that. So, this is what would be my, this $c(t,x)$ function, where y is given above, which is normal(0,1), follows the normal standard variable and the probability measure $\tilde{P}$.

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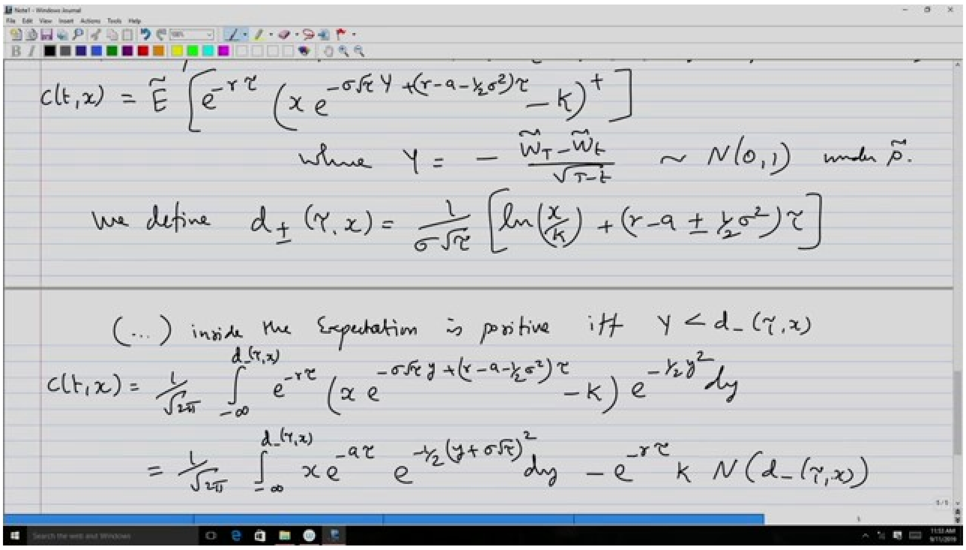

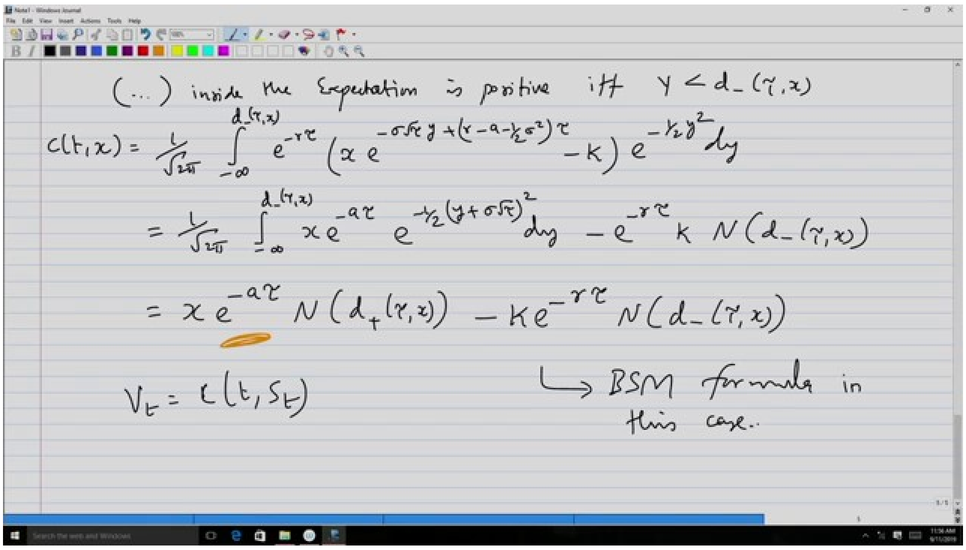

So, we also define as usual earlier, d_{+-}. d_+ involve the plus here d_- involve minus here, that is what is the expression, ok. Now, by that same argument that this is the, the quantity here inside this bracket, we have this positive part of it. This is this quantity, ok. So, the this quantity inside the expectation is positive, just like in the earlier case that we have observed, if and only if my $y < d_-(\tau,x)$, ok, (if I write).

So that means, then I can write my $c(t,x)$ to be in this particular form. You just to go through the similar step as to what we have done in the earlier case, ok. Except that there is a small minor change in the expression. This particular quantity, you can again rewrite, right.

So, if you just carry out the similar steps, what you would find is the first expression you can write it as this and the second expression will give you this expression. And if you make the change of variable $y + \sigma\tau$ as some dz and then simplify this. So, what you will end up with this, $e^{-a\tau}$, because this whole expression if we take out this minus a e to the power minus a times tau out then it is exactly same as the previous expression. Previous time when we did with the case.

$-Ke^{-rT}N(d_-(\tau,x))$, so, this is what the expression that you would get. So, hence my V_t is $c(t,S_t)$ where the X will get replaced by this. So, what is the change that you notice here, you notice the change in 2 places 1 is in the definition of T d plus and minus. So, you have an extra term here, which is minus a. And you also have an extra term here in the form, right.

So, this is the effect of that, but you have to re derive otherwise, you cannot simply replace R by r minus a. because that is not true in general, because in the second term and the expression that r will remain the same. So, it is not that way. You have to re-derive and but the deviation has this effect that this is what it is. So, this is the formula that you would find normally is applicable when there is a dividend rate is being applied, which means the continuously paying dividend rate is the case that you are considering.

Then you will have this idea and this is the, the pricing formula for the European call option. When the underlying stock pays, continuously pays a dividend at a continuous rate of a, then the time t arbitrage price is given by this BSM formula. So, this is the BSM formula in this case, right. So, this is the case of continuously paying dividends that we hear. So, what is the change that it has made? We have seen it here, ok.

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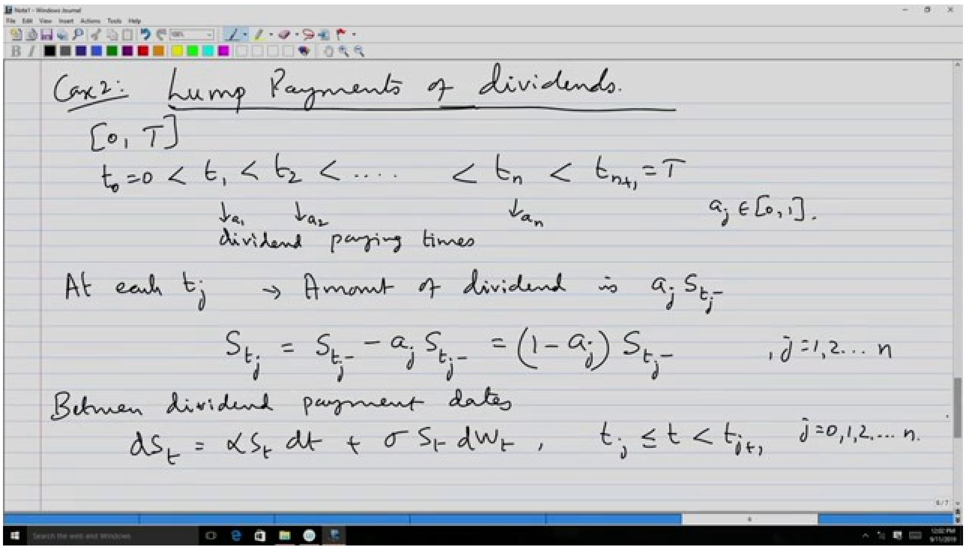

Now, look at the case 2, where discrete dividends or being paid are what 1 would call as lump sum or lump payments of dividends, ok. So, what you have you have $0 = t_0 < t_1 < t_2 < t_3 < \ldots < t_n < t_{n+1} = T$. So, these are the dividend paying dates or times, dividend paying times and dividend this paid at the rate, say here a_1, here a_2 and here a_n.

So, assume that in your interval of 0 to T that you are considering, there are some end time points at which dividends are being paid and the dividend rate at that point of time of payment are the, you know, the proportion is this a_1 and my a_j is as usual. Now, this is not the rate this is the lump payment of dividend case. So, in this particular case this is the proportion, right, is what then you have denoting it as a_1, a_2 dividend paying times $t_1, t_2, \ldots, t_n$ and these are the dividend proportions that you are paying at this point, right.

Now, between the dividend payments, ok, so, what happens at time? So, at each time, ok, at each time t_j, what happens? The amount of dividend is essentially $a_j S_{t_j^-}$. So, the amount of dividend is this and hence, we get $S_{t_j} = (1 - a_j) S_{t_j^-}$.

And between dividend payment dates what you have my dS_t your walls as this for $t_j \leq t \leq t_{j+1}$. So, between dividend payment dates the evolution of stock price process is the usual GBM, Geometric Brownian Motion dynamics. At t_j what happens, when the dividend is paid amount of dividend is we denote it as $a_j S_{t_j^-}$. So, you determine your proportion, right, between 0 and 1 and multiply by the stock that is the amount of dividends.

The stock price is reduced by the amount of dividend at time t_j, ok. So, now, this completely describes the dynamics of the stock price process at all times between 0 and t. What you have you started 0, 0 is not a dividend paying time. So, from 0 up to, but not including t1, the dynamics is given by the GBM dynamics, but at t_1 then the stock price process is given by this X dividend price which is S_{t_j} at t_j.

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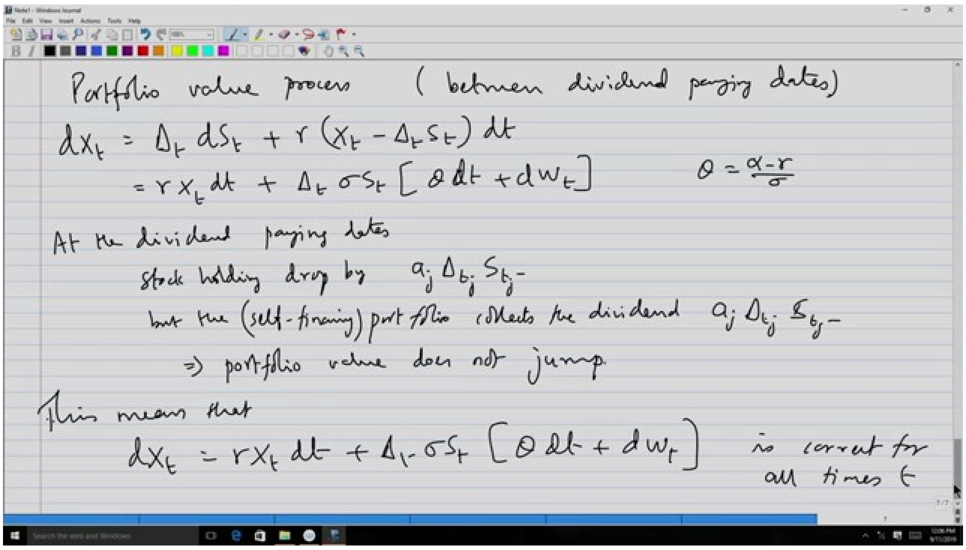

So, now, in this particular case, let us look at what happens or how do we write the stock price process, ok. So, what we have, dSt is this, ok. Now, what we do? we do the usual between different division promotion, so, suppose you start with the portfolio value process, suppose if you are looking at it corresponding to some delta t starting with some X_0, then this would be $\Delta_t dS_t + r(X_t - \Delta_t S_t)dt$, at all times, right.

So, this is essentially the process between dividend paying dates. So, this is again you can see that by substituting all those because there is no change with respect to the classical case that we have got it. $\Delta_t \sigma S_t [\theta dt + dWt]$, right. So, again the marketplace of risk is alpha minus r by sigma, ok. This is the portfolio value process between dividend payment dates, ok.

Now, at the dividend paying dates what happens, the stock price is reduced by, right. The value of the portfolio stock holdings drops by which amount, $a_j \Delta_{t_j} S_{t_j^-}$. But the portfolio collects the dividends because this amount is getting reinvested, ok. But the self financing portfolio collects the dividend. How much? Exactly the same amount; because nothing that is t_j^-. So, this implies that the portfolio value does not jump at this point, at the stock dividend, right.

So, that means, that implies. So, what does this means, this means that, this is correct for all times t, whether it is dividend paying times are non dividend paying times. So, this formula then is correct for all times t, because self-financing portfolio. The stock holdings dropped by a proportionate, a certain amount, but the portfolio collects the dividend. So, the portfolio value does not change exactly the same principle as in the other case that they have used, right.

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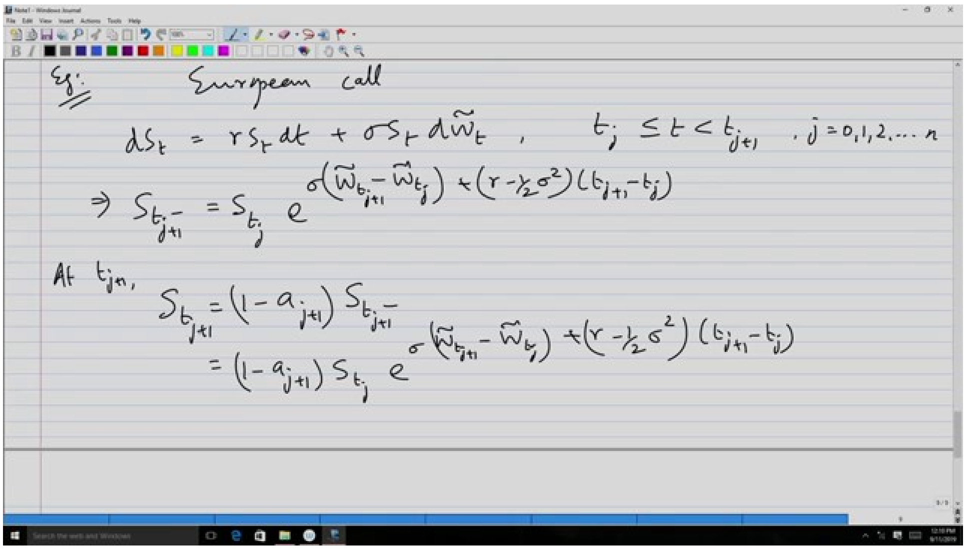

So, which means the usual risk neutral pricing formula holds, ok. Only thing is now we need to look at how the stock price behaves at various time points, how do we do the stock price in terms of by incorporating α or σ and these a_js for j is equal to 1 to n, ok. So, (what we have so, we have) the example case that we look at it, ok. So, now we will take the European call the the same one will take, we know, $dS_t = rS_tdt + \sigma S_t d\tilde{W}_t$.

Now, we can determine easily $S_{t_{j+1}^-}$. Also , at $S_{t_{j+1}} = (1-a)S_{t_{j+1}^-}$. The dynamics is the GBM dynamics.

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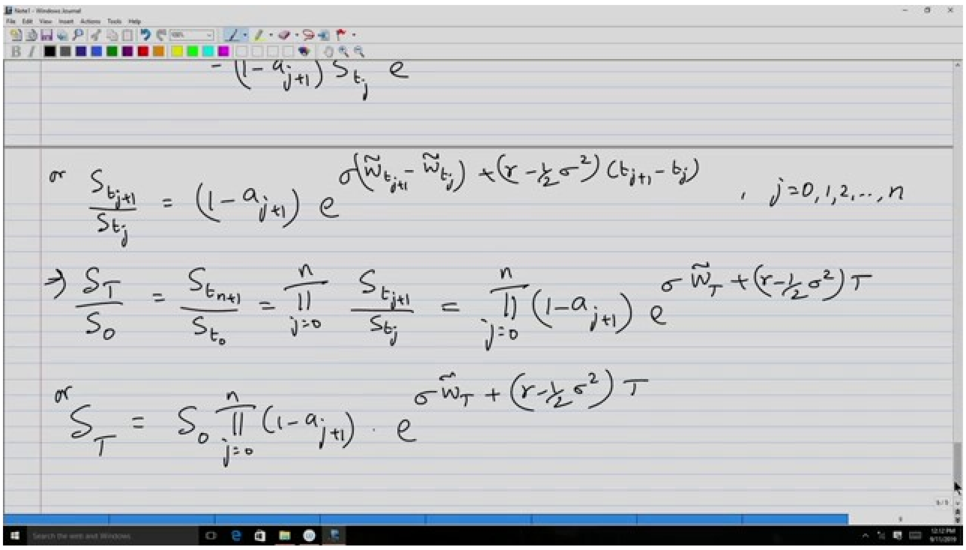

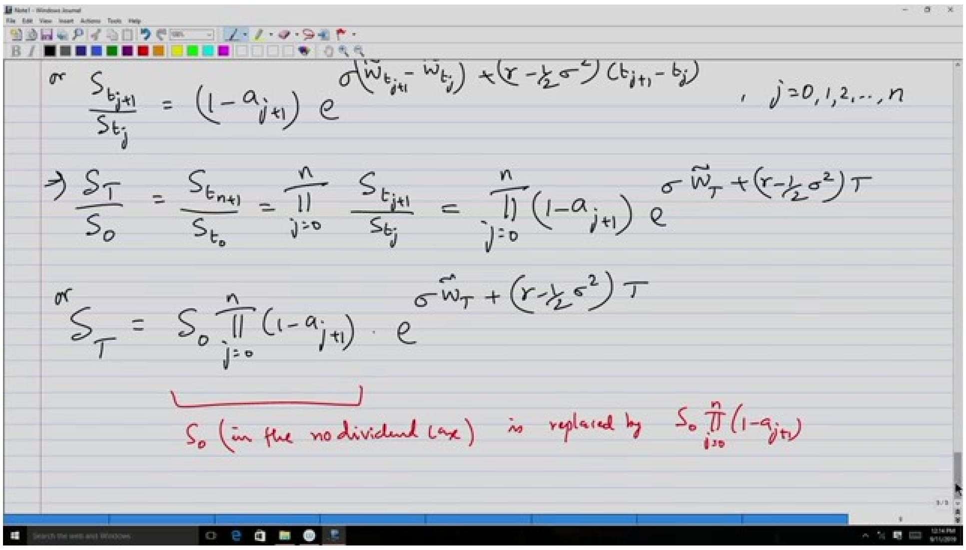

Or you can write in general, right, or $\frac{S_{t_{j+1}}}{S_{t_j}}$ is essentially that. And this is true for all of this 0 to n because $t_0 = 0$ and the $t_{n+1} = T$, that is what you have. So, then this implies my S_T/S_0 at 0 if I am looking at it, which is essentially $S_{t_{n+1}}/S_{t_0}$, which is nothing but my product of 0 to n $S_{t_{j+1}}/S_{t_j}$.

So, what we have done between 2 dividends dates it is GBM, but dividends dates the stock price makes a jump, right. It is it drops by a certain amount, a jump it makes, means something up and jump down. It makes a jump and by incorporating those jumps, now, we are given a final expression for S_T starting from 0 up to T, and how the stock price process can be represented in terms of all the parameters of the model, right. So, that is what we have seen.

So, what we are seeing here is, this is similar to the no dividend case with the change that my S_0 here, ok. So, in the, the classical case S_0 in the no dividend case is actually replaced by S_0 product of $(1-a_{j+1})$ where j is from 0 to n, right. This is what you observed, right, otherwise everything remain the same. So, this is what you are observing.

As if, you start with n S_0, which is reduced by this quantity, this product of $(1-a_{j+1})$because $(1-a_{j+1})$ is something which is less than or equal to 1 typically less than 1. Now, you are making a product of all such quantities, so, as if you are starting with the S_0 if you start with the S_0 multiplied by this quantity which is less than 1, typically less than 1 or less than an equal to 1 to be precise. Then this is what the stock price evolution is given by in this particular case.

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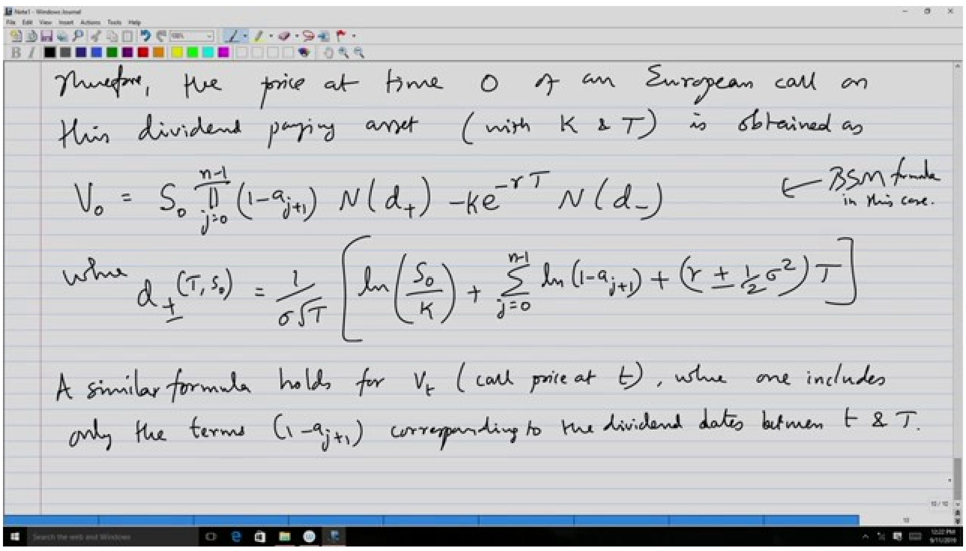

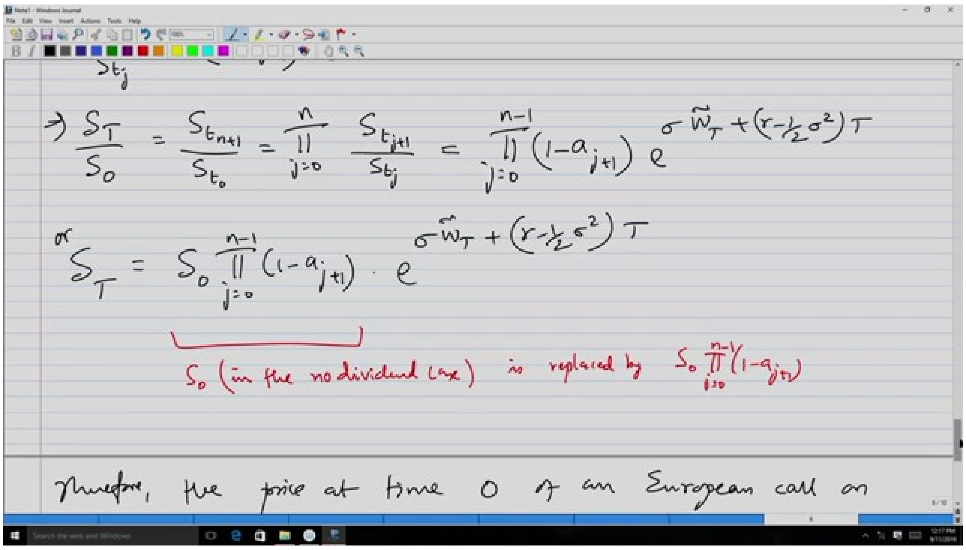

So, therefore, suppose if I adopt this, the price at time 0, (now for conveniently we take 0 for the time being), right. So, it is as if you are replacing S_0 by this quantity. so therefore, the price at times 0 of a European call on this dividend paying asset, ok, with K and T as the usual parameters, K is the strike and T is the maturity, right, is obtained as the original S_0 in that is replaced by this new S_0 because if you What it says is the evolution, that as if your S_0 is reduced by the quantity, then the complete evolution takes place. So, that is what coming out. So, this means that V at 0 in that particular case will be replaced by S_0 product of j is equal to 0 to,(sorry, there is some change, ok. There is some typo here,) essentially, 0 to 1, $S_{t_{j+1}}$, 0 to n if I take, this will be then I write this there is nothing like a $n+1$. So, it is an.

So, the last term is actually capital T tn plus so, I should get this product as a n minus 1. So, and hence this will also be a n minus 1, there is a typo correction that you may need to look at. So, this I think just note the correction that they have just made that see a up to ta the dividend paying times like there is t_{n+1} is not a dividend paying time which is equal to the capital T. Hence, this product j is equal to 1 to n running is essentially, if I write in this form then it is 0 to n minus 1 that is what you need to note, ok.

So, come back, so this is what you will have 1 minus aj plus 1, this is the new S_0 times N of d plus, minus e to the power minus rtK times of this is $N(d_{-})$, where my d plus minus essentially this would be equal to. So, anyway this is T and S_0 essentially, right. This 1 over sigma under root t because time 0 is what you are looking at, ok.

Log of S_0 by k the usual term, but plus now the old S_0 is replaced by the new S_0. So, the log of S_0 into product of this, so that becomes sum of the log of sum and again the product becomes the sum. So, ultimately you will get log of 1 minus aj plus 1 plus r plus or minus half sigma square times capital T. So, this is the time 0 price, ok. A similar formula holds for V_t, means the call price at t where 1 includes only the terms $1-a_{j+1}$ corresponding to the dividend dates between t and capital T, right.

So, here type 0 we have written for simplicity, which includes all the terms a 1 to a n. Now, see if you are writing this formula, right, so, you can use exactly the same thing instead of T now, I can write in this particular step instead of T, I can pick any t. Then I will look at only those products, which are up to that point of time or after that whatever is happening, right. So, up to that point of time whatever the dividends are there, which will not come into the picture.

So, this whole formula in the sum and in the product that we have written here, it will include only those terms of this $1-a_{j+1}$, which falls, which corresponds to the dividend paying dates, which fall between small t and capital T. Because it is 0 to capital T in this written dates, which includes all dates. Suppose if, it is falls after t1 for example, if you are looking at some time between t_1 and t_2, then the first term will not be part of this product and this sum that is a chain that you would find, ok.

So, no matter whatever it is, now, we could arrive at risk neutral pricing formula or the Black Scholes formula in this particular case, right. So, this is the BSM formula. In this case, with discrete dividend case or lump sum dividend case is what then you have here, ok. So, this is the modification or the change that you will make, when you want to incorporate dividends into the your analysis into your model for a dividend paying stock, what is the change that will happen, that is, ok.

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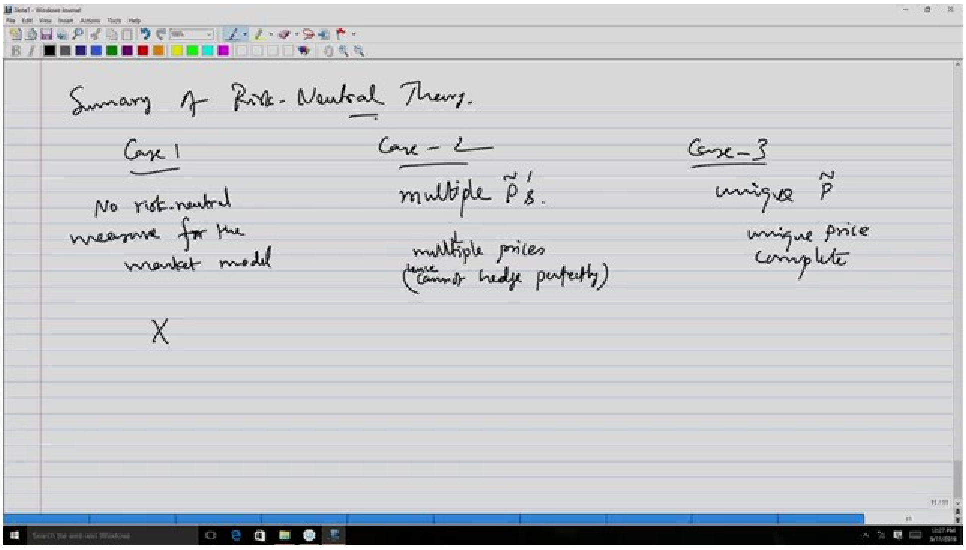

So, with this of course, then we may conclude our discussion on this preliminary ideas of the mathematical finance course. and as a summary of risk neutral theory, what you see is that there are 3 cases; case 1, case 2, case 3, if you want to look at. So, this comes from where from fundamental theorem of BSM pricing that is what, is the major importance of this, so no, in the model. So, no risk neutral measure in the or for the model, for the market model that you are considering. So, models should not be used. Case 2 you have multiple $\tilde{P}$'s, ok.

So, that makes multiple prices for the risk neutral measure. So, you cannot perfectly hedge, ok. You cannot hedge completely or perfectly in the underlying asset. but this is reality, ok. For example, we have not explored but of course, if you have idea you will know that the credit derivatives what one talks about, where you are bringing inside the model that underlying assets may are the defaultness, means that the default or not, when you bring inside then what will happen?

So, this is in that particular case for example, you will have multiple $\tilde{P}$'s, but then because you cannot hedge completely with the underlying asset itself because that itself is a default double asset. So, such cases will happen. So, this is what is multiple P and the ideal case, so to say is the unique $\tilde{P}$s, right. So, where there is a unique price and the model is completed, ok.

So, this is what you know we have with respect to, for example, the models that we have considered binominal model and the classical BSM model with 1 non-dividend paying asset or dividend paying asset, but that is tradable. Important point that it is tradable. so that you can take positions in the underlying asset. Suppose, that asset becomes a non tradable asset, which does happen in reality in the underlying need not be a tradable asset, then the model becomes incomplete, but dynamics but when we say classical BSM we assume the tradable asset, right.

So, all those models are complete models binominal, classical, BSM. The multi-dimensional model, of course, if you have a unique solution to those theta marketplace of risk equations, then you will have a unique $\tilde{P}$, ok. So, all these no risk neutral measures exist are the multiple $\tilde{P}$, unique $\tilde{P}$ essentially mean the marketplace of risky equations, whether it has no solution case or multiple solution case, or unique solution case, right you go back to that system of equations.

And completeness again as you as you understood based upon the hedging equations, whether it has a solution, ok which is what the second fundamental theorem guarantees that if it has a unique solution to the marketplace of this equation, then there is at least 1 solution to the hedging equations and hence the marketplace is completed. So, that is the ideal case, which is case 3. Case 2 is what but normally you would see in most real life situations, case 1 the model should not be used and hence, you have to change the model, because this model is bad for any usage, ok.

So, in these cases, the model could be used, but then the calibration and other issues will come case to case. But that is any way more, if you want to explore further you can look at all those issues that you have not here. So, this is a summary of risk neutral theory for you would have. So, with this, we end our discussion on this Black Scholes model. This is the continuous time this neutral pricing theory case. Hope that you know you have understood at least the very basic preliminary ideas behind the mathematical finance, which you have been explored to throughout this course. Thank you.